Exercícios de limites envolvendo funções trigonmétricas:

1. $\displaystyle\lim_{x\to 0} \frac{\operatorname{sen}(kx)}{x}$

2. $\displaystyle\lim_{x\to 1} \frac{\cos\left(\frac{\pi x}{2}\right)}{1-x}$

3. $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+\operatorname{sen} x}-\sqrt{1-\operatorname{sen} x}}{x}$

4. $\displaystyle\lim_{x\to \infty} x \operatorname{sen}\left(\frac{1}{x}\right)$

5. $\displaystyle\lim_{x\to 0} x \operatorname{sen}\left(\frac{1}{x}\right)$

6. $\displaystyle\lim_{x\to 1} (1-x)\tan\left(\frac{\pi x}{2}\right)$

7. $\displaystyle\lim_{x\to 0} \cot(2x)\cdot\cot\left(\frac{\pi}{2}-x\right)$

8. $\displaystyle\lim_{x\to 0} \frac{\operatorname{sen} ax-\operatorname{sen} bx}{x}$

1. $L = \lim_{x\to 0} \frac{\text{sen}\,(kx)}{x} = \lim_{x\to 0} k \frac{\text{sen}\,(kx)}{kx}$

   Usando a transformação de variáveis $t = kx$ com $\lim_{x\to 0} t = 0$,

   $L = k \underbrace{\lim_{t\to 0} \frac{\text{sen}\,(t)}{t}}_{=1 (L.f.)} = k$

2. $L = \lim_{x\to 1} \frac{\cos\left(\frac{\pi x}{2}\right)}{1-x} = \frac{\text{“}0\text{”}}{0}$

   Ideia: $t = 1-x$, $\lim_{x\to 1} t = 0$,

   $L = \lim_{t\to 0} \frac{\cos\left(\frac{\pi(1-t)}{2}\right)}{t} = \lim_{t\to 0} \frac{\cos\left(\frac{\pi}{2} - \frac{\pi t}{2}\right)}{t} = \lim_{t\to 0} \frac{\text{sen}\left(\frac{\pi t}{2}\right)}{t}$

   $s = \frac{\pi t}{2}$, $\lim_{t\to 0} s = 0$

   $L = \lim_{s\to 0} \frac{\text{sen}\,(s)}{\frac{2s}{\pi}} = \frac{\pi}{2} \underbrace{\lim_{s\to 0} \frac{\text{sen}\,(s)}{s}}_{=1} = \frac{\pi}{2}$

3. $L = \lim_{x\to 0} \frac{\sqrt{1+\text{sen}\,x} - \sqrt{1-\text{sen}\,x}}{x} = \frac{\text{“}0\text{”}}{0}$

   $L = \lim_{x\to 0} \frac{\sqrt{1+\text{sen}\,x} - \sqrt{1-\text{sen}\,x}}{x} \cdot \frac{\sqrt{1+\text{sen}\,x} + \sqrt{1-\text{sen}\,x}}{\sqrt{1+\text{sen}\,x} + \sqrt{1-\text{sen}\,x}}$

   $= \lim_{x\to 0} \frac{(1+\text{sen}\,x) - (1-\text{sen}\,x)}{x(\sqrt{1+\text{sen}\,x} + \sqrt{1-\text{sen}\,x})} = \lim_{x\to 0} \frac{2\,\text{sen}\,x}{x(\sqrt{1+\text{sen}\,x} + \sqrt{1-\text{sen}\,x})}$

   $= \lim_{x\to 0} 2 \underbrace{\frac{\text{sen}\,x}{x}}_{\to 1} \cdot \underbrace{\frac{1}{(\sqrt{1+\text{sen}\,x} + \sqrt{1-\text{sen}\,x})}}_{\to \frac{1}{1+1}} = 1$

4. $L = \lim_{x\to\infty} x\,\text{sen}\left(\frac{1}{x}\right) = \text{“}\infty \cdot 0\text{”}$

   $t = \frac{1}{x}$ $\lim_{x\to\infty} t = 0^+$

   $L = \underbrace{\lim_{t\to 0^+} \frac{\text{sen}\,t}{t}}_{=1} = 1$

5. $\lim_{x\to 0} x\,\text{sen}\left(\frac{1}{x}\right)$ não é fatorável pois $\lim_{x\to 0} \text{sen}\left(\frac{1}{x}\right)$ não existe.

   Utilizamos que $-1 \leq \text{sen}\left(\frac{1}{x}\right) \leq 1$.

   Multiplicando essa desigualdade por x, obtemos para $x > 0$

   $-x \leq x\,\text{sen}\left(\frac{1}{x}\right) \leq x$ e para $x < 0$ $-x \geq x\,\text{sen}\left(\frac{1}{x}\right) \geq x$

Em ambos os casos, observamos que os limites das funções dos lados direito e esquerdo são nulos, i.e.,

$$\lim_{x\to 0} -x = \lim_{x\to 0} x = 0$$

o que implica pelo Teorema do Confronto que $\lim\limits_{x\to 0} x \operatorname{sen}\left(\dfrac{1}{x}\right) = 0$

6. $L = \lim\limits_{x\to 1}(1-x)\tan\left(\dfrac{\pi x}{2}\right) = \text{“}0\cdot\infty\text{”}$

$$L = \lim_{x\to 1}(1-x)\frac{\operatorname{sen}\left(\frac{\pi x}{2}\right)}{\cos\left(\frac{\pi x}{2}\right)} = \lim_{x\to 1}\underbrace{\frac{1-x}{\cos\left(\frac{\pi x}{2}\right)}}_{\to\frac{2}{\pi}(Ex.2)}\underbrace{\operatorname{sen}\left(\frac{\pi x}{2}\right)}_{\to 1} = \frac{2}{\pi}$$

7. $L = \lim\limits_{x\to 0}\cot(2x)\cdot\cot\left(\dfrac{\pi}{2}-x\right) = \text{“}\infty\cdot 0\text{”}$

$$L = \lim_{x\to 0}\frac{\cos(2x)}{\operatorname{sen}(2x)}\cdot\frac{\cos\left(\frac{\pi}{2}-x\right)}{\operatorname{sen}\left(\frac{\pi}{2}-x\right)} = \lim_{x\to 0}\underbrace{\frac{\cos(2x)}{\operatorname{sen}\left(\frac{\pi}{2}-x\right)}}_{\to 1}\cdot\underbrace{\frac{\cos\left(\frac{\pi}{2}-x\right)}{\operatorname{sen}(2x)}}_{\to\frac{\text{“}0\text{”}}{0}}$$

$$\lim_{x\to 0}\frac{\cos\left(\frac{\pi}{2}-x\right)}{\operatorname{sen}(2x)} = \lim_{x\to 0}\frac{\operatorname{sen}(x)}{\operatorname{sen}(2x)} = \lim_{x\to 0}\underbrace{\frac{\operatorname{sen}(x)}{x}}_{\to 1}\cdot\underbrace{\frac{x}{\operatorname{sen}(2x)}}_{\to\frac{1}{2}} = \frac{1}{2}$$

Assim, $L = 1\cdot\dfrac{1}{2} = \dfrac{1}{2}$

8. $L = \lim\limits_{x\to 0}\dfrac{\operatorname{sen} ax - \operatorname{sen} bx}{x} = a\underbrace{\lim\limits_{x\to 0}\dfrac{\operatorname{sen} ax}{ax}}_{=1} - b\underbrace{\lim\limits_{x\to 0}\dfrac{\operatorname{sen} bx}{bx}}_{=1} = a-b$