Partial Differentiation

12

Joseph Louis Lagrange
(1736–1813)

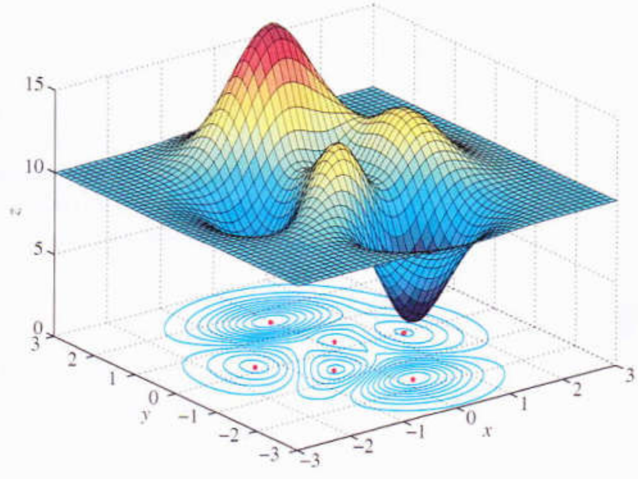
Joseph Louis Lagrange is remembered for his great treatises on analytical mechanics and on the theory of functions that summarized much of eighteenth-century pure and applied mathematics. These treatises—*Mécanique analytique* (1788), *Théorie des fonctions analytiques* (1797), and *Leçons sur le calcul des fonctions* (1806)—systematically developed and applied widely the differential and integral calculus of multivariable functions expressed in terms of the rectangular coordinates x, y, z in three-dimensional space. They were written and published in Paris during the last quarter-century of Lagrange's career. But he grew up and spent his first 30 years in Turin, Italy. His father pointed Lagrange toward the law, but by age 17 Lagrange had decided on a career in science and mathematics. Based on his early work in celestial mechanics (the mathematical analysis of the motions of the planets and satellites in our solar system), Lagrange in 1766 succeeded Leonhard Euler as director of the Berlin Academy in Germany.

Lagrange regarded his far-reaching work on maximum-minimum problems as his best work in mathematics. This work, which continued throughout his long career, dated back to a letter to Euler that Lagrange wrote from Turin when he was only 19. This letter outlined a new approach to a certain class of optimization problems that comprise the calculus of variations. A typical example is the *isoperimetric problem,* which asks what curve of a given arc length encloses a plane region with the greatest area. (The answer: a circle.) In the *Mécanique analytique,* Lagrange applied his "method of multipliers" to investigate the motion of a particle in space that is constrained to move on a surface defined by an equation of the form $g(x, y, z) = 0$. Section 12.9 applies the Lagrange multiplier method to the problem of maximizing or minimizing a function $f(x, y, z)$ subject to a "constraint" of the form

$$g(x, y, z) = 0.$$

Today this method has applications that range from minimizing the fuel required for a spacecraft to achieve its desired trajectory to maximizing the productivity of a commercial enterprise limited by the availability of financial, natural, and personnel resources.

Modern scientific visualization often employs computer graphic techniques to present different interpretations of the same data simultaneously in a single figure. The following color graphic shows both a graph of a surface $z = f(x, y)$ and a contour map showing *level curves* that appear to encircle points (x, y) corresponding to *pits and peaks* on the surface. In Section 12.5 we learn how to locate multivariable maximum-minimum points like those visible on this surface.



12.1 INTRODUCTION

We turn our attention here and in Chapters 13 and 14 to the calculus of functions of more than one variable. Many real-world functions depend on two or more variables. For example:

- In physical chemistry the ideal gas law $pV = nRT$ (where n and R are constants) is used to express any one of the variables p (pressure), V (volume), and T (temperature) as a function of the other two.
- The altitude above sea level at a particular location on the earth's surface depends on the latitude and longitude of the location.
- A manufacturer's profit depends on sales, overhead costs, the cost of each raw material used, and in many cases, additional variables.
- The amount of usable energy a solar panel can gather depends on its efficiency, its angle of inclination to the sun's rays, the angle of elevation of the sun above the horizon, and other factors.

A typical application may call for us to find an extreme value of a function of several variables. For example, suppose that we want to minimize the cost of making a rectangular box with a volume of 48 ft^3, given that its front and back cost $\$1/\text{ft}^2$, its top and bottom cost $\$2/\text{ft}^2$, and its two ends cost $\$3/\text{ft}^2$. Figure 12.1.1 shows such a box of length x, width y, and height z. Under the conditions given, its total cost will be

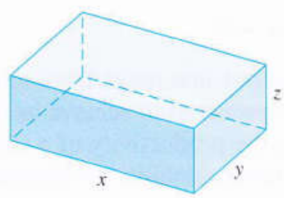


FIGURE 12.1.1 A box whose total cost we want to minimize.

$$C = 2xz + 4xy + 6yz \quad \text{(dollars)}.$$

But x, y, and z are not independent variables, because the box has fixed volume

$$V = xyz = 48.$$

We eliminate z, for instance, from the first formula by using the second; because $z = 48/(xy)$, the cost we want to minimize is given by

$$C = 4xy + \frac{288}{x} + \frac{96}{y}.$$

Because neither of the variables x or y can be expressed in terms of the other, the single-variable maximum-minimum techniques of Chapter 3 cannot be applied here. We need new optimization techniques applicable to functions of two or more independent variables. In Section 12.5 we shall return to this problem.

The problem of optimization is merely one example. We shall see in this chapter that many of the main ingredients of single-variable differential calculus—limits, derivatives and rates of change, chain rule computations, and maximum-minimum techniques—can be generalized to functions of two or more variables.

12.2 FUNCTIONS OF SEVERAL VARIABLES

Recall from Section 1.1 that a real-valued *function* is a rule or correspondence f that associates a unique real number with each element of a set D. The domain D has always been a subset of the real line for the functions of a single variable that we have studied up to this point. If D is a subset of the plane, then f is a function of *two* variables—for, given a point P of D, we naturally associate with P its rectangular coordinates (x, y).

DEFINITION Functions of Two or Three Variables

A **function of two variables,** defined on the **domain** D in the plane, is a rule f that associates with each point (x, y) in D a unique real number, denoted by $f(x, y)$. A **function of three variables,** defined on the **domain** D in space, is a rule f that associates with each point (x, y, z) in D a unique real number $f(x, y, z)$.

We can typically define a function f of two (or three) variables by giving a formula that specifies $f(x, y)$ in terms of x and y (or $f(x, y, z)$ in terms of x, y, and z). In case the domain D of f is not explicitly specified, we take D to consist of all points for which the given formula is meaningful.

