Quasi-Newton methods for Order-Value Optimization and Value-at-Risk calculations

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Abstract

The OVO (Order-Value Optimization) problem consists in the minimization of the order-value function $F_p(x)$, defined by

$$F_p(x) = f_{i_p(x)}(x),$$

where

$$f_{i_1(x)}(x) \leq \ldots \leq f_{i_m(x)}(x).$$

The functions $f_1, \ldots, f_m$ are defined on $\Omega \subset I\!R^n$ and p is an integer between 1 and m.

When x is a vector of portfolio positions and $f_i(x)$ is the predicted loss under the scenario i, the order-value function is the discrete Value-at-Risk (VaR) function, which is largely used in risk evaluations.

The OVO problem is continuous but nonsmooth. A Cauchy-like method with guaranteed convergence to points that satisfy a first order optimality condition was recently introduced by Andreani, Dunder

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and Martínez. In this paper a quasi-Newton method is introduced that generalizes the Cauchy method. Convergence proofs are given. The new method is applied to a Brazilian-oriented variation of Stiglitz's Ideal Banking System.

Key words: Order-value optimization, Value-at-Risk, quasi-Newton methods, Ideal Banking System.

1 Order-Value Optimization problem

Given m continuous functions $f_1, \ldots, f_m$, defined in a domain $\Omega \subset I\!R^n$ and an integer $p \in \{1, \ldots, m\}$, the ($p-$) Order-Value (OVO) function F_p is given by

$$F_p(x) = f_{i_p(x)}(x)$$

for all $x \in \Omega$, where

$$f_{i_1(x)}(x) \leq f_{i_2(x)}(x) \leq \ldots \leq f_{i_p(x)}(x) \leq \ldots \leq f_{i_m(x)}(x).$$

The OVO function is continuous. However, even if the functions f_i are differentiable, the function f may not be smooth.

The OVO problem consists in the minimization of the Order-Value function:

$$\text{Minimize } F_p(x) \quad \text{subject to} \quad x \in \Omega. \tag{1}$$

This problem has many applications:

1. Assume that Ω is a space of decisions and, for each $x \in \Omega$, $f_i(x)$ represents the cost of decision x under the scenario i. The Minimax decision corresponds to choose x in such a way that the maximum possible cost is minimized. This is a very pessimistic alternative and decision-makers may prefer to discard the worst possibilities in order to proceed in a more realistic way. When $f_i(x)$ represents the predicted loss for the set x of portfolio positions under the scenario i, the function $F_p(x)$ is the Value-at-Risk function, which is largely used in the finance industry [10].

2. Assume that we have a parameter-estimation problem where the space of parameters is Ω and $f_i(x)$ is the error corresponding to the observation i when the parameter x is adopted. The Minimax estimation problem corresponds to minimize the maximum error. As it is well

known this estimate is very sensitive to the presence of outliers [9]. Sometimes, we want to eliminate (say) the 15% larger errors because they can represent wrong observations. This leads to minimize the $p-$Order-Value function with $p \approx 0.85 \times m$. The OVO strategy is adequate to avoid the influence of systematic errors [1].

3. In the examples above, the useful OVO problems correspond to large values of p. In [4] applications of the "small p" OVO problem were given. They correspond to situations in which one wants to identify hidden patterns in the presence of a noisy set of data. In this application, a pattern is defined as a subset $S(x) \subset I\!R^n$. A set of data points $\{P_1, \ldots, P_m\} \subset I\!R^n$ is given. For each $i = 1, \ldots, m$, $f_i(x)$ is a "pseudo-distance" between P_i and $S(x)$. Only a small number of points belong to the correct hidden pattern $S(x_*)$, therefore the parameters x_* are obtained by solving the OVO problem for rather small values of p.

The OVO problem is difficult because the objective function is not smooth (even when the functions f_i are) and because the problem generally exhibits many local minimizers. In [1] a Cauchy-like method was introduced for solving OVO. At each iteration of the Cauchy method the functions f_i are approximated by linear functions. When Ω is a polytope a Linear Programming subproblem must be approximately solved at each iteration and the new point is obtained using a line search.

In [2] a reformulation of the OVO problem was introduced that transforms it into a large nonlinear programming problem. In this paper it was proved that local minimizers of the OVO problems are necessarily KKT points of the reformulation.

The cases $p = m$ and $p = 1$ deserve some comments. The OVO problem with $p = m$ is the classical Min-Max problem, largely used in applications. When $p = 1$ we have the Min-Min problem, in which one wants to minimize the minimum value of a set of m functions $f_1, \ldots, f_m$.

In this paper we generalize the results of [1]. Instead of approximating the functions f_i by linear functions, we approximate them by quadratics taking positive-semidefinite quasi-Newton Hessians. We obtain the same global convergent results of [1]. In addition, under some more restrictive assumptions, we obtain superlinear and quadratic convergence results.

We aim to apply OVO techniques to macroeconomic models where averse-to-risk behavior is expected. Here we consider the Ideal Banking System (IBS) of J. Stiglitz [14] (Nobel Prize of Economics in 2001). This model uses a portfolio approach to explain how banks decide on how much to lend, and how changes in regulations affect the supply of credit. We apply the quasi-Newton method to the IBS model and we show that the technique is generally effective, thus providing useful tools for simulation of bank behavior.

This paper is organized as follows. The main algorithm is presented in Section 2. In Section 3 we give a global convergence proof. Local convergence is discussed in Section 4. The application to the optimization of decisions in the Ideal Banking System of Stiglitz is given in Section 5. The description of the computer implementation and numerical results are in Section 6. In Section 7 we state some conclusions, we state open problems and we suggest lines for future research.

Notation.

$\|\cdot\|$ is the 2-norm of vectors and matrices;

$\mathcal{B}(z, \delta) = \{x \in I\!R^n \mid \|x - z\| < \delta\}$;

$I\!N = \{0, 1, 2, \ldots\}$;

$I\!R^n_+ = \{x \in I\!R^n \mid x \geq 0\}$;

$I\!R^n_{++} = \{x \in I\!R^n \mid x > 0\}$.

The symbol $[v]_i$ denotes the i-th component of the vector v in the cases in which the usual notation v_i can lead to confusion.

We denote $g_j(x) = \nabla f_j(x)$.

$\mathcal{N}(A)$ denotes the null-space of the matrix A.

$\mathcal{R}(Z)$ denotes the column-space of the matrix Z.

2 Main algorithm

We will consider the OVO problem (1) in the case in which Ω is closed and convex and the functions f_i are continuously differentiable. From now on, p will be fixed and, in order to simplify the notation, we write:

$$f(x) = F_p(x) \;\forall\; x \in \Omega.$$

As in [1], let us define, for all $\varepsilon \geq 0$, $x \in \Omega$:

$$I_\varepsilon(x) = \{j \in \{1, \ldots, m\} \mid f(x) - \varepsilon \le f_j(x) \le f(x) + \varepsilon\}.$$

Definition. We say that x is $\varepsilon-$stationary (called ε-optimal in [1]) if

$$\mathcal{D}(x) \equiv \{d \in I\!R^n \mid x + d \in \Omega \text{ and } g_j(x)^T d < 0 \ \ \forall\, j \in I_\varepsilon(x)\} = \emptyset.$$

In [1] it was proved that any local minimizer x of (1) is $\varepsilon-$stationary for all $\varepsilon \ge 0$.

The main algorithm considered in this paper is given below. The algorithm generates feasible points $x_k \in \Omega$. Given the current approximation x_k, the idea is to define quadratic approximations of all the functions belonging to $I_\varepsilon(x_k)$. Then, the maximum of these quadratics is approximately minimized on a restricted feasible set. In this way a search direction is obtained, along which we seek sufficient decrease of $f(x)$.

