Lista 3 - Gabarito

1)

$$\begin{aligned}
P(N(s)=k|N(t)=n) &= \frac{P(N(t)=n|N(s)=k)P(N(s)=k)}{P(N(t)=n)} = \frac{P(N(t-s)=n-k)P(N(s)=k)}{P(N(t)=n)} \\
&= \frac{\dfrac{e^{-\lambda(t-s)}(\lambda(t-s))^{n-k}}{(n-k)!}\dfrac{e^{-\lambda s}(\lambda s)^k}{k!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}} \\
&= \frac{n!}{k!(n-k)!}e^{-\lambda(t-s+s-t)}\lambda^{n-k+k-n}\frac{(t-s)^{n-k}s^k}{t^n} \\
&= C_k^n(\frac{t-s}{t})^{n-k}(\frac{s}{t})^k = C_k^n(1-\frac{s}{t})^{n-k}(\frac{s}{t})^k
\end{aligned}$$

2)

Homens 2/min, P(H)=$\frac{1}{3}$, S^1

Mulheres 4/min, P(M)=$\frac{2}{3}$, S^2

Corrida de Poisson

$$\begin{aligned}
P(S_2^1 < S_3^2) &= \sum_{k=2}^{4} C_k^4(\frac{1}{3})^k(\frac{2}{3})^{4-k} = 6(\frac{1}{3})^2(\frac{2}{3})^2 + 4(\frac{1}{3})^3(\frac{2}{3})^1 + (\frac{1}{3})^4 \\
&= \frac{24}{81} + \frac{8}{81} + \frac{1}{81} \quad = \quad \frac{33}{81} \quad = \quad \frac{11}{27}
\end{aligned}$$

3)

Sabemos que se X_i v.a. Poi(λ_i), $\sum_{i=1}^n X_i \sim Poi(\sum_{i=1}^n \lambda_i)$, então como $N_1(t) \sim Poi(\lambda_1)$ e $N_2(t) \sim Poi(\lambda_2)$, $N(t) = N_1(t) + N_2(t) \sim Poi(\lambda), \lambda = \lambda_1 + \lambda_2$

$$\begin{aligned}
P(N_1(t)=1|N(t)=1) &= \frac{P(N(t)=1|N_1(t)=1)P(N_1(t)=1)}{P(N(t)=1)} = \frac{P(N_2(t)=0)P(N_1(t)=1)}{P(N(t)=1)} \\
&= \frac{e^{-\lambda_2 t}(\lambda_2 t)^0 e^{-\lambda_1 t}(\lambda_1 t)^1}{e^{-\lambda t}(\lambda t)^1} \\
&= e^{-t(\lambda-\lambda_1-\lambda_2)}\frac{\lambda_1 t}{(\lambda_1+\lambda_2)t} \quad = \quad \frac{\lambda_1}{(\lambda_1+\lambda_2)}
\end{aligned}$$

4)
a)$P(N(T)=0)=e^{-\lambda T}$
b)X(t) = tempo de espera

$$\begin{aligned} E[X(t)] &= E[X(t)\mathbb{I}_{T_1<T}+X(t)\mathbb{I}_{T_1>T}] = E[X(t)\mathbb{I}_{T_1<T}] = E[T_1+E[X(t)]] \\ &= \frac{\int_0^T \lambda t e^{-\lambda t}dt}{P(T_1>T)} = \frac{\frac{1-e^{-\lambda t}(1+\lambda t)}{\lambda}}{e^{-\lambda T}} \quad = \quad \frac{1-e^{-\lambda t}}{\lambda} - T \end{aligned}$$

5)
a)$E[N(T)] = \quad E[E[N(T)|T]] = E[\lambda T] = \lambda E[T] = \quad \lambda\mu$
b)$E[TN(T)] = \quad E[E[TN(T)|T]] = E[T\lambda T] = \quad \lambda E[T^2]$
$Cov(T,N(T)) = \quad E[TN(T)] - E[T]E[N(T)] = \lambda E[T^2] - E[T]\lambda E[T] = \quad \lambda\sigma^2$
c)$E[N^2(T)] = \quad EE[N^2(T)|T] = E[\lambda T + (\lambda T)^2] = \quad \lambda E[T] + \lambda^2 E[T^2]$
$Var(N(T)) = \quad E[N(T)^2] - E[N(T)]^2 = \lambda E[T] + \lambda^2 E[T^2] - (\lambda E[T])^2 = \quad \lambda\mu + \lambda^2\sigma^2$

6)
$X(t) \sim Poi(3\frac{2}{3}) = Poi(2)$
$P(X(t)=0) = e^{-2t}$
$E[X(t)] = 2$

7)
$N_1(t) \sim Poi(\lambda P_1(t)) = Poi(\lambda e^{-t})$

8)
a)
Tipo 1 - cliente "antigo"chegar antes de s e saiu depois de t+s
Tipo 2 - cliente "novo"chegar depois de s e saiu depois de t+s
Tipo 3 - chegar antes de s e sair entre s e s+t
Tipo 4 - outros

$$\begin{aligned} p_1(u) &= P(servico > t+s-u) = e^{-\mu(t+s-u)}, u<s \\ p_2(u) &= e^{-\mu(t+s-u)}, s<u<t+s \\ p_3(u) &= P(s-u < servico < t+s-u) = 1-e^{-\mu(t+s-u)} - (1-e^{-\mu(s-u)}) = \\ &= e^{-\mu(s-u)} - e^{-\mu(t+s-u)}, u<s \end{aligned}$$

$X(s) = N_1(t+s) + N_3(t+s)$
$X(t+s) = N_1(t+s) + N_2(t+s)$

$$
\begin{aligned}
E[X(t+s)|X(s)] &= E[X(s) - N_3(t+s) + N_2(t+s)|X(s)] = \\
&= X(s) + E[N_2(t+s)] - E[N_3(t+s)|X(s)] \\
&= X(s) - X(s)(1-e^{-\mu t}) + \frac{\lambda}{\mu}(1-e^{-\mu t}) \\
&= X(s)e^{-\mu t} + \frac{\lambda}{\mu}(1-e^{-\mu t})
\end{aligned}
$$

Pois:
$E[N_3(t+s)|X(s)=k] = k(1-e^{-\mu t})$
$E[N_2(t+s)] = \int_s^{t+s} \lambda e^{-\mu(t+s-u)} du = \lambda e^{-\mu(t+s)} \dfrac{e^{\mu(t+s)} - e^{\mu s}}{\mu} = \dfrac{\lambda}{\mu}(1-e^{-\mu t})$

b)
Tipo 1 - chegou antes de t, saiu depois de t
Tipo 2 - chegou antes de t, saiu antes de t
$X(t) = N_1(t) \perp Y(t) = N_2(t)$
$P_1(u) = P(servico > t-u) = e^{-\mu(t-u)}$
$E[X(t)|Y(t)=n] = E[X(t)] = \lambda \int_0^t P_1(u)du \quad = \quad \dfrac{\lambda}{\mu}(1-e^{-\mu t})$