EXAMPLE 1 The domain of the function f with formula

$$f(x, y) = \sqrt{25 - x^2 - y^2}$$

is the set of all (x, y) such that $25 - x^2 - y^2 \geqq 0$—that is, the circular disk $x^2 + y^2 \leqq 25$ of radius 5 centered at the origin. Similarly, the function g defined as

$$g(x, y, z) = \frac{x + y + z}{\sqrt{x^2 + y^2 + z^2}}$$

is defined at all points in space where $x^2 + y^2 + z^2 > 0$. Thus its domain consists of all points in three-dimensional space $\boldsymbol{R}^3$ other than the origin $(0, 0, 0)$.

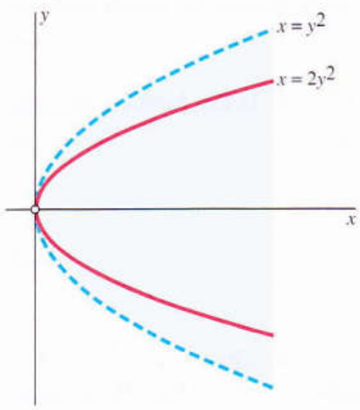


FIGURE 12.2.1 The domain of $f(x, y) = \dfrac{y}{\sqrt{x - y^2}}$ (Example 2).

EXAMPLE 2 Find the domain of definition of the function with formula

$$f(x, y) = \frac{y}{\sqrt{x - y^2}}. \tag{1}$$

Find also the points (x, y) at which $f(x, y) = \pm 1$.

Solution For $f(x, y)$ to be defined, the *radicand* $x - y^2$ must be positive—that is, $y^2 < x$. Hence the domain of f is the set of points lying strictly to the right of the parabola $x = y^2$. This domain is shaded in Fig. 12.2.1. The parabola in the figure is dashed to indicate that it is not included in the domain of f; any point for which $x = y^2$ would entail division by zero in Eq. (1).

The function $f(x, y)$ has the value ± 1 whenever

$$\frac{y}{\sqrt{x - y^2}} = \pm 1;$$

that is, when $y^2 = x - y^2$, so $x = 2y^2$. Thus $f(x, y) = \pm 1$ at each point of the parabola $x = 2y^2$ [other than its vertex $(0, 0)$, which is not included in the domain of f]. This parabola is shown as a solid curve in Fig. 12.2.1.

In a geometric, physical, or economic situation, a function typically results from expressing one descriptive variable in terms of others. As we saw in Section 12.1, the cost C of the box discussed there is given by the formula

$$C = 4xy + \frac{288}{x} + \frac{96}{y}$$

in terms of the length x and width y of the box. The value C of this function is a variable that depends on the values of x and y. Hence we call C a **dependent variable,** whereas x and y are **independent variables.** And if the temperature T at the point (x, y, z) in space is given by some formula $T = h(x, y, z)$, then the dependent variable T is a function of the three independent variables x, y, and z.

We can define a function of four or more variables by giving a formula that includes the appropriate number of independent variables. For example, if an amount A of heat is released at the origin in space at time $t = 0$ in a medium with thermal diffusivity k, then—under appropriate conditions—the temperature T at the point (x, y, z) at time $t > 0$ is given by

$$T(x, y, z, t) = \frac{A}{(4\pi k t)^{3/2}} \exp\left(-\frac{x^2 + y^2 + z^2}{4kt} \right).$$

If A and k are constants, then this formula gives the temperature T as a function of the four independent variables x, y, z, and t.

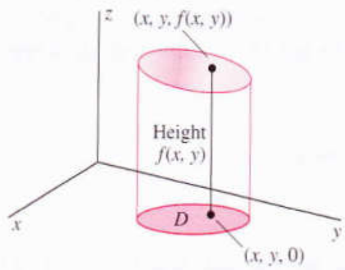


FIGURE 12.2.2 The graph of a function of two variables is typically a surface "over" the domain of the function.

We shall see that the main differences between single-variable and multivariable calculus show up when only two independent variables are involved. Hence many of our results will be stated in terms of functions of two variables. Most of these results readily generalize by analogy to the case of three or more independent variables.

Graphs and Level Curves

We can visualize how a function f of two variables "works" in terms of its graph. The **graph** of f is the graph of the equation $z = f(x, y)$. Thus the graph of f is the set of all points in space with coordinates (x, y, z) that satisfy the equation $z = f(x, y)$. (See Fig. 12.2.2.) The planes and quadric surfaces of Sections 11.4 and 11.7 provide some simple examples of graphs of functions of two variables.

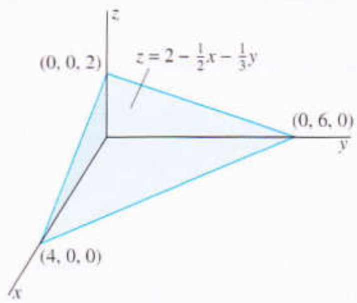


FIGURE 12.2.3 The planar graph of Example 3.

EXAMPLE 3 Sketch the graph of the function $f(x, y) = 2 - \frac{1}{2}x - \frac{1}{3}y$.

Solution We know from Section 11.4 that the graph of the equation $z = 2 - \frac{1}{2}x - \frac{1}{3}y$ is a plane, and we can visualize it by using its intercepts with the coordinate axes to plot the portion in the first octant of space. Clearly $z = 2$ if $x = y = 0$. Also the equation gives $y = 6$ if $x = z = 0$ and $x = 4$ if $y = z = 0$. Hence the graph looks as pictured in Fig. 12.2.3.

EXAMPLE 4 The graph of the function $f(x, y) = x^2 + y^2$ is the familiar circular paraboloid $z = x^2 + y^2$ (Section 11.7) shown in Fig. 12.2.4.

EXAMPLE 5 Find the domain of definition of the function

$$g(x, y) = \tfrac{1}{2}\sqrt{4 - 4x^2 - y^2} \tag{2}$$

and sketch its graph.

Solution The function g is defined wherever $4 - 4x^2 - y^2 \geqq 0$—that is, $x^2 + \frac{1}{4}y^2 \leqq 1$—so that Eq. (2) does not involve the square root of a negative number. Thus the domain of g is the set of points in the xy-plane that lie on and within the ellipse $x^2 + \frac{1}{4}y^2 = 1$ (Fig. 12.2.5). If we square both sides of the equation $z = \frac{1}{2}\sqrt{4 - 4x^2 - y^2}$ and simplify the result, we get the equation

$$x^2 + \tfrac{1}{4}y^2 + z^2 = 1$$

of an ellipsoid with semiaxes $a = 1$, $b = 2$, and $c = 1$ (Section 11.7). But $g(x, y)$ as defined in Eq. (2) is nonnegative wherever it is defined, so the graph of g is the upper half of the ellipsoid (Fig. 12.2.6).