Algorithm 2.1

Let $x_0 \in \Omega$ be an arbitrary initial point. Let $\theta \in (0,1)$, $\Delta > 0$, $\varepsilon > 0$, $0 < \sigma_{min} < \sigma_{max} < 1$, $\eta \in (0,1]$, $c_B > 0$.

Assume that $x_k \in \Omega$ and let $\{B_{k,1}, \ldots, B_{k,m}\}$ be symmetric and positive semidefinite matrices such that $\|B_{k,j}\| \le c_B$ for all $j = 1, \ldots, m$. The steps of the $k-$th iteration are:

Step 1. (Solving the subproblem)

Define, for all $d \in I\!R^n$,

$$M_k(d) \equiv \max_{j \in I_\varepsilon(x_k)} \{g_j(x_k)^T d + \frac{1}{2} d^T B_{k,j} d\}.$$

Consider the subproblem

$$\text{Minimize } M_k(d) \text{ subject to } x_k + d \in \Omega, \ \|d\|_\infty \le \Delta. \tag{2}$$

Let $\bar{d}_k$ be a solution of (2). Let $d_k \in I\!R^n$ be such that $x_k + d_k \in \Omega$, $\|d_k\|_\infty \le \Delta$ and

$$M_k(d_k) \le \eta M_k(\bar{d}_k). \tag{3}$$

If $M_k(d_k) = 0$ stop.

Step 2. (Steplength calculation)

Set $\alpha \leftarrow 1$.

If

$$f(x_k + \alpha d_k) \le f(x_k) + \theta\alpha M_k(d_k) \tag{4}$$

set $\alpha_k = \alpha$, $x_{k+1} = x_k + \alpha_k d_k$ and finish the iteration. Otherwise, choose $\alpha_{new} \in [\sigma_{min}\alpha, \sigma_{max}\alpha]$, set $\alpha \leftarrow \alpha_{new}$ and repeat the test (4).

Remarks

- The subproblem (2) is equivalent to the convex optimization problem

$$\text{Minimize } w$$
$$\text{subject to}$$
$$g_j(x_k)^T d + \frac{1}{2} d^T B_{k,j} d \le w \ \forall\, j \in I_\varepsilon(x_k),$$
$$x_k + d \in \Omega, \ \ \|d\|_\infty \le \Delta.$$

  Since we assume that the matrices $B_{k,j}$ are positive semidefinite (not necessarily positive definite), the step control $\|d\|_\infty \le \Delta$ is necessary to ensure that the subproblem has a solution. Clearly, if Ω is bounded this constraint is not necessary, since the requirement $x_k + d \in \Omega$ forces its fulfillment for Δ large enough.

- The computation of the exact minimizer of the subproblem $\bar{d}_k$ is not necessary. The fulfillment of (3) can be obtained using duality arguments as in [1].

3 Global convergence

In this section we prove that every limit point of a sequence generated by Algorithm 2.1 is ε-stationary. We will use the following general assumption.

Assumption A0. *The set Ω is convex, closed and bounded. Moreover, there exist $c > 0, L > 0$ such that, for all $x, y \in \Omega$, $j = 1, \dots, m$,*

$$\|g_j(x)\|_\infty \le c, \ \ \|g_j(y) - g_j(x)\|_\infty \le L\|y - x\|_\infty. \tag{5}$$

From (5) it follows that, for all $x, y \in \Omega$,

$$f_j(y) \le f_j(x) + g_j(x)^T(y - x) + \frac{L}{2}\|y - x\|_\infty^2 \ \forall\, j = 1, \dots, m. \tag{6}$$

Clearly, this assumption is satisfied if Ω is bounded and the functions f_j have continuous second derivatives on Ω.

The following theorem says that Algorithm 2.1 is well defined. In other words, given an iterate $x_k \in \Omega$, either the algorithm stops at x_k because x_k is ε-stationary or the iteration finishes after a finite number of reductions of α with a point that satisfies (4). Moreover, we give a lower bound for the steplength α_k. The proof is similar to the one of Theorem 2.3 of [1].

Theorem 3.1. *Assume that* **A0** *holds and* $x_k \in \Omega$ *is the* $k-th$ *iterate of Algorithm 2.1. Then:*

(a) The algorithm stops at x_k *if, and only if,* x_k *is* $\varepsilon-$*stationary.*

(b) If the algorithm does not stop at x_k*, then the iteration is well defined and*

$$\alpha_k \geq \min\left\{\frac{2\sigma_{min}\gamma_k(1-\theta)}{L\Delta^2}, \frac{\varepsilon\sigma_{min}}{3c\Delta}\right\},$$

where

$$\gamma_k = -M_k(d_k) > 0.$$

Proof. If the algorithm stops at x_k, then $M_k(d_k) = 0$. Therefore, by (3), $M_k(\bar{d}_k) = 0$. So, $M_k(d) \geq 0$ for all $d \in \mathcal{D}(x_k)$ such that $\|d\|_\infty \leq \Delta$. Thus, by the convexity of the function M_k, $M_k(d) \geq 0$ for all $d \in \mathcal{D}(x_k)$. This implies that x_k is ε-stationary.

Reciprocally, if x_k is ε-stationary, we must have that $M_k(\bar{d}_k) = 0$, so $M_k(d_k) = 0$ and the algorithm stops at x_k.

If the algorithm does not stop at x_k, then $M_k(d_k) < 0$. Therefore,

$$-\gamma_k = \max_{j \in I_\varepsilon(x_k)} \{g_j(x_k)^T d_k + \frac{1}{2} d_k^T B_{k,j} d_k\} < 0.$$

Assume that

$$\alpha \in \left[0, \frac{2\gamma_k(1-\theta)}{L\Delta^2}\right].$$

Then,

$$\frac{L\alpha\Delta^2}{2} \leq (1-\theta)\gamma_k.$$

Thus,

$$\frac{L\alpha\Delta^2}{2} \leq (\theta - 1)\left[g_j(x_k)^T d_k + \frac{1}{2} d_k^T B_{k,j} d_k\right] \ \forall \ j \in I_\varepsilon(x_k).$$

Therefore,

$$g_j(x_k)^T d_k + \frac{1}{2} d_k^T B_{k,j} d_k + \frac{L\alpha\Delta^2}{2} \leq \theta\left[g_j(x_k)^T d_k + \frac{1}{2} d_k^T B_{k,j} d_k\right] \ \forall \ j \in I_\varepsilon(x_k).$$

Then, since $\|d_k\|_\infty \leq \Delta$, $\alpha \leq 1$ and $B_{k,j} \geq 0$,

$$\alpha g_j(x_k)^T d_k + \frac{1}{2}(\alpha d_k)^T B_{k,j}(\alpha d_k) + \frac{L\alpha^2\|d_k\|_\infty^2}{2} \leq$$
$$\alpha\theta\left[g_j(x_k)^T d_k + \frac{1}{2} d_k^T B_{k,j} d_k\right] \ \forall \ j \in I_\varepsilon(x_k).$$

Thus,

$$\begin{aligned} &f_j(x_k) + g_j(x_k)^T(\alpha d_k) + \frac{1}{2}(\alpha d_k)^T B_{k,j}(\alpha d_k) + \frac{L}{2}\|\alpha d_k\|_\infty^2 \\ &\leq f_j(x_k) + \alpha\theta\left[g_j(x_k)^T d_k + \frac{1}{2} d_k^T B_{k,j} d_k\right] \ \forall \ j \in I_\varepsilon(x_k). \end{aligned} \tag{7}$$

But, by (6),

$$f_j(x_k + \alpha d_k) \leq f_j(x_k) + g_j(x_k)^T(\alpha d_k) + \frac{L}{2}\|\alpha d_k\|_\infty^2. \tag{8}$$

