CLASSIFICATION AND EXISTENCE OF DOUBLY-PERIODIC INSTANTONS

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Abstract

We present a classification of SU(2) instantons on \(T^2 \times \mathbb{R}^2\) according to their asymptotic behaviour. We then study the existence of such instantons for different values of the asymptotic parameters, uncovering some surprising non-existence results. We also describe explicitly the moduli space for unit charge.

1. Introduction

Anti-self-dual connections on \(T^2 \times \mathbb{R}^2\) with quadratic curvature decay, so-called doubly-periodic instantons, have been studied by various authors both from the mathematical \([1, 9, 10]\) and from the physical \([4, 7, 12]\) points of view.

For physicists, the motivation to study periodic instantons on \(\mathbb{R}^4\), that is, anti-self-dual connections on \(S^1 \times \mathbb{R}^3\) (calorons), \(T^2 \times \mathbb{R}^2\) (doubly-periodic instantons) and \(T^3 \times \mathbb{R}\) (spatially-periodic instantons), comes from the observation that these can be viewed as limiting cases of configurations on the four-dimensional torus (regarded as instantons periodic on all four directions) where either three, two or one directions are taken to be very large when compared to the others \([6]\).

In particular, doubly-periodic instantons are closely related to self-dual, vortex-like configurations on \(\mathbb{R}^4\), that is, configurations that are concentrated near a two-dimensional plane \([7]\).

One striking feature is that even though there are no charge-one instantons on \(T^4\) \([3]\), partially periodic instantons of unit charge can be shown to exist. Therefore, there must be obstructions to folding instantons along the non-compact directions (that is, passing from an instanton on \(T^d \times \mathbb{R}^{4-d}\) to an instanton on \(T^{d+1} \times \mathbb{R}^{3-d}\)). However, the precise mechanism underlying such phenomena is yet to be understood. The idea is that the folding of charge-one partially-periodic instantons inevitably leads to a singular instanton on \(T^4\) \([4]\).

Charge-one calorons have been studied in some detail by Kraan and van Baal \([13]\), among others. A closer study of spatially-periodic instantons, though, remains as a glaring gap in the mathematical literature; see, however, \([15]\).

The careful study of doubly-periodic instantons with unit charge is the main motivation for the present work. We begin by summarizing the classification of rank-2 instantons and their correspondence with certain holomorphic vector bundles, as established in \([1]\). We then use the Fourier–Mukai transform to relate doubly-periodic instantons and rational maps \(\mathbb{P}^1 \rightarrow \mathbb{P}^1\), providing a more detailed description of the moduli space.
Although the existence of doubly-periodic instantons was guaranteed in [9], not all possible values of the asymptotic parameters can be realized. Surprisingly, global topological obstructions for particular values do arise. In section 5, we describe all such obstructions for doubly-periodic instantons with unit charge. We argue that there exist a basic doubly-periodic instanton, out of which all others are obtained via translations. Finally, in section 6, one existence and one non-existence result for instantons with higher charge and given asymptotic parameters are discussed.

2. Classification

Consider an SU(2) bundle $E \to T^2 \times \mathbb{R}^2$. As in [1, 9, 10], let $A$ be a connection on $E$ such that $|F_A| = O(r^{-2})$ with respect to the Euclidean metric on $T^2 \times \mathbb{R}^2$ normalized so that $T^2$ has unit volume. As usual, we define the charge of $A$ by the formula

$$k = \frac{1}{8\pi^2} \int_{T^2 \times \mathbb{R}^2} |F_A|^2.$$

Special solutions of the anti-self-duality equations may be obtained by restricting to torus invariant connections. Such instantons come from solutions $(B, \psi)$ of Hitchin’s equations on $\mathbb{R}^2$:

$$F_B + [\psi, \psi^*] = 0, \quad \bar{\partial}_B \psi = 0$$

in the following way. Recall that $B$ is an SU(2) connection on $\mathbb{C}$, and $\psi$ is a $(1,0)$-form with values in $\mathfrak{sl}(2)$. Let $\psi = \frac{1}{2}(\psi_0 + i\psi_1)dw$, and consider the connection (where $x$ and $y$ are coordinates on $T^2$):

$$A_0 = B + \psi_0 dx + \psi_1 dy$$

which is a torus invariant instanton connection. Assuming that $|F_{A_0}| = O(r^{-2})$, the asymptotic behaviour of solutions $(B, \psi)$ is given by one of the following models:

$$B = d, \quad \psi = \left( \begin{array}{cc} \lambda & 0 \\ 0 & -\lambda \end{array} \right) dw,$$

(1)

$$B = d + i \left( \begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right) d\theta, \quad \psi = \left( \begin{array}{cc} \mu & 0 \\ 0 & -\mu \end{array} \right) \frac{dw}{w},$$

(2)

$$B = d + i \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) \frac{d\theta}{\ln r^2}, \quad \psi = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \frac{dw}{w \ln r^2},$$

(3)

where $\lambda, \mu \in \mathbb{C}$ and $-\frac{1}{2} \leq \alpha < \frac{1}{2}$. The solutions of examples (1) and (2) can be superimposed, and such superpositions are called the semisimple solutions. On the other hand, solutions of example (3) cannot be superimposed with the others; these are called the nilpotent solutions, and can only exist when $\lambda = \mu = \alpha = 0$.