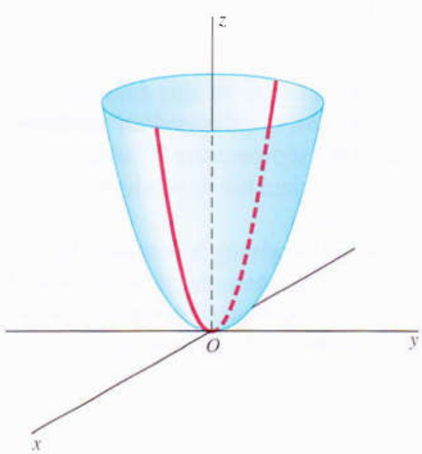


FIGURE 12.2.4 The paraboloid is the graph of the function $f(x, y) = x^2 + y^2$.

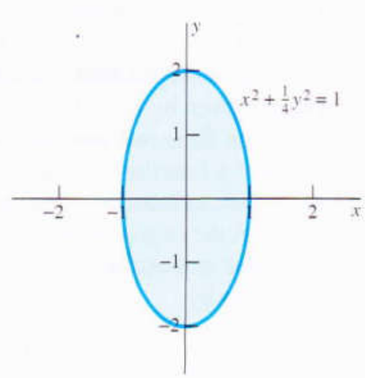


FIGURE 12.2.5 The domain of the function $g(x, y) = \frac{1}{2}\sqrt{4 - 4x^2 - y^2}$.

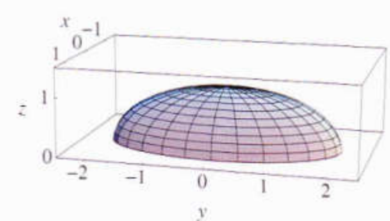


FIGURE 12.2.6 The graph of the function g is the upper half of the ellipsoid.

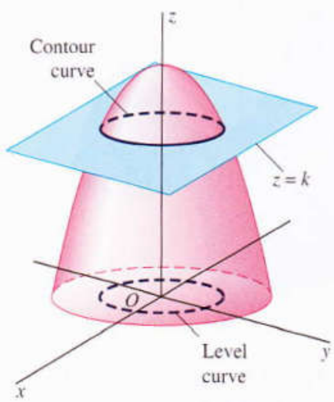


FIGURE 12.2.7 A contour curve and the corresponding level curve.

The intersection of the horizontal plane $z = k$ with the surface $z = f(x, y)$ is called the **contour curve** of **height** k on the surface (Fig. 12.2.7). The vertical projection of this contour curve into the xy-plane is the **level curve** $f(x, y) = k$ of the function f. Thus a level curve of f is simply a set in the xy-plane on which the value $f(x, y)$ is *constant*. On a topographic map, such as the one in Fig. 12.2.8, the level curves are curves of constant height above sea level.

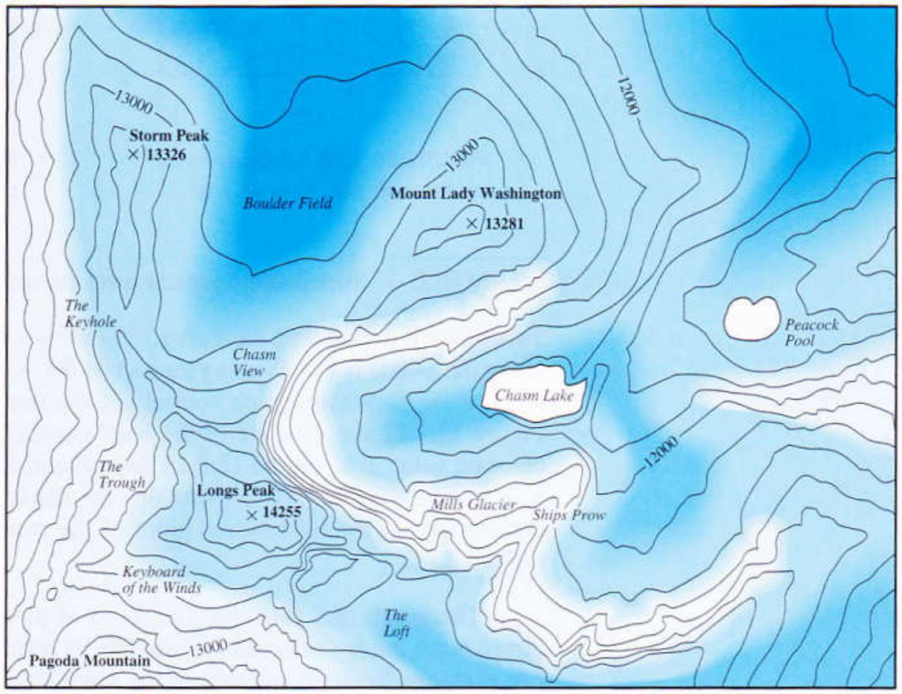


FIGURE 12.2.8 The region near Longs Peak, Rocky Mountain National Park, Colorado, showing contour lines at intervals of 200 feet.

Level curves give a two-dimensional way of representing a three-dimensional surface $z = f(x, y)$, just as the two-dimensional map in Fig. 12.2.8 represents a three-dimensional mountain range. We do this by drawing typical level curves of $z = f(x, y)$ in the xy-plane, labeling each with the corresponding (constant) value of z. Figure 12.2.9 illustrates this process for a simple hill.

EXAMPLE 6 Figure 12.2.10 shows some typical contour curves on the paraboloid $z = 25 - x^2 - y^2$. Figure 12.2.11 shows the corresponding level curves.

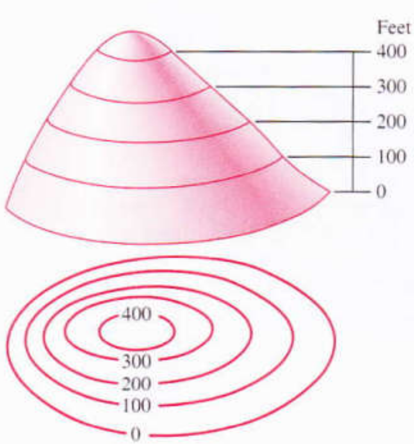


FIGURE 12.2.9 Contour curves and level curves for a hill.

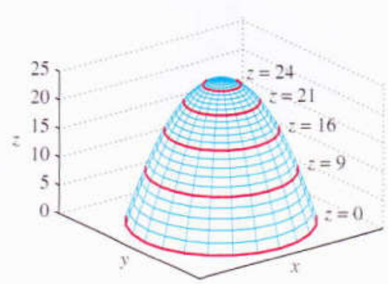


FIGURE 12.2.10 Contour curves on the surface $z = 25 - x^2 - y^2$.

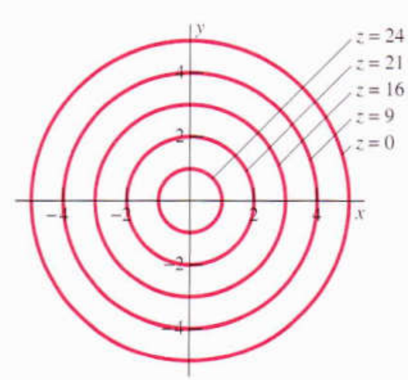


FIGURE 12.2.11 Level curves of the function $f(x, y) = 25 - x^2 - y^2$.