Therefore, by (7) and (8),

$$f_j(x_k + \alpha d_k) \leq f_j(x_k) + \alpha\theta\left[g_j(x_k)^T d_k + \frac{1}{2} d_k^T B_{k,j} d_k\right] \ \forall \ j \in I_\varepsilon(x_k).$$

We proved that, if $\alpha \in [0, 2\gamma_k(1-\theta)/(L\Delta^2)]$,

$$f_j(x_k + \alpha d_k) \leq f_j(x_k) + \alpha\theta M_k(d_k) \ \forall \ j \in I_\varepsilon(x_k).$$

If, in addition, $\alpha \in [0, \varepsilon/(3c\Delta)]$ we deduce, exactly as in Theorem 2.3 of [1], that

$$f(x_k + \alpha d_k) \leq f(x_k) + \alpha\theta M_k(d_k).$$

Therefore, if $\alpha \in [0, \min\{2\gamma_k(1-\theta)/(L\Delta^2), \varepsilon/(3c\Delta)]$, the test (4) must hold. This means that a value of α that does not satisfy (4) cannot be smaller than $\min\{2\gamma_k(1-\theta)/(L\Delta^2), \varepsilon/(3c\Delta)\}$. So, the accepted α must satisfy:

$$\alpha_k \geq \min\left\{\frac{2\sigma_{min}\gamma_k(1-\theta)}{L\Delta^2}, \frac{\varepsilon\sigma_{min}}{3c\Delta}\right\},$$

as we wanted to prove. $\square$

The main convergence result is given in Theorem 3.2. First, we need to prove a simple preparatory lemma.

Lemma 3.1. *Assume that* **A0** *holds and* $\{x_k\}$ *is an infinite sequence generated by Algorithm 2.1, then either*

$$\lim_{k\to\infty} f(x_k) = -\infty \tag{9}$$

or

$$\lim_{k\to\infty} M_k(d_k) = 0. \tag{10}$$

Proof. If (9) does not hold, then, by (4), we have that

$$\lim_{k\to\infty} \alpha_k M_k(d_k) = 0. \tag{11}$$

If $\lim_{k\to\infty} M_k(d_k) = 0$ we are done. Otherwise, there exists an infinite sequence of indices $K_1 \subset I\!N$ such that $\{M_k(d_k)\}_{k\in K_1}$ is bounded away from zero. Then, by (11),

$$\lim_{k\in K_1} \alpha_k = 0.$$

So, by Theorem 3.1,

$$\lim_{k\in K_1} \gamma_k = 0.$$

By the definition of γ_k this implies that $\lim_{k\in K_1} M_k(d_k) = 0$. So, the Lemma is proved. $\square$

Theorem 3.2. *Assume that* **A0** *holds and* $x_* \in \Omega$ *is a limit point of a sequence generated by Algorithm 2.1. Then* x_* *is* ε*-stationary.*

Proof. Since $f(x_{k+1}) \le f(x_k)$ for all k and x_* is a limit point of $\{x_k\}$ then

$$\lim_{k\to\infty} f(x_k) = f(x_*).$$

Therefore, by Lemma 3.1,

$$\lim_{k\to\infty} M_k(d_k) = 0.$$

So, by (3),

$$\lim_{k\to\infty} M_k(\bar{d}_k) = 0.$$

Let K be an infinite sequence of indices such that

$$\lim_{k \in K} x_k = x_*.$$

Suppose, by contradiction, that x_* is not ε-stationary. Then, there exists $\gamma > 0$ and $d \in I\!R^n$ such that $x_* + d \in \Omega$ and

$$g_j(x_*)^T d \leq -\gamma \;\; \forall \;\; j \in I_\varepsilon(x_*).$$

Let $R = \max\{1, \Delta^2 n c_B/(4\gamma)\}$. So,

$$\frac{\Delta^2 n c_B}{8R^2} \leq \frac{\gamma}{2R}. \tag{12}$$

Without loss of generality we may assume that $\|d\|_\infty \leq \Delta/2$. Then, by (12),

$$g_j(x_*)^T \left(\frac{d}{R}\right) + \frac{n}{2}\left\|\frac{d}{R}\right\|_\infty^2 c_B \leq -\frac{\gamma}{2R} \;\; \forall \;\; j \in I_\varepsilon(x_*), k = 0, 1, 2, \ldots.$$

Define $\hat{\gamma} = \dfrac{\gamma}{2R} > 0$ and $\hat{d} = \dfrac{1}{R} d \in I\!R^n$. Since Ω is convex and $R \geq 1$ we have that $x_* + \hat{d} \in \Omega$, $\|\hat{d}\|_\infty \leq \Delta/2$, and

$$g_j(x_*)^T \hat{d} + \frac{n\|\hat{d}\|_\infty^2 c_B}{2} \leq -\hat{\gamma} \;\; \forall \;\; j \in I_\varepsilon(x_*). \tag{13}$$

By continuity, for k large enough, $k \in K$, defining

$$\hat{d}_k = \hat{d} + x_* - x_k,$$

we have that $\|\hat{d}_k\|_\infty \leq \Delta$, $x_k + \hat{d}_k \in \Omega$ and

$$\lim_{k \in K} \hat{d}_k = \hat{d}.$$

By (10), we have that $\lim_{k\to\infty} M_k(\bar{d}_k) = 0$. Therefore,

$$\liminf_{k\to\infty} M_k(\hat{d}_k) \geq 0. \tag{14}$$

Since $\{1, \ldots, m\}$ is finite, there exists $j_0 \in I_\varepsilon(x_k)$ such that

$$g_{j_0}(x_k)^T \hat{d}_k + \frac{1}{2}\hat{d}_k^T B_{k,j_0} \hat{d}_k = M_k(\hat{d}_k) \tag{15}$$

for all k belonging to an infinite sequence of indices $K_1 \subset K$.

Clearly,

$$f(x_k) - \varepsilon \leq f_{j_0}(x_k) \leq f(x_k) + \varepsilon$$

for all $k \in K_1$. Taking limits in these inequalities and using the continuity of f, we obtain:

$$f(x_*) - \varepsilon \leq f_{j_0}(x_*) \leq f(x_*) + \varepsilon.$$

Therefore,

$$j_0 \in I_\varepsilon(x_*). \tag{16}$$

Since $\|B_{k,j}\| \leq c_B$ for all k, j, there exists an infinite sequence of indices $K_2 \subset K_1$ and a symmetric matrix B, $\|B\| \leq c_B$, such that

$$\lim_{k \in K_2} B_{k,j_0} = B. \tag{17}$$

By (14) and (17), taking limits in (15) for $k \in K_2$, we have:

$$g_{j_0}(x_*)^T \hat{d} + \frac{1}{2} \hat{d}^T B \hat{d} \geq 0. \tag{18}$$

Since $\|B\| \leq c_B$, the inequalities (13) and (18) imply that $j_0 \notin I_\varepsilon(x_*)$. This contradicts (16). $\square$

4 Local convergence

In this section we prove that, under suitable assumptions, the sequence generated by Algorithm 2.1 converges superlinearly (or quadratically) to a limit point x_*. We will restrict ourselves to the case in which Ω is described by a set of linear inequalities. Different assumptions will be necessary for different results along this section. These assumptions will be stated as soon as they are needed in the proofs.

Assumption **A1** says that $\{x_k\}$ is an infinite sequence generated by Algorithm 2.1, x_* is a limit point and x_* is isolated in two senses. On one hand, it is the unique stationary point in a ball $\mathcal{B}(x_*, \delta)$. On the other hand, for all the points on this ball, the set $I_\varepsilon(x)$ contains only one index.

The technique used in this section consists of reducing, as much as possible, the OVO problem to the minimization of a single smooth function and to mimic the results proved in [3] for a variable metric method with linear constraints. See, for completeness, [5].