The torus invariant instanton is then given, in the semisimple case, by

$$A_0 = d + i \left( \begin{array}{cc} a_0 & 0 \\ 0 & -a_0 \end{array} \right),$$

(4)

Take $f : T^2 \times \mathbb{R}^2 \to \mathbb{R}$; then $f = O(r^{-p})$ if $\lim_{r \to \infty} r^p \cdot f$ exists, where $r$ is the radial coordinate in $\mathbb{R}^2$. 

with
\[ a_0 = \lambda_1dx + \lambda_2dy + (\mu_1 \cos \theta - \mu_2 \sin \theta) \frac{dx}{r} + (\mu_1 \sin \theta + \mu_2 \cos \theta) \frac{dy}{r} + \alpha d\theta, \]
while in the nilpotent case we have
\[ A_0 = d + i \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ \ln r & \ln r & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{-i\theta}(dx - idy) \\ e^{i\theta}(dx + idy) & 0 \end{pmatrix}. \] (5)

Remark that the connection \( A_0 \) has a flat limit over the torus at infinity, that is, as \( r \to \infty \):
\[ d + i \begin{pmatrix} \lambda_1dx + \lambda_2dy & 0 \\ 0 & -\lambda_1dx - \lambda_2dy \end{pmatrix}. \] (6)

This limiting flat connection underlies a holomorphic vector bundle \( \xi_0 \oplus -\xi_0 \), where the elements \( \pm \xi_0 \) of the dual torus \( \hat{T}^2 \) are called the asymptotic states of the connection. Moreover, \( \mu \) and \( \alpha \) are called the residue and the limiting holonomy of \( A \), respectively.

As was shown in [1], the three standard examples above completely describe the behaviour at infinity of doubly-periodic instantons with quadratic curvature decay.

**Theorem 2.1** Let \( A \) be a doubly-periodic instanton connection such that \( |F_A| = O(r^{-2}) \). Then there is a gauge near infinity such that
\[ A = A_0 + a, \]
where \( A_0 \) is one of the previous models (4) and (5), and, for some \( \delta > 0 \), in the semisimple case,
\[ |a| = O\left(\frac{1}{r^{1+\delta}}\right), \quad |\nabla A_0 a| = O\left(\frac{1}{r^{2+\delta}}\right); \]
in the nilpotent case,
\[ |a| = O\left(\frac{1}{r(\ln r)^{1+\delta}}\right), \quad |\nabla A_0 a| = O\left(\frac{1}{r^2(\ln r)^{2+\delta}}\right). \]

In short, doubly-periodic instantons can be classified according to their asymptotic parameters \((\pm \xi_0, \mu, \alpha)\).

**A remark on asymptotic decay.** As was mentioned in [1], the semisimple part of Theorem 2.1 can be proved under the weaker hypothesis that the curvature is \( O(r^{-1-\epsilon}) \). Thus if we begin with the assumption that \( |F_A| = O(r^{-1-\epsilon}) \) and apply the Nahm transform, we end up with a singular Higgs pair as described in [10]. We can then Nahm transform this Higgs pair back into a doubly-periodic instanton \( A' \) which is gauge equivalent to \( A \), but such that \( |F_{A'}| = O(r^{-2}) \); see [9]. Therefore we have proved the following statement.

**Proposition 2.2** Let \( A \) be a doubly-periodic instanton connection such that \( |F_A| = O(r^{-1-\epsilon}) \) for some \( \epsilon > 0 \) with non-trivial asymptotic state. Then actually \( |F_A| = O(r^{-2}) \).

The condition \( |F_A| = O(r^{-1-\epsilon}) \) is close to the finite action condition \( ||F_A||_{L^2} < \infty \). Indeed, it seems reasonable to pose the following conjecture.

**Conjecture.** Every doubly-periodic instanton connection of finite action \( ||F_A||_{L^2} < \infty \) has quadratic curvature decay, that is, \( |F_A| = O(r^{-2}) \).
Summary of results. Recall that the Nahm transform associates instantons on $T^d \times \mathbb{R}^{4-d}$ to (possibly singular) solutions of the dimensionally reduced anti-self-duality equations on $\hat{T}^d$, the torus dual to $T^d$. The case of doubly-periodic instantons ($d = 2$) was carefully described by the author in [9, 10], where it has been shown that doubly-periodic instantons are equivalent to certain singular solutions of Hitchin’s equations on a two-dimensional torus. The asymptotic parameters described in the previous section are translated into data describing the singularities of the Higgs pair; see [1] for the detailed statement. Since a theorem of Simpson [14] guarantees the existence of singular Higgs pairs for generic values of the singularities’ parameters, the existence of doubly-periodic instantons for all $k$ and generic values of the asymptotic parameters is also guaranteed via the Nahm transform.