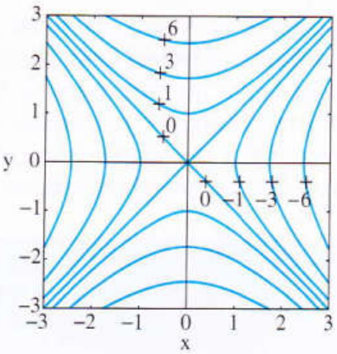


FIGURE 12.2.12 Level curves for the function $f(x, y) = y^2 - x^2$.

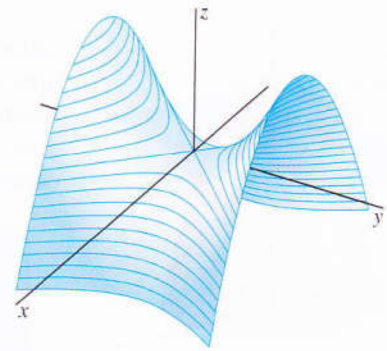


FIGURE 12.2.13 Contour curves on $z = y^2 - x^2$ (Example 7).

EXAMPLE 7 Sketch some typical level curves for the function $f(x, y) = y^2 - x^2$.

Solution If $k \neq 0$ then the curve $y^2 - x^2 = k$ is a hyperbola (Section 9.6). It opens along the y-axis if $k > 0$, along the x-axis if $k < 0$. If $k = 0$ then we have the equation $y^2 - x^2 = 0$, whose graph consists of the two straight lines $y = x$ and $y = -x$. Figure 12.2.12 shows some of the level curves, each labeled with the corresponding constant value of z. Figure 12.2.13 shows contour curves on the hyperbolic paraboloid $z = y^2 - x^2$ (Section 11.7). Note that the saddle point at the origin on the paraboloid corresponds to the intersection point of the two level curves $y = x$ and $y = -x$ in Fig. 12.2.12.

The graph of a function $f(x, y, z)$ of three variables cannot be drawn in three dimensions, but we can readily visualize its **level surfaces** of the form $f(x, y, z) = k$. For example, the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$ are spheres (spherical surfaces) centered at the origin. Thus the level surfaces of f are the sets in space on which the value $f(x, y, z)$ is constant.

If the function f gives the temperature at the location (x, y) or (x, y, z), then its level curves or surfaces are called **isotherms.** A weather map typically includes level curves of the ground-level atmospheric pressure; these are called **isobars.** Even though you may be able to construct the graph of a function of two variables, that graph might be so complicated that information about the function (or the situation it describes) is obscure. Frequently the level curves themselves give more information, as in weather maps. For example, Fig. 12.2.14 shows level curves for the annual numbers of days of

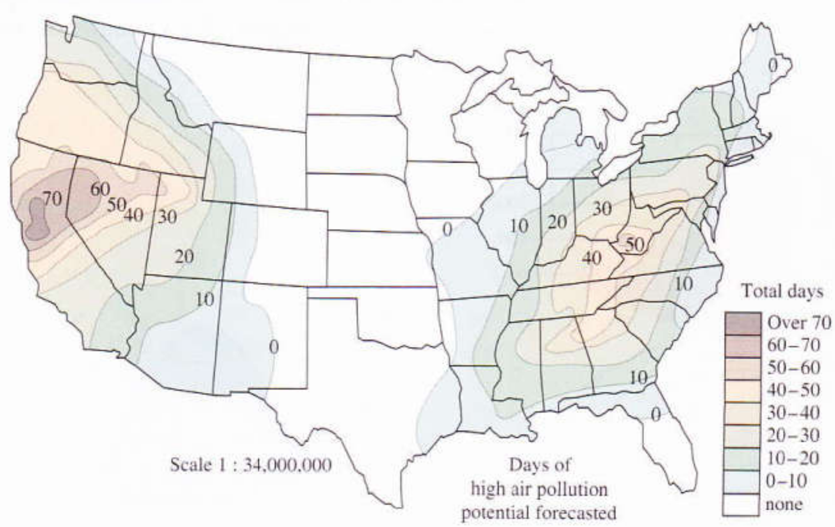


FIGURE 12.2.14 Days of high air pollution forecast in the United States (from National Atlas of the United States, U.S. Department of the Interior, 1970).

high air pollution forecast at different localities in the United States. The scale of this figure does not show local variations caused by individual cities. But a glance indicates that western Colorado, south Georgia, and central Illinois all expect the same number (10, in this case) of high-pollution days each year.

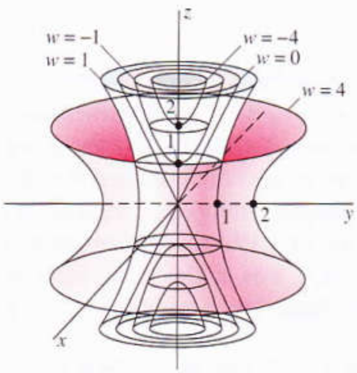


FIGURE 12.2.15 Some level surfaces of the function $w = f(x, y, z) = x^2 + y^2 - z^2$ (Example 8).

EXAMPLE 8 Figure 12.2.15 shows some level surfaces of the function

$$f(x, y, z) = x^2 + y^2 - z^2.$$

If $k > 0$, then the graph of $x^2 + y^2 - z^2 = k$ is a hyperboloid of one sheet, whereas if $k < 0$ it is a hyperboloid of two sheets. The cone $x^2 + y^2 - z^2 = 0$ lies between these two types of hyperboloids.

Computer Plots

Many computer systems have surface and contour plotting routines like the *Maple* commands

```
plot3d(y∧2 - x∧2, x = -3..3, y = -3..3);
with(plots):   contourplot(y∧2 - x∧2, x = -3..3, y = -3..3);
```

and the *Mathematica* commands

```
Plot3D[ y∧2 - x∧2, {x,-3,3}, {y,-3,3} ]
ContourPlot[ y∧2 - x∧2, {x,-3,3}, {y,-3,3} ]
```

for the function $f(x, y) = y^2 - x^2$ of Example 7.

EXAMPLE 9 Figure 12.2.16 shows both the graph and some projected contour curves of the function

$$f(x, y) = (x^2 - y^2)\exp(-x^2 - y^2).$$

Observe the patterns of nested level curves that indicate "pits" and "peaks" on the surface. In Fig. 12.2.17, the level curves that correspond to surface contours above the xy-plane are shown in red, while those that correspond to contours below the xy-plane are shown in blue. In this way we can distinguish between peaks and pits. It appears likely that the surface has peaks above the points $(\pm 1, 0)$ on the x-axis in the xy-plane, and has pits below the points $(0, \pm 1)$ on the y-axis. Because $f(x, \pm x) \equiv 0$, the two 45° lines $y = \pm x$ in Fig. 12.2.17 are also level curves; they intersect at the point $(0, 0)$ in the plane that corresponds to a saddle point or "pass" (as in *mountain pass*) on the surface.