Assumption A1. *There exists a point* $x_* \in \Omega$*, a sequence* $\{x_k\}$ *generated by Algorithm 2.1 and an infinite set of indices* $K \subset \{0, 1, 2 \ldots\}$ *such that*

$$\lim_{k \in K} x_k = x_*.$$

Moreover, there exists $\varepsilon > 0$*,* $\delta > 0$*,* $j_0 \in \{1, \ldots, m\}$ *such that* x_* *is the unique* ε*-stationary point of (1) that belongs to* $\mathcal{B}(x_*, \delta)$ *and, for all* $x \in \Omega \cap \mathcal{B}(x_*, \delta)$*,*

$$I_\varepsilon(x) = \{j_0\}.$$

The second assumption says that the matrices $B_{k,j}$ are positive definite (not merely positive semidefinite as assumed in the previous section) and both $\|B_{k,j}\|$ and $\|B_{k,j}^{-1}\|$ are bounded.

Assumption A2. *For all* $k = 0, 1, 2, \ldots$*,* $j = 1, \ldots, m$*, the matrix* $B_{k,j}$ *is positive definite,* $\|B_{k,j}\| \leq c_B$ *and* $\|B_{k,j}^{-1}\| \leq c_B$*.*

Theorem 4.1. *Assume that* **A0 − A2** *hold. Then,*

$$\lim_{k \in K} \|d_k\| = \lim_{k \in K} \|\bar{d}_k\| = \lim_{k \in K} \|x_{k+1} - x_k\| = 0.$$

and

$$\lim_{k \to \infty} x_k = x_*. \tag{19}$$

Proof. Suppose that $\lim_{k \in K} \|d_k\| = 0$ is not true. Since $\|d_k\|_\infty \leq \Delta$ for all $k \in I\!N$, we can take a subsequence $K_1 \subset K$ and $\gamma > 0$ such that

$$x_k + d_k \in \Omega \ \ \forall\, k \in K_1,$$

$$\|d_k\| \geq \gamma > 0 \ \ \forall\, k \in K_1,$$

$$\lim_{k \in K_1} B_{k,j_0} = B. \tag{20}$$

Moreover, B is symmetric and positive definite and

$$\lim_{k \in K_1} d_k = d \neq 0. \tag{21}$$

Since $\{x_k\}$ admits a limit point, by Lemma 3.1 we have that

$$\lim_{k \to \infty} M_k(d_k) = 0.$$

But, by Assumption **A1**, for $k \in K$ large enough,

$$M_k(d_k) = \frac{1}{2} d_k^T B_{k,j_0} d_k + g_{j_0}(x_k)^T d_k.$$

Therefore, taking limits for $k \in K_1$ and using (20) and (21), we get:

$$\frac{1}{2} d^T B d + g_{j_0}(x_*)^T d = 0.$$

Since B is positive definite and $d \neq 0$ this implies that

$$g_{j_0}(x_*)^T d < 0.$$

By Assumption **A1**, this means that $\mathcal{D}(x_*) \neq \emptyset$. This contradicts the fact that x_* is ε-stationary.

Therefore, $\lim_{k \in K} \|d_k\| = 0$. The fact that $\lim_{k \in K} \|x_{k+1} - x_k\| = 0$ follows immediately from the definition of Algorithm 2.1. Moreover, by (3),

$$0 \geq M_k(\bar{d}_k) \geq \frac{1}{\eta} M_k(d_k),$$

so $\lim_{k \to \infty} M_k(\bar{d}_k) = 0$. Using this fact, the proof that $\lim_{k \to \infty} \|\bar{d}_k\| = 0$ follows as the proof of $\lim_{k \to \infty} \|d_k\| = 0$.

Since $\lim_{k \in K} \|x_{k+1} - x_k\| = 0$, the proof of (19) follows as in Theorem 3.1 of [3]. $\square$

Assumption A3. *The feasible set* Ω *is given by*

$$\Omega = \{x \in \mathbb{R}^n \mid Ax \leq b\}.$$

The matrix A *has* q *rows and* n *columns. Without loss of generality, we assume that the active constraints at* x_* *are the ones corresponding to the first* s *rows of* A.

From now on, we define:

$$A = (a_1, \ldots, a_q)^T, \quad b = (b_1, \ldots, b_q)^T,$$

$$\bar{A} = (a_1, \ldots, a_s)^T, \quad \bar{b} = (b_1, \ldots, b_s)^T.$$

The following lemma says that $\bar{d}_k$, the solution of the subproblem, must satisfy KKT conditions if k is large enough. The reason is that, by Assumption **A1**, the subproblem consists in the minimization of a quadratic with linear constraints in a neighborhood of x_*.

Lemma 4.1. *Assume that* **A0** $-$ **A3** *hold. Then, for all* $k \in I\!N$ *large enough, there exists* $\lambda_k \in I\!R^q_+$ *such that*

$$B_{k,j_0}\bar{d}_k + g_{j_0}(x_k) + A^T\lambda_k = 0, \tag{22}$$

$$A(x_k + \bar{d}_k) \leq b, \tag{23}$$

$$\lambda_k^T[A(x_k + \bar{d}_k) - b] = 0. \tag{24}$$

Proof. By **A1**, for k large enough, $\bar{d}_k$ is the solution of

$$\text{Minimize } \frac{1}{2}d^T B_{k,j_0} d + g_{j_0}(x_k)^T d$$

subject to $A(x_k + d) \leq b$, $\|d\|_\infty \leq \Delta$.

But, by Theorem 4.1, $\|\bar{d}_k\|_\infty < \Delta$ for k large enough. Therefore, $\bar{d}_k$ must satisfy the optimality conditions of the problem

$$\text{Minimize } \frac{1}{2}d^T B_{k,j_0} d + g_{j_0}(x_k)^T d$$

subject to $A(x_k + d) \leq b$.

These conditions are (22)–(24). □

With the same assumptions of Lemma 4.1 we are able to prove the following result. It says that the point x_* satisfies KKT conditions related to the minimization of f_{j_0}. Again, the isolation Assumption **A1** is the crucial argument that justifies this fact.

Lemma 4.2. *Assume that* **A0** $-$ **A3** *hold. Then, there exists* $\bar{\lambda}_* \in I\!R^s_+$ *such that*

$$g_{j_0}(x_*) + \bar{A}^T\bar{\lambda}_* = 0,$$

$$\bar{A}x_* = \bar{b},$$

$$a_i^T x_* < b_i, i = s+1, \ldots, q.$$

Proof. By Assumption **A1**, the optimality condition implies that x_* must be a KKT point of

$$\text{Minimize } f_{j_0}(x)$$

subject to $Ax \leq b$. Therefore, the thesis follows from **A3**. □

The next two assumptions say, respectively, that strict complementarity conditions hold and that the gradients of the active constraints are linearly independent at x_* .

Assumption A4. *The multipliers at the solution are such that*

$$[\bar{\lambda}_*]_i > 0 \ \ \forall \ \ i = 1, \dots, s.$$

Assumption A5. *The vectors $a_1, \dots, a_s$ are linearly independent. (In particular, $s \leq n$.)*

The following lemma says that, under the given assumptions, the active constraints at the points $x_k + \bar{d}_k$ coincide with the active constraints at x_*. Moreover, the Lagrange multipliers at the solution can be approximated by the Lagrange multipliers associated to $\bar{d}_k$.

Lemma 4.3. *Assume that* **A0–A5** *hold. Then,*

1. *There exist $k_0 \in I\!N$ and $\{\bar{\lambda}_k\}_{k \geq k_0} \subset I\!R^s_{++}$ such that, for all $k \geq k_0$,*

$$B_{k,j_0}\bar{d}_k + g_{j_0}(x_k) + \bar{A}^T\bar{\lambda}_k = 0, \tag{25}$$

$$\bar{A}(x_k + \bar{d}_k) = \bar{b} \tag{26}$$

and

$$a_i^T(x_k + \bar{d}_k) < b_i \ \ \forall \, i = s+1, \dots, m. \tag{27}$$

2.

$$\lim_{k \to \infty} [\bar{\lambda}_k]_i = [\bar{\lambda}_*]_i \ \ \forall \, i = 1, \dots, s.$$

Proof. Since $\bar{d}_k \to 0$ and $a_i^T x_* < b_i$ for all $i = s+1, \dots, m$, we have that (27) holds for k large enough. Therefore, by (24),

$$[\lambda_k]_i = 0 \, \forall \, i = s+1, \dots, m.$$

By (22), defining $\bar{\lambda}_k = ([\lambda_k]_1, \dots, [\lambda_k]_s)^T$, the equation (25) also holds.

Let us prove that (26) holds for k large enough. Observe first that, by **A5**,

$$\bar{\lambda}_* = -(\bar{A}\bar{A}^T)^{-1}\bar{A}g_{j_0}(x_*). \tag{28}$$

By (25),

$$\bar{A}B_{k,j_0}\bar{d}_k + \bar{A}g_{j_0}(x_k) + \bar{A}\bar{A}^T\bar{\lambda}_k = 0.$$

Moreover, by the nonsingularity of $\bar{A}\bar{A}^T$:

$$(\bar{A}\bar{A}^T)^{-1}\bar{A}B_{k,j_0}\bar{d}_k + (\bar{A}\bar{A}^T)^{-1}\bar{A}g_{j_0}(x_k) + \bar{\lambda}_k = 0.$$

Taking limits and using that $\bar{d}_k \to 0$ we obtain that the sequence $\bar{\lambda}_k$ is convergent and

$$\lim_{k\to\infty} \bar{\lambda}_k = -(\bar{A}\bar{A}^T)^{-1}\bar{A}g_{j_0}(x_*).$$

By (28), this implies that

$$\lim_{k\to\infty} \bar{\lambda}_k = \bar{\lambda}_*.$$

By **A4**, $\bar{\lambda}_* > 0$. Then, $\bar{\lambda}_k > 0$ for all k large enough. By (24) the equality (26) also holds. □

The following assumption concerns the implementation of the algorithm. It says that we choose $d_k = \bar{d}_k$ as the solution of each subproblem, at least when k is large enough.

Assumption A6. *For all k large enough, $d_k = \bar{d}_k$.*

In the following lemma we prove that, if the active constraints at some iterate k are the same as the active constraints at x_* then this property will be preserved throughout the iterative process.

Lemma 4.4. *Assume that* **A0** − **A6** *hold and, for some $\bar{k} \geq k_0$ we have that*

$$\bar{A}x_{\bar{k}} = \bar{b}. \tag{29}$$

Suppose that the columns of $Z \in \mathbb{R}^{n\times(n-s)}$ form a basis of $\mathcal{N}(\bar{A})$.

then

$$\bar{A}x_k = \bar{b} \tag{30}$$

and

$$\bar{d}_k = -Z(Z^T B_{k,j_0} Z)^{-1} Z^T g_{j_0}(x_k) \tag{31}$$

for all $k \geq \bar{k}$.

Proof. The fact that (30) holds for all $k \geq \bar{k}$ follows by induction from (26) and (29) due to the line-search procedure. The formula (31) comes from (25) and (26). □

Lemma 4.5. *Assume that* **A0** − **A6** *hold. Then, every limit point of the sequence $\{\bar{d}_k/\|\bar{d}_k\|\}$ belongs to $\mathcal{N}(\bar{A})$.*

Moreover, for all $k \in I\!N$,

$$\frac{\bar{d}_k}{\|\bar{d}_k\|} = \tilde{d}_k^{(1)} + \tilde{d}_k^{(2)},$$

where $\tilde{d}_k^{(1)} \in \mathcal{N}(\bar{A})$, $\tilde{d}_k^{(2)} \in \mathcal{R}(\bar{A}^T)$ and

$$\lim_{k\to\infty} \tilde{d}_k^{(2)} = 0.$$

Proof. By (25), there exists $k_0 \in I\!N$ such that

$$\bar{d}_k^T B_{k,j_0} \bar{d}_k + g_{j_0}(x_k)^T \bar{d}_k + \lambda_k^T \bar{A} \bar{d}_k = 0 \;\; \forall\, k \geq k_0. \tag{32}$$

By Theorem 4.1 and Assumption **A1**, for k large enough we have:

$$f(x_k) = f_{j_0}(x_k) \text{ and } f(x_k + \bar{d}_k) = f_{j_0}(x_k + \bar{d}_k).$$

Therefore, by Assumption **A0**,

$$f(x_k + \bar{d}_k) = f(x_k) + \theta M_k(\bar{d}_k) + (1-\theta) g_{j_0}(x_k)^T \bar{d}_k - \frac{\theta}{2} \bar{d}_k^T B_{k,j_0} \bar{d}_k + o(\|\bar{d}_k\|).$$

Thus, by (32), and the positive-definiteness of B_{k,j_0},

$$f(x_k+\bar{d}_k) - f(x_k) - \theta M_k(\bar{d}_k) = -(1-\frac{\theta}{2}) \bar{d}_k^T B_{k,j_0} \bar{d}_k - (1-\theta)\lambda_k^T \bar{A}\bar{d}_k + o(\|\bar{d}_k\|)$$

$$\leq (\theta - 1)\lambda_k^T \bar{A}\bar{d}_k + o(\|\bar{d}_k\|).$$

Then,

$$\frac{f(x_k + \bar{d}_k) - f(x_k) - \theta M_k(\bar{d}_k)}{\|\bar{d}_k\|} \leq (\theta - 1)\lambda_k^T \bar{A} \frac{\bar{d}_k}{\|\bar{d}_k\|} + o(1).$$

The rest of the proof is identical to the part of the proof of Lemma 3.3 of [3] that follows formula (24) of that paper. □

Assumption **A7** imposes additional smoothness conditions on the function f_{j_0}. We will assume that f_{j_0} admits second derivatives and that these derivatives are Lipschitz-continuous on a neighborhood of x_*.

Assumption A7. *The Hessian $\nabla^2 f_{j_0}(x)$ exists and is continuous for all x in a neighborhood of x_*. Moreover, there exists $L_2 > 0$ such that, for all x, y on this neighborhood,*

$$\|\nabla^2 f_{j_0}(y) - \nabla^2 f_{j_0}(x)\| \leq L_2 \|y - x\|.$$

Assumption **A8** is a Dennis-Moré condition [7]. It states the minimal requirement for proving that convergence is superlinear.

Assumption A8. *There exists $Z \in I\!R^{n\times(n-s)}$ such that $\mathcal{R}(Z) = \mathcal{N}(\bar{A})$ and*

$$\lim_{k\to\infty} \frac{\|Z^T[B_{k,j_0} - \nabla^2 f_{j_0}(x_k)]\bar{d}_k\|}{\|\bar{d}_k\|} = 0.$$

Assumption **A9** states a sufficient condition for x_* being a local minimizer of f_{j_0} restricted to the active constraints.

Assumption A9. *The Hessian $\nabla^2 f_{j_0}(x_*)$ is positive definite on $\mathcal{N}(\bar{A})$.* In other words, $Z^T\nabla^2 f_{j_0}(x_*)Z$ is positive definite if Z is as in Assumption **A8**.

The following lemma states that, eventually, the first trial point at each iteration satisfies the sufficient descent condition (4). Therefore, backtracking is not necessary at all and, due to the isolation hypothesis, the method behaves as a nonlinear-system solver.