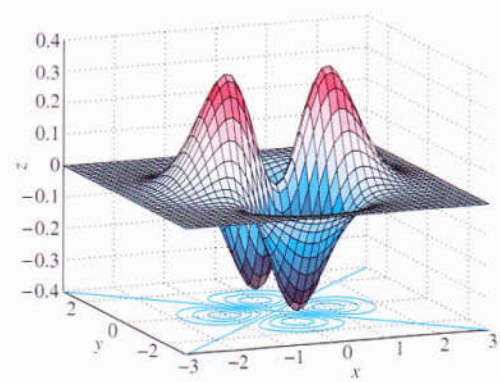


FIGURE 12.2.16 The graph and projected contour curves of the function $f(x, y) = (x^2 - y^2)e^{-x^2-y^2}$.

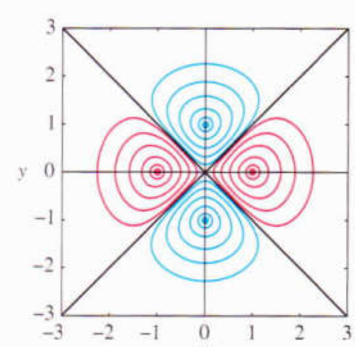


FIGURE 12.2.17 Level curves for the function $f(x, y) = (x^2 - y^2)e^{-x^2-y^2}$.

REMARK In Section 12.5 we will study analytic methods for locating maximum and minimum points of functions of two variables *exactly*. But Example 9 indicates that plots of level curves provide a valuable tool for locating them *approximately*.

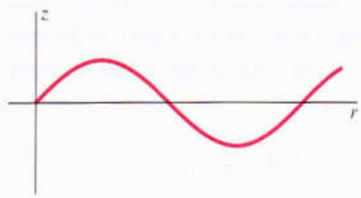


FIGURE 12.2.18 The curve $z = \sin r$ (Example 10).

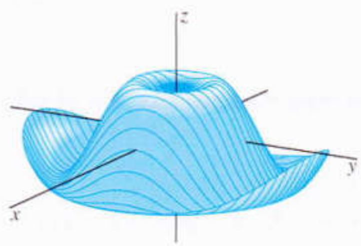


FIGURE 12.2.19 The hat surface $z = \sin\sqrt{x^2 + y^2}$ (Example 10).

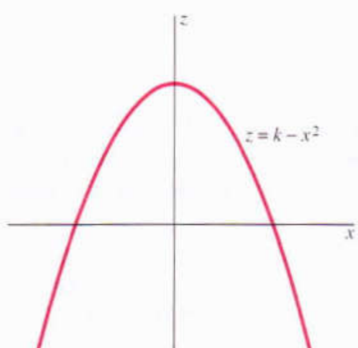


FIGURE 12.2.20 The intersection of $z = f(x, y)$ and the plane $y = y_0$ (Example 11).

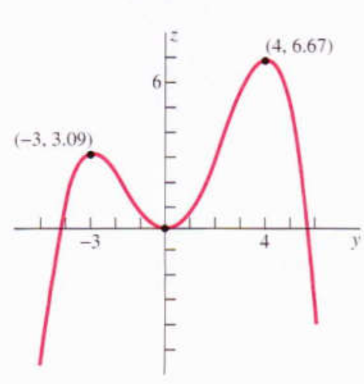


FIGURE 12.2.21 The curve $z = \frac{3}{4}y^2 + \frac{1}{24}y^3 - \frac{1}{32}y^4$ (Example 11).

EXAMPLE 10 The surface

$$z = \sin\sqrt{x^2 + y^2} \tag{3}$$

is symmetrical with respect to the z-axis, because Eq. (3) reduces to the equation $z = \sin r$ (Fig. 12.2.18) in terms of the radial coordinate $r = \sqrt{x^2 + y^2}$ that measures perpendicular distance from the z-axis. The *surface* $z = \sin r$ is generated by revolving the curve $z = \sin x$ around the z-axis. Hence its level curves are circles centered at the origin in the xy-plane. For instance, $z = 0$ if r is an integral multiple of π, whereas $z = \pm 1$ if r is any odd integral multiple of $\pi/2$. Figure 12.2.19 shows traces of this surface in planes parallel to the yz-plane. The "hat effect" was achieved by plotting (x, y, z) for those points (x, y) that lie within a certain ellipse in the xy-plane.

Given an arbitrary function $f(x, y)$, it can be quite a challenge to construct by hand a picture of the surface $z = f(x, y)$. Example 11 illustrates some special techniques that may be useful. Additional surface-sketching techniques will appear in the remainder of this chapter.

EXAMPLE 11 Investigate the graph of the function

$$f(x, y) = \tfrac{3}{4}y^2 + \tfrac{1}{24}y^3 - \tfrac{1}{32}y^4 - x^2. \tag{4}$$

Solution The key feature in Eq. (4) is that the right-hand side is the *sum* of a function of x and a function of y. If we set $x = 0$, we get the curve

$$z = \tfrac{3}{4}y^2 + \tfrac{1}{24}y^3 - \tfrac{1}{32}y^4 \tag{5}$$

in which the surface $z = f(x, y)$ intersects the yz-plane. But if we set $y = y_0$ in Eq. (4), we get

$$z = \left(\tfrac{3}{4}y_0^2 + \tfrac{1}{24}y_0^3 - \tfrac{1}{32}y_0^4\right) - x^2;$$

that is,

$$z = k - x^2, \tag{6}$$

which is the equation of a parabola in the xz-plane. Hence the trace of $z = f(x, y)$ in each plane $y = y_0$ is a parabola of the form in Eq. (6) (Fig. 12.2.20).

We can use the techniques of Section 4.5 to sketch the curve in Eq. (5). Calculating the derivative of z with respect to y, we get

$$\frac{dz}{dy} = \frac{3}{2}y + \frac{1}{8}y^2 - \frac{1}{8}y^3 = -\frac{1}{8}y(y^2 - y - 12) = -\frac{1}{8}y(y + 3)(y - 4).$$

Hence the critical points are $y = -3$, $y = 0$, and $y = 4$. The corresponding values of z are

$$f(0, -3) = \tfrac{99}{32} \approx 3.09, \quad f(0, 0) = 0, \quad \text{and} \quad f(0, 4) = \tfrac{20}{3} \approx 6.67.$$

Because $z \to -\infty$ as $y \to \pm\infty$, it follows readily that the graph of Eq. (5) looks like that in Fig. 12.2.21.