Lemma 4.6. *Assume that* **A0** $-$ **A9** *hold. Then, there exists $k_1 \in I\!N$ such that, for all $k \geq k_1$,*

$$x_{k+1} = x_k + \bar{d}_k.$$

Proof. By Assumption **A7**,

$$f(x_k + \bar{d}_k) = f_{j_0}(x_k) + g_{j_0}(x_k)^T\bar{d}_k + \frac{1}{2}\bar{d}_k^T\nabla^2 f_{j_0}(x_k)\bar{d}_k + o(\|\bar{d}_k\|^2)$$

$$= f_{j_0}(x_k)+\theta M_k(\bar{d}_k)+(1-\theta)g_{j_0}(x_k)^T\bar{d}_k^T-\frac{\theta}{2}\bar{d}_k^T B_{k,j_0}\bar{d}_k+\frac{1}{2}\bar{d}_k^T\nabla^2 f_{j_0}(x_k)\bar{d}_k+o(\|\bar{d}_k\|^2)$$

$$= f_{j_0}(x_k) + \theta M_k(\bar{d}_k) + (1-\theta)[g_{j_0}(x_k) + B_{k,j_0}\bar{d}_k]^T\bar{d}_k$$

$$-(1-\frac{\theta}{2})\bar{d}_k^T B_{k,j_0}\bar{d}_k + \frac{1}{2}\bar{d}_k^T\nabla^2 f_{j_0}(x_k)\bar{d}_k + o(\|\bar{d}_k\|^2)$$

$$= f_{j_0}(x_k)+\theta M_k(\bar{d}_k)+(1-\theta)[g_{j_0}(x_k)+B_{k,j_0}\bar{d}_k]^T\bar{d}_k+(1-\frac{\theta}{2})\bar{d}_k^T[\nabla^2 f_{j_0}(x_k)-B_{k,j_0}]\bar{d}_k$$

$$+\frac{1}{2}(\theta-1)\bar{d}_k^T\nabla^2 f_{j_0}(x_k)\bar{d}_k + o(\|\bar{d}_k\|^2).$$

Therefore,

$$\frac{f(x_k+\bar{d}_k) - f_{j_0}(x_k) - \theta M_k(\bar{d}_k)}{\|\bar{d}_k\|^2} = (1-\theta)\frac{[g_{j_0}(x_k) + B_{k,j_0}\bar{d}_k]^T\bar{d}_k}{\|\bar{d}_k\|^2}$$

$$+\left(1-\frac{\theta}{2}\right)\frac{\bar{d}_k^T[\nabla^2 f_{j_0}(x_k)-B_{k,j_0}]\bar{d}_k}{\|\bar{d}_k\|^2}+\frac{1}{2}(\theta-1)\frac{\bar{d}_k^T\nabla^2 f_{j_0}(x_k)\bar{d}_k}{\|\bar{d}_k\|^2}+\frac{o(\|\bar{d}_k\|^2)}{\|\bar{d}_k\|^2}. \tag{33}$$

Now, $\bar{d}_k$ is the exact solution of (2) and $\nabla M_k(\bar{d}_k) = B_{k,j_0}\bar{d}_k + g_{j_0}(x_k)$. By the convexity of $M_k(d)$, the function $M_k(d)$ decreases monotonically along the segment $\{t\bar{d}_k,\ t \in [0,1]\}$. This implies that, for all $k \in I\!N$,

$$(g_{j_0}(x_k) + B_{k,j_0}\bar{d}_k)^T\bar{d}_k = \nabla M_k(\bar{d}_k)^T\bar{d}_k \leq 0. \tag{34}$$

Let $\tilde{d}_k^{(1)}$ and $\tilde{d}_k^{(2)}$ be as in the thesis of Lemma 4.5. Then,

$$\frac{\bar{d}_k^T[\nabla^2 f_{j_0}(x_k)-B_{k,j_0}]\bar{d}_k}{\|\bar{d}_k\|^2} = \frac{\bar{d}_k^T}{\|\bar{d}_k\|}[\nabla^2 f_{j_0}(x_k)-B_{k,j_0}]\frac{\bar{d}_k}{\|\bar{d}_k\|}$$

$$= (\tilde{d}_k^{(1)} + \tilde{d}_k^{(2)})^T[\nabla^2 f_{j_0}(x_k)-B_{k,j_0}]\frac{\bar{d}_k}{\|\bar{d}_k\|}$$

$$= (\tilde{d}_k^{(1)})^T[\nabla^2 f_{j_0}(x_k)-B_{k,j_0}]\frac{\bar{d}_k}{\|\bar{d}_k\|} + (\tilde{d}_k^{(2)})^T[\nabla^2 f_{j_0}(x_k)-B_{k,j_0}]\frac{\bar{d}_k}{\|\bar{d}_k\|}. \tag{35}$$

The first term of (35) tends to zero by the Dennis-Moré condition and the second term tends to zero since, by Lemma 4.5, $\tilde{d}_k^{(2)}$ tends to zero. Thus,

$$\lim_{k\to\infty}\frac{\bar{d}_k^T[\nabla^2 f_{j_0}(x_k)-B_{k,j_0}]\bar{d}_k}{\|\bar{d}_k\|^2} = 0. \tag{36}$$

By Lemma 4.5 and Assumption A9, there exist $k_2 \in I\!N$ and $\varrho > 0$ such that

$$\frac{\bar{d}_k^T\nabla^2 f_{j_0}(x_k)\bar{d}_k}{\|\bar{d}_k\|^2} > \varrho \tag{37}$$

for all $k \geq k_2$.

Using (33), (34), (36) and (37) we obtain, for $k \geq k_2$,

$$\frac{f(x_k+\bar{d}_k) - f_{j_0}(x_k) - M_k(\bar{d}_k)}{\|\bar{d}_k\|^2} \leq \frac{1}{2}(\theta-1)\varrho + o(1)$$

So, taking $k_1 \geq k_2$ large enough and $k \geq k_1$,

$$\frac{f(x_k+\bar{d}_k) - f_{j_0}(x_k) - \theta M_k(\bar{d}_k)}{\|\bar{d}_k\|^2} \leq 0.$$

This implies the desired result. □

The final theorem of this section states that, under assumptions **A0–A8**, the convergence is superlinear. The convergence is quadratic if the matrices B_{k,j_0} are true Hessians.

Theorem 4.2. *Assume that* $\mathbf{A0} - \mathbf{A9}$ *hold. Then, for* k *large enough, we have that* $\alpha_k = 1$ *and* $\bar{A}x_k = \bar{b}$*. Moreover,* $\{x_k\}$ *converges superlinearly to* x_**. Finally, if, for* k *large enough,*

$$B_{k,j_0} = \nabla^2 f_{j_0}(x_k),$$

the convergence is quadratic.

Proof. This proof is identical to the proofs of Theorems 3.2 and 3.3 of [3], replacing G_k by B_{k,j_0}, f by f_{j_0} and g_k by $g_{j_0}(x_k)$. □

5 Application

The central features of the ideal banking system (IBS) of [14] are: (a) government-insured deposits; (b) government-imposed reserve requirements with reserves held at the central bank in interest-free accounts; and (c) no transaction costs. In the (slightly modified) IBS two-period model, the decision variables are:

1. M: investments in T-bills.
2. N: loans.
3. r: interest rate charged to loans.
4. e: expenditure in screening and monitoring.
5. D: deposits borrowed by the bank.
6. s: interest rate paid by banks to their depositors.

The data of the problem are:

1. a: initial net worth.
2. τ: reserve requirement.
3. ρ: interest rate of T-bills.

The variables of the model are subject to the budget constraint:

$$a + (1-\tau)D = M + N + e. \tag{38}$$

In the IBS model defined in [14] the interest rate paid by banks to their depositors must equal that on government T-bills ($s = \rho$). This restriction will not be imposed here. We use the same notation as [14] except for the constant τ, which is called k in [14].