Now we can see what the surface $z = f(x, y)$ looks like. Each vertical plane $y = y_0$ intersects the curve in Eq. (5) at a single point, and this point is the vertex of a parabola that opens downward like that in Eq. (6); this parabola is the intersection of the plane and the surface. Thus the surface $z = f(x, y)$ is generated by translating the vertex of such a parabola along the curve

$$z = \tfrac{3}{4}y^2 + \tfrac{1}{24}y^3 - \tfrac{1}{32}y^4,$$

as indicated in Fig. 12.2.22.

Figure 12.2.23 shows some typical contour curves on this surface. They indicate that the surface resembles two peaks separated by a mountain pass. Figure 12.2.24 shows a computer plot of level curves of the function $f(x, y)$. The nested level curves enclosing the points $(0, -3)$ and $(0, 4)$ correspond to the peaks at the point $(0, -3, \frac{99}{32})$ and $(0, 4, \frac{20}{3})$ on the surface $z = f(x, y)$. The level figure-eight curve through $(0, 0)$

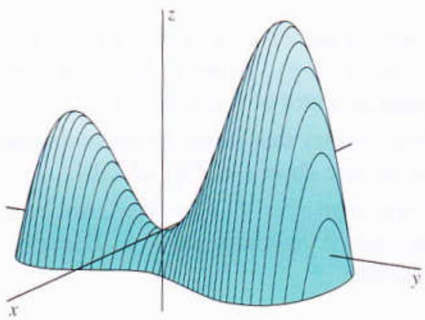


FIGURE 12.2.22 Trace parabolas of $z = f(x, y)$ (Example 11).

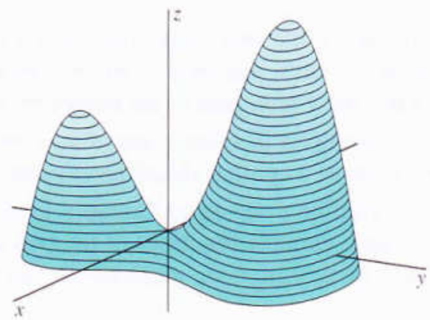


FIGURE 12.2.23 Contour curves on $z = f(x, y)$ (Example 11).

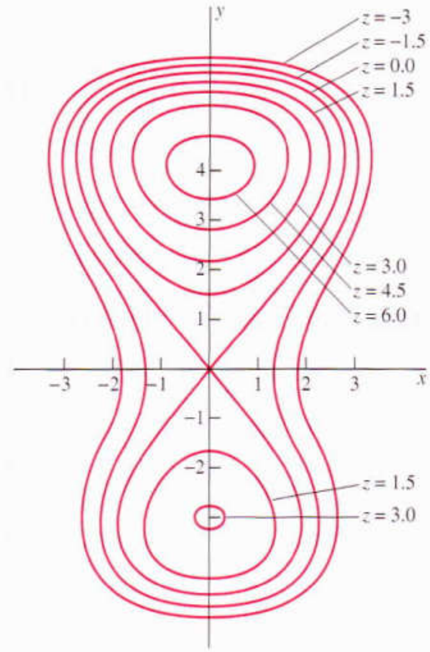


FIGURE 12.2.24 Level curves of the function $f(x, y) = \frac{3}{4}y^2 + \frac{1}{24}y^3 - \frac{1}{32}y^4 - x^2$ (Example 11).

marks the saddle point (or pass) that we see at the origin on the surface in Figs. 12.2.22 and 12.2.23. Extreme values and saddle points of functions of two variables are discussed in Sections 12.5 and 12.10.

12.2 TRUE/FALSE STUDY GUIDE

Use the following true/false items to check your reading and review of this section. You may consult the hints provided in the answer section.

1. Suppose the function f of two variables is defined by a formula giving the value $f(x, y)$ in terms of x and y. If the domain D is not explicitly specified, then we take D to consist of all points for which the given formula is meaningful.
2. The domain of the function f defined by the formula
$$f(x, y) = \sqrt{25 - x^2 - y^2}$$
is the set of all points (x, y) whose distance from the origin $(0, 0)$ is less than 5.
3. If the cost $C(x, y)$ of a box with base of length x and height y is given by
$$C = 4xy + \frac{288}{x} + \frac{96}{y},$$
then C is an independent variable and x and y are dependent variables.
4. The *graph* of the function f of two variables is the set of all points in space with coordinates of the form $(x, y, f(x, y))$.
5. The graph of the function $f(x, y) = 2 - \frac{1}{2}x - \frac{1}{3}y$ is a plane.
6. The graph of the function $g(x, y) = \frac{1}{2}\sqrt{4 - 4x^2 - y^2}$ is an ellipsoid.
7. A *level curve* of a function f of two variables is precisely the same thing as a *contour curve* of f.
8. If k is a constant, then the graph of the equation $x^2 + y^2 - z^2 = k$ is a hyperboloid of one sheet, because there is one minus sign on the left-hand side of the equation.
9. The pattern of level curves of a function $f(x, y)$ looks essentially the same near a point (x, y) corresponding to a "peak" on the surface $z = f(x, y)$ as near a

point corresponding to a saddle point or "pass." In particular, in either case we see level curves encircling the point in question.

10. Every level curve of the function

$$f(x, y) = \frac{3}{4}y^2 + \frac{1}{24}y^3 - \frac{1}{32}y^4 - x^2$$

is a closed curve that encircles either the point $(0, -3)$ or the point $(0, 4)$.

12.2 CONCEPTS: QUESTIONS AND DISCUSSION

1. Summarize the relationship between the level curves of a function $f(x, y)$ and the pits, peaks, and passes on the surface $z = f(x, y)$. In short, how can you locate likely pits, peaks, and passes by looking at a plot of level curves?
2. Give examples of other types of data for your country that might be presented in the form of a contour (level curve) map like the one shown in Fig. 12.2.14.
3. The function graphed in Example 11 is of the form $z = f(x) + g(y)$, the sum of single-variable functions of the two independent variables x and y. Describe a way of sketching the graph of any such function.

12.2 PROBLEMS

In Problems 1 through 20, state the largest possible domain of definition of the given function f.

1. $f(x, y) = 4 - 3x - 2y$

2. $f(x, y) = \sqrt{x^2 + 2y^2}$

3. $f(x, y) = \dfrac{1}{x^2 + y^2}$

4. $f(x, y) = \dfrac{1}{x - y}$

5. $f(x, y) = \sqrt[3]{y - x^2}$

6. $f(x, y) = \sqrt{2x} + \sqrt[3]{3y}$

7. $f(x, y) = \sin^{-1}(x^2 + y^2)$

8. $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$

9. $f(x, y) = \exp(-x^2 - y^2)$ (Fig. 12.2.25)

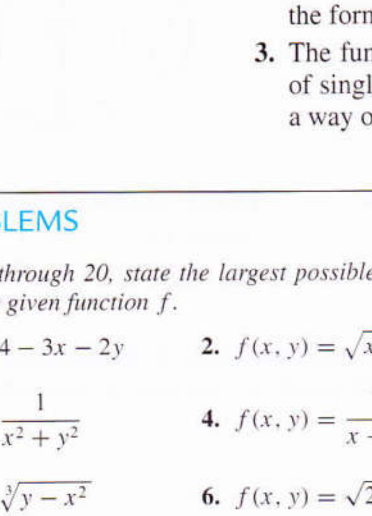


FIGURE 12.2.25 The graph of the function of Problem 9.

10. $f(x, y) = \ln(x^2 - y^2 - 1)$

11. $f(x\ y) = \ln(y - x)$

12. $f(x, y) = \sqrt{4 - x^2 - y^2}$

13. $f(x, y) = \dfrac{1 + \sin xy}{xy}$

14. $f(x, y) = \dfrac{1 + \sin xy}{x^2 + y^2}$ (Fig. 12.2.26)

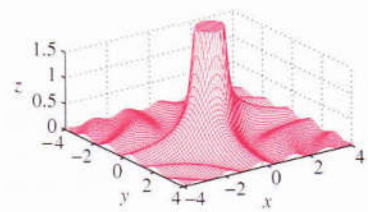


FIGURE 12.2.26 The graph $z = \dfrac{1 + \sin(xy)}{x^2 + y^2}$ of Problem 14.

15. $f(x, y) = \dfrac{xy}{x^2 - y^2}$

16. $f(x, y, z) = \dfrac{1}{\sqrt{z - x^2 - y^2}}$

17. $f(x, y, z) = \exp\left(\dfrac{1}{x^2 + y^2 + z^2}\right)$

18. $f(x, y, z) = \ln(xyz)$

19. $f(x, y, z) = \ln(z - x^2 - y^2)$

20. $f(x, y, z) = \sin^{-1}(3 - x^2 - y^2 - z^2)$

In Problems 21 through 30, describe the graph of the function f.

21. $f(x, y) = 10$

22. $f(x, y) = x$

23. $f(x, y) = x + y$

24. $f(x, y) = \sqrt{x^2 + y^2}$

25. $f(x, y) = x^2 + y^2$

26. $f(x, y) = 4 - x^2 - y^2$

27. $f(x, y) = \sqrt{4 - x^2 - y^2}$

28. $f(x, y) = 16 - y^2$

29. $f(x, y) = 10 - \sqrt{x^2 + y^2}$

30. $f(x, y) = -\sqrt{36 - 4x^2 - 9y^2}$

In Problems 31 through 40, sketch some typical level curves of the function f.

31. $f(x, y) = x - y$

32. $f(x, y) = x^2 - y^2$

33. $f(x, y) = x^2 + 4y^2$

34. $f(x, y) = y - x^2$

35. $f(x, y) = y - x^3$

36. $f(x, y) = y - \cos x$

37. $f(x, y) = x^2 + y^2 - 4x$

38. $f(x, y) = x^2 + y^2 - 6x + 4y + 7$

39. $f(x, y) = \exp(-x^2 - y^2)$

40. $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$

In Problems 41 through 46, describe the level surfaces of the function f.

41. $f(x, y, z) = x^2 + y^2 - z$

42. $f(x, y, z) = z + \sqrt{x^2 + y^2}$

43. $f(x, y, z) = x^2 + y^2 + z^2 - 4x - 2y - 6z$

44. $f(x, y, z) = z^2 - x^2 - y^2$

45. $f(x, y, z) = x^2 + 4y^2 - 4x - 8y + 17$

46. $f(x, y, z) = x^2 + y^2 + 25$

In Problems 47 through 52, the function $f(x, y)$ is the sum of a function of x and a function of y. Hence you can use the method of Example 11 to construct a sketch of the surface $z = f(x, y)$. Match each function with its graph among Figs. 12.2.27 through 12.2.32.

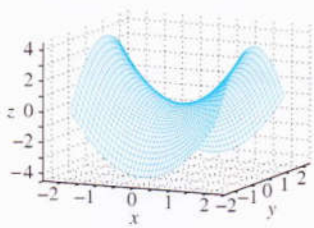


FIGURE 12.2.27

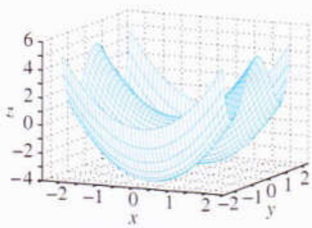


FIGURE 12.2.28

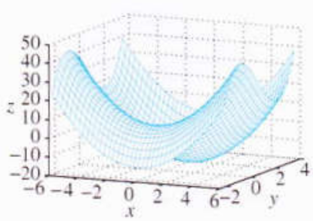


FIGURE 12.2.29

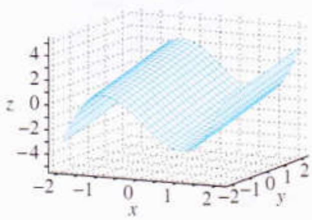


FIGURE 12.2.30

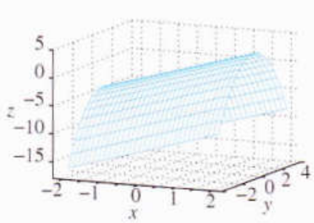


FIGURE 12.2.31

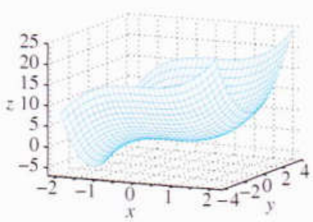


FIGURE 12.2.32

47. $f(x, y) = x^3 + y^2$

48. $f(x, y) = 2x - y^2$

49. $f(x, y) = x^3 - 3x + \frac{1}{2}y$

50. $f(x, y) = x^2 - y^2$

51. $f(x, y) = x^2 + y^4 - 4y^2$

52. $f(x, y) = 2y^3 - 3y^2 - 12y + x^2$

Problems 53 through 58 show the graphs of six functions $z = f(x, y)$. Figures 12.2.39 through 12.2.44 show level curve plots for the same functions but in another order; the level curves in each figure correspond to contours at equally spaced heights on the surface $z = f(x, y)$. Match each surface with its level curves.

53. $z = \dfrac{1}{1 + x^2 + y^2}$, $|x| \leqq 2$, $|y| \leqq 2$ (Fig. 12.2.33)

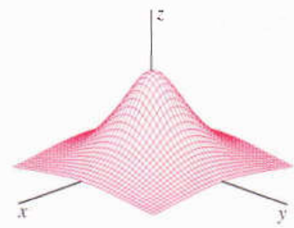


FIGURE 12.2.33 $z = \dfrac{1}{1 + x^2 + y^2}$, $|x| \leqq 2$, $|y| \leqq 2$.