A crucial constraint of the model is that N cannot exceed a function of demand for loans N^d, which depends on the interest rate r:

$$N \leq N^d(r). \tag{39}$$

Another similar constraint of the model is that the deposits D cannot exceed a function of demand for deposits D^o, that depends on the interest rate s:

$$D \leq D^o(s). \tag{40}$$

Other constraints of the model involve lower and upper bounds of the variables:

$$\ell_D \leq D \leq u_D, \quad \ell_M \leq M \leq u_M, \quad \ell_N \leq N \leq u_N, \tag{41}$$

$$\ell_s \leq s \leq u_s, \quad \ell_e \leq e \leq u_e, \quad \ell_r \leq r \leq u_r. \tag{42}$$

Roughly speaking, the bank aims to maximize the end-of-the-period net worth (EW), given by

$$\bar{a} = M(1+\rho) - sD - (1-\tau)D + V, \tag{43}$$

where V represents the gross returns from loans. However, V is not a fixed quantity but a random variable with m possible values $V_1, \ldots, V_m$, bounded above by $(1+r)N$. In fact, the upper bound is smaller, since the chance of non-devolution of loans exists even at very small interest rates. Here we use the bound

$$V_i \leq 0.99995(1+r)N, \; \forall\, i = 1, \ldots, m.$$

The maximum gross return from loans is even smaller. The bound above must be multiplied by a coefficient between 0 and 1 that decreases with r, vanishes at $r = \infty$ and is equal to 1 if $r = 0$. This is because the chance of obtaining whole devolution (with interests) of the loans decreases with r and tends to zero if r tends to ∞. So, we define:

$$\mathrm{GRL}_{max} = 0.99995(1+r)N \times \exp(-\kappa r^2)$$

for a given $\kappa > 0$ and we postulate that

$$V_i \leq \mathrm{GRL}_{max}, \ \forall\, i = 1, \ldots, m.$$

The actual gross return from loans under scenario i is thus defined as:

$$V_i = \mathrm{GRL}_{max} - z_i \Delta(e, N)\mathrm{GRL}_{max}, \tag{44}$$

where $\Delta(e, N) \in [0, 1]$ decreases when e increases and, for all $i = 1, \ldots, m$ the number $z_i \in [0, 1]$ defines the scenario i.

We define:

$$\Delta(e, N) = \frac{\sqrt{\tilde{\gamma}_1^2(e - \tilde{\gamma}_2)^2 + 10^{-4}} - \tilde{\gamma}(e - \tilde{\gamma}_2)}{2} + \Delta_{\min} \tag{45}$$

where $\tilde{\gamma}_1 = \frac{5}{N}$, $\tilde{\gamma}_2 = \frac{N}{5}(1 - \Delta_{min})$. The expression (45) is a smooth approximation of $\max\{1 - 5e/N, \Delta_{min}\}$.

The definitions above are enough to state the OVO optimization problem. Define $n = 6$, $x_1 = M$, $x_2 = N$, $x_3 = e$, $x_4 = r$, $x_5 = s$, $x_6 = D$. Then, the functions f_i, $i = 1, \ldots, m$ are given by:

$$f_i(x) = -\left[M(1 + \rho) - sD - (1 - \tau)D + V_i\right],$$

where V_i is defined by (44) for $i = 1, \ldots, m$.

Given $p \in \{1, \ldots, m\}$, the OVO problem associated to this application is

$$\text{Minimize } F_p(x) \text{ subject to (38)-(42).} \tag{46}$$

Therefore, (46) is a constrained OVO optimization problem. With the appropriate definitions of N^d and D^o the constraints will be linear and Algorithm 2.1 will be fully applicable.

In our practical application we use the following data:

- $a = 54.6$ billions of reals (brazilian currency), $\tau = 0.5$, $\rho = 0.03$, $\kappa = 0.6$, $\Delta_{min} = 0.1$.
- The constraints (41) and (42) that define lower and upper bounds of the variables are:

$$
\begin{aligned}
0 &\leq M \leq 5a, && (47)\\
0.1a &\leq N \leq 5a, && (48)\\
0 &\leq e \leq 5a, && (49)\\
0 &\leq r \leq 0.8, && (50)\\
0.02 &\leq s \leq 0.1, && (51)\\
a &\leq D \leq 5a. && (52)
\end{aligned}
$$

- The functions N^d and D^o are:

$$N^d(r) = 5a - \frac{4.9a}{0.8}r, \quad (53)$$
$$D^o(s) = 50as. \quad (54)$$

- In fully realistic applications the vector z is estimated using classification of borrowers. So, the analytic distribution for z_i is not known. In these cases, a closed analytic representation of the objective function is not available and only a simulation approach is possible. However, in order to simplify our presentation, we adopted here the following definition for $z_i,\ i = 1, \ldots, m$:

$$z_i = \varphi^{-1}(w_i),\ i = 1, \ldots, m,$$

where

$$\varphi(t) = 3t^2 - 2t^3,\ \forall\, t \in [0, 1]$$

and $w_1, \ldots, w_m$ are random and uniformly distributed in $[0, 1]$.

6 Numerical experience

In this section we first describe the computer implementation of Algorithm 2.1.

6.1 Hessian approximations and algorithmic parameters

If the number of scenarios m is small it is possible to compute or update a different matrix $B_{k,j}$ for each $j = 1, \ldots, m$. We cannot afford this computation if m is large. If a Newtonian approach is adopted ($B_{k,j} = \nabla^2 f_j(x_k)$) one can compute $B_{k,j}$ only when $j \in I_\varepsilon(x_k)$. This is an interesting alternative, but it does not guarantee positive semidefiniteness of the Hessian approximations. In our implementation we adopted a BFGS approach [8]. Moreover, in order to decrease computational cost, we postulate that $B_k = B_{k,j}$ for all $j = 1, \ldots, m$, $k \in I\!N$. The matrices B_k were generated taking $B_0 = I$ and, for all $k \in I\!N$,

$$s_k = x_{k+1} - x_k,\ y_k = g_{i_p(x_k)}(x_{k+1}) - g_{i_p(x_k)}(x_k),$$

$$B_{k+1} = B_k + \frac{y_k y_k^T}{y_k^T s_k} - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k}$$

if $y_k^T s_k > 0$, whereas $B_{k+1} = B_k$ if $y_k^T s_k \leq 0$. As it is well known, this guarantees that B_k is positive definite for all $k \in I\!N$ [8].

Finally, if $\|B_k\|_\infty > 10^3 n$ we replace B_k by the Identity matrix.

The algorithmic parameters used for running Algorithm 2.1 were:

$$\Delta = 1.0,\ \theta = 0.5,\ \sigma_{\min} = 0.1,\ \sigma_{\max} = 0.9,\ \eta = 1,\ \varepsilon = 10^{-6}.$$

The practical convergence criterion was $M_k(d_k) \geq -10^{-6}$.

For solving the subproblem at Step 1 of the algorithm we used the IMSL Library routine DNCONG [12, 13].

6.2 Choice of the initial point

Given $p \in \{1, \ldots, m\}$, the OVO problem associated to the IBS model is the one defined in the previous section. Before starting Algorithm 2.1 we generate 1000 random points in Ω and evaluate the p-Order-Value function on these points. The point x_0 having the smallest p-Order-Value functional value is taken as the initial approximation for Algorithm 2.1. In our implementation the set Ω is defined by the constraints (38), (47)-(54). For generating random points in Ω we proceed as follows:

1. Choose r randomly between 0 and 0.8. In this way the constraint (50) is satisfied.

2. Compute N randomly between $0.1a$ and $N^d(r)$. In this way the constraints (48) and (53) are satisfied.

3. Compute e randomly between 0 and $\frac{N}{5}(1 - \Delta_{\min})$. In this way the constraint (49) is satisfied.

4. (a) If $2(N + e - a) < a$, compute D randomly between a and $5a$.

   (b) If $a \leq 2(N+e-a) \leq 5a$, compute D randomly between $2(N+e-a)$ and $5a$.

   (c) If $2(N + e - a) > 5a$, it is impossible to satisfy the constraints. In this case, return to the computation of N and e.

5. Compute $s = \dfrac{D}{50a}$ which is the smallest rate that satisfies the constraints (51) and (54).

6. Finally, compute $M = a + (1 - \tau)D - N - e$ (so, the budget constraint (38) is satisfied). Since $a + (1 - \tau)D \geq N + e$, M also satisfies the constraint (47).

p	M	N	e	r	s	D	Iter	Eval	-$f(x_0)$	-$f(x_*)$
10	0.	110.	0.	0.488	0.040	110.	30	31	70.1	71.6
50	0.	90.8	0.	0.541	0.027	72.5	22	25	61.6	62.3
100	0.	69.5	12.4	0.541	0.020	54.6	19	29	58.6	59.4
500	0.	69.4	12.5	0.541	0.020	54.6	45	55	56.2	56.7
700	75.5	5.46	0.989	0.541	0.020	54.6	22	33	55.4	55.9
900	75.5	5.46	0.991	0.541	0.020	54.6	19	29	55.2	55.8
950	75.4	5.46	0.991	0.541	0.020	54.6	12	22	54.9	55.8
990	75.4	5.46	0.992	0.541	0.020	54.6	25	37	55.0	55.7
995	75.4	5.46	0.993	0.541	0.020	54.6	22	29	54.7	55.7
999	75.4	5.46	0.993	0.541	0.020	54.6	20	29	55.0	55.7

Table 1: Results obtained by Algorithm 2.1

6.3 Computer results

The computations were done on a 2.4Ghz Intel Pentium IV Computer with 1Gb of RAM in double precision Fortran.

Taking $m = 1000$, the solutions $x^* = (M, N, e, r, s, D)$ obtained by Algorithm 2.1 for different values of p are shown in Table 1. In this table “Iter” is the number of iterations used by Algorithm 2.1 and “Eval” is the number of evaluations of F_p.

Observe that the solution is essentially the same for all values of $p \geq 700$. Averse-to-risk behavior under the boundary conditions given by the data of this problem produces small availability of credit and very high interest rates (N small, r high). This result was expected, but the surprise is that the degree of aversion to risk (measured by the VaR parameter p/m) did not need to be close to 1 for getting it. It is interesting to report, for each value of p, the average end-of-the-period net worth (43), the standard deviation of this magnitude, its maximum value and its minimum value. These quantities are defined below and reported in Table 2.

$$\text{Average of EW} = \mu = \frac{1}{m}\sum_{i=1}^{m} -f_i(x^*),$$

$$\text{Standard deviation of EW} = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}[-f_i(x^*) - \mu]^2}\ ,$$

p	Maximum EW	Minimum EW	Average EW	Deviation EW
10	79.7	-57.7	-2.05	24.4
50	77.4	-36.6	9.60	20.2
100	61.3	51.7	55.6	1.70
500	61.1	52.1	55.8	1.60
700	56.4	55.7	56.0	0.125
900	56.4	55.7	55.9	0.125
950	56.4	55.7	55.9	0.125
990	56.4	55.7	55.9	0.124
995	56.4	55.7	55.9	0.124
999	56.4	55.7	55.9	0.124

Table 2: Statistics of End-of-period net worth

$$\text{Maximum of EW} = \max\{-f_1(x_*), \ldots, -f_m(x_*)\},$$
$$\text{Minimum of EW} = \min\{-f_1(x_*), \ldots, -f_m(x_*)\}.$$

In Table 2 we observe that, as expected, when aversion to risk increases (p large) the maximum end-of-period net worth decreases. However this decrease is negligible for $p \geq 700$. Reciprocally, the minimum end-of-period net worth increases when p grows. Moreover, for $p \geq 700$ the averse-to-risk behavior produces a very small standard deviation of the final distribution. Consequently, the difference between the maximum and the minimum end-of-period net worth is small. It is interesting to observe that the average end-of-period net worth increases from $p = 10$ to $p = 700$ and decreases for $p = 700$ on.

With the aim of providing a better insight on the behavior of Algorithm 2.1, we investigated the individual iterations of the method in the case $p = 900$. We observed that:

- For all the iterations it was verified that the set $I_\varepsilon(x_k)$ had only one element (which was always the same). At the last 15 iterations we got $\alpha_k = 1$ (the first trial point was accepted). Observe in Table 1 that, according to the number of functional evaluations, the first trial point was accepted at most iterations for all the problems.

- The constraint (40) was active at the solution found. The variables N, s and D were at their lower bounds. These four constraints were active at the five final iterations.

- The last 10 final quotients $\|x_{k+1} - x_*\| / \|x_k - x_*\|$ were:

$$0.75, 0.71, 0.83, 0.94, 0.56, 0.04, 0.59, 0.06, 0.53, 0.005.$$

These observations are compatible with theoretically superlinear convergence of the algorithm.

Again in the case $p = 900$ we tried to solve the problem (46) with the Cauchy method proposed in [1] with the same parameters, initial point and stopping criteria used in Algorithm 2.1. After 50000 iterations the stopping criteria $M_k(d_k) > -10^{-6}$ was not attained. We observed that:

- As in Algorithm 2.1, for all the iterations of the Cauchy method it was verified that the set $I_\varepsilon(x_k)$ had only one element (always the same).
- The Euclidian distance between the solution found by the quasi-Newton method and the points obtained by the Cauchy method after 10000, 20000, 30000, 40000 and 50000 iterations were:

$$1.0346,\ 0.58938,\ 0.46413,\ 0.39863,\ \text{and } 0.35249,\ \text{respectively.}$$

- The difference between the minimum value obtained by the quasi-Newton method and the value of F_p at the points obtained by the Cauchy method after 10000, 20000, 30000, 40000 and 50000 iterations were:

$$0.01865,\ 0.00601,\ 0.00372,\ 0.00274,\ \text{and } 0.00214,\ \text{respectively.}$$

- The number of evaluations of F_p after 50000 iterations of the Cauchy method was 741663.
- For all iterations of the Cauchy method $0 < \alpha_k \leq 0.0625 < 1$. So, the first trial point was always rejected.

These observations show that, although the Cauchy method converges to the solution, its convergence is very slow when compared with Algorithm 2.1.

7 Conclusions

We introduced a new algorithm of quasi-Newton type for convex-constrained Order-Value Optimization problems. The new algorithm can be used for optimization of the Value-at-Risk, especially when the loss-functions f_i are nonlinear and the constraints are linear. We proved global convergence in the sense that every limit point satisfies an optimality condition. Local (superlinear and quadratic) convergence was proved under some additional restrictive assumptions.

We tested our method using the IBS model of Stiglitz. In the experiments we used a BFGS approach. The results were compatible with economic analysis and somewhat surprising in the sense that very restrictive-to-credit decisions were taken even in situations in which aversion to risk was not high.

The performance of the new method was much better than the performance of the primal Cauchy method introduced recently in [1]. Moreover, the restrictive assumptions used to prove rapid convergence seem to hold, at least in a particular illustrative example.

The Order-Value Optimization problem is relevant, not only for its financial applications, but also for its applications to robust estimation and data analysis [4]. Moreover, it is a challenging global and nonsmooth optimization problem. Therefore, much research is expected both in the introduction and analysis of methods for this problem. The method introduced in this paper deserves future investigation along several lines. From the theoretical point of view the most important question is whether the assumptions made for local convergence can be weakened or not. In particular, we do not know whether the isolation assumption is essential for superlinear convergence, even in the unconstrained case. It must be warned, however, that, according to the counter-examples obtained recently by Dai [6] and Mascarenhas [11] a fully global convergence theorem for the BFGS without safeguards cannot be expected.

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