54. $z = r^2 \exp(-r^2) \cos^2\left(\frac{3}{2}\theta\right)$, $|x| \leqq 3$, $|y| \leqq 3$ (Fig. 12.2.34)

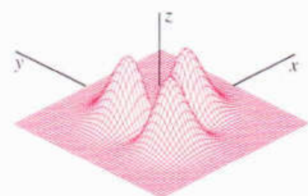


FIGURE 12.2.34 $z = r^2 \exp(-r^2) \cos^2\left(\frac{3}{2}\theta\right)$, $|x| \leqq 3$, $|y| \leqq 3$, $r \geqq 0$.

55. $z = \cos\sqrt{x^2 + y^2}$, $|x| \leqq 10$, $|y| \leqq 10$ (Fig. 12.2.35)

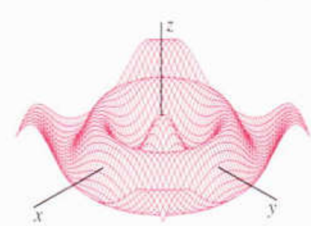


FIGURE 12.2.35 $z = \cos\sqrt{x^2 + y^2}$, $|x| \leqq 10$, $|y| \leqq 10$.

56. $z = x \exp(-x^2 - y^2)$, $|x| \leqq 2$, $|y| \leqq 2$ (Fig. 12.2.36)

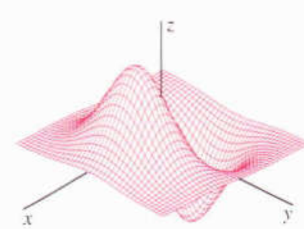


FIGURE 12.2.36 $z = x \exp(-x^2 - y^2)$, $|x| \leqq 2$, $|y| \leqq 2$.

57. $z = 3(x^2 + 3y^2)\exp(-x^2 - y^2)$, $|x| \leqq 2.5$, $|y| \leqq 2.5$ (Fig. 12.2.37)

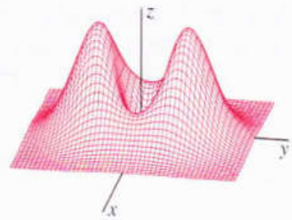


FIGURE 12.2.37 $z = 3(x^2 + 3y^2)\exp(-x^2 - y^2)$, $|x| \leqq 2.5$, $|y| \leqq 2.5$.

58. $z = xy\exp\left(-\frac{1}{2}(x^2 + y^2)\right)$, $|x| \leqq 3.5$, $|y| \leqq 3.5$ (Fig. 12.2.38)

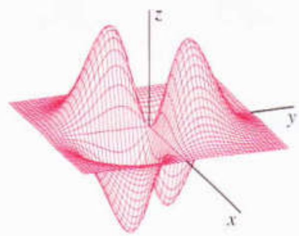


FIGURE 12.2.38 $z = xy\exp\left(-\frac{1}{2}(x^2 + y^2)\right)$, $|x| \leqq 3.5$, $|y| \leqq 3.5$.

59. Use a computer to investigate surfaces of the form $z = (ax + by)\exp(-x^2 - y^2)$. How do the number and locations of apparent peaks and pits depend on the values of the constants a and b?

60. Use a computer to graph the surface $z = (ax^2 + 2bxy + cy^2)\exp(-x^2 - y^2)$ with different values of the parameters a, b, and c. Describe the different types of surfaces that are obtained in this way. How do the number and locations of apparent peaks and pits depend on the values of the constants a, b, and c?

61. Use a computer to investigate surfaces of the form $z = r^2\exp(-r^2)\sin n\theta$. How do the number and locations of apparent peaks and pits depend on the value of the integer n?

62. Repeat Problem 61 with surfaces of the form $z = r^2\exp(-r^2)\cos^2 n\theta$.

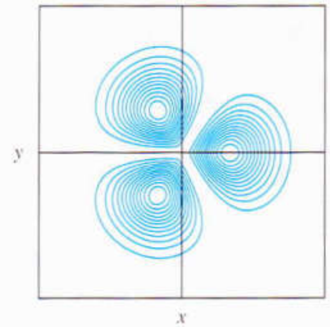


FIGURE 12.2.39

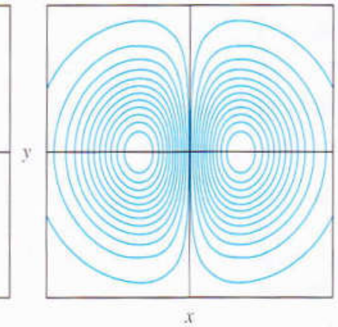


FIGURE 12.2.40

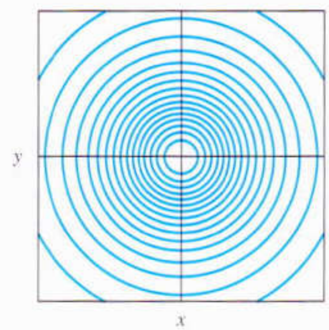


FIGURE 12.2.41

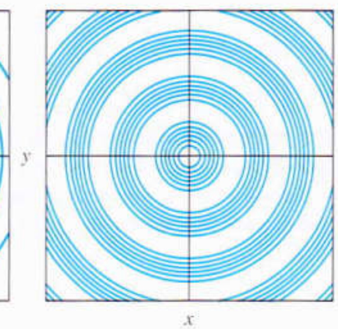


FIGURE 12.2.42

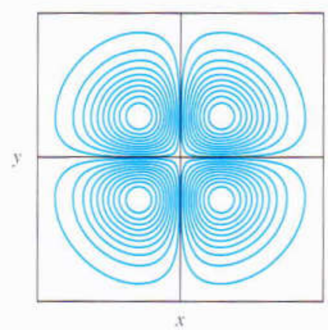


FIGURE 12.2.43

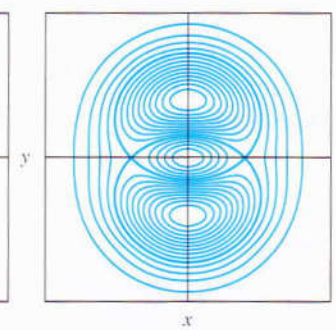


FIGURE 12.2.44

12.3 LIMITS AND CONTINUITY

We need limits of functions of several variables for the same reasons that we needed limits of functions of a single variable—so that we can discuss continuity, slopes, and rates of change. Both the definition and the basic properties of limits of functions of several variables are essentially the same as those that we stated in Section 2.2 for functions of a single variable. For simplicity, we shall state them here only for functions of two variables x and y; for a function of three variables, the pair (x, y) should be replaced with the triple (x, y, z).

For a function f of two variables, we ask what number (if any) the values $f(x, y)$ approach as (x, y) approaches the fixed point (a, b) in the coordinate plane. For a function f of three variables, we ask what number (if any) the values $f(x, y, z)$ approach as (x, y, z) approaches the fixed point (a, b, c) in space.