More generally, the Nahm transform has so far been the single most powerful tool to prove existence of invariant instantons on $\mathbb{R}^4$ (for example, the ADHM construction of instantons, Hitchin’s approach to monopoles, etc.). On the other hand, the Nahm transform can also be used to prove non-existence results as well. A typical example is the non-existence of charge-one instantons over $T^4$ mentioned on the Introduction; see also Theorem 2.5 below for a similar result concerning doubly-periodic instantons.

In this paper, however, we adopt a different point of view, which seems better suited for our purposes. As suggested in [1], we will use a Hitchin–Kobayashi correspondence, that is, a correspondence between instantons on $T^2 \times \mathbb{R}^2$ and certain holomorphic vector bundles over $\mathbb{T} \times \mathbb{P}^1$, as our main tool to obtain more precise existence and non-existence results. Here, $\mathbb{T}$ denotes the elliptic curve coming from a choice of complex structure on $T^2$.

Since instantons are known to exist for generic values of the asymptotic parameters, the task becomes to determine what happens at non-generic values. We will now summarize the relevant results proved in this paper.

**Theorem 2.3** There are no instantons of unit charge with either $\xi_0 = -\xi_0$ or $\mu = 0$. These are the only obstructions to unit charge, that is, there exist charge-one instantons for all $(\pm \xi_0, \alpha, \mu)$ provided $\xi_0 \neq -\xi_0$ and $\mu \neq 0$.

The main lesson we take from this theorem is that asymptotic behaviour does impose obstructions to the existence of instantons! The occurrence of such obstructions is indeed rather surprising, since the asymptotic models (4) and (5) are well defined for all $(\pm \xi_0, \mu, \alpha)$. Thus we conclude that this must be a global, topological phenomenon. The author is unaware of the occurrence of similar phenomena in gauge theory.

**Theorem 2.4** Doubly-periodic instantons with charge $k \geq 2$ and $\mu \neq 0$ do exist.

In particular, we reobtain the fact that doubly-periodic instantons exist for all $k$ and generic values of the asymptotic parameters.

Finally, we state one last result that follows from the Nahm transform, but we do not prove it; see [1, Lemma 5.6].

**Theorem 2.5** There are no instantons with $\xi_0 \neq -\xi_0$ and $\mu = 0$, for all $k$.

In other words, asymptotic obstructions also occur at higher charge.

### 3. A Hitchin–Kobayashi correspondence

Before we can state precisely the Hitchin–Kobayashi correspondence for doubly-periodic instantons, a few definitions are necessary. Let $\mathcal{E}$ be a holomorphic vector bundle of rank two
over $T \times P^1$ such that $E|_{T_\infty} = \xi_0 \oplus -\xi_0$. Let $\mathcal{F} \subset \mathcal{E}$ be a locally-free subsheaf of rank 1; its $\alpha$-degree is defined as follows:

$$\alpha - \text{deg} \mathcal{F} = \begin{cases} c_1(\mathcal{F}|T) + \alpha & \text{if } \mathcal{F}|_{T_\infty} \subset -\xi_0, \\ c_1(\mathcal{F}|T) - \alpha & \text{if } \mathcal{F}|_{T_\infty} \subset \xi_0. \end{cases}$$

where $[T]$ is the fundamental class of $T$.

Define $\alpha$-stability of $\mathcal{E}$ as the condition that any (locally-free, rank-1) subsheaf $\mathcal{F} \subset \mathcal{E}$ such that either $\mathcal{F}|_{T_\infty} \subset -\xi_0$ or $\mathcal{F}|_{T_\infty} \subset \xi_0$ has negative $\alpha$-degree (we shall omit the $\alpha$ when there is no ambiguity).

**Theorem 3.1** There is a 1–1 correspondence between the following objects:

- SU(2) doubly-periodic instanton connections of charge $k$ with quadratic curvature decay and fixed asymptotic parameters $(\pm \xi_0, \alpha)$;
- $\alpha$-stable, rank-2 holomorphic vector bundles $\mathcal{E} \to T \times P^1$ with trivial determinant such that $c_2(\mathcal{E}) = k$ and $\mathcal{E}|_{T_\infty} = \xi_0 \oplus -\xi_0$.

The theorem translates the problem of studying doubly-periodic instantons into an exercise in algebraic geometry: it is enough to study instanton bundles, that is, rank-2 holomorphic vector bundles $\mathcal{E} \to T \times P^1$ with trivial determinant that are $\alpha$-stable for some $\alpha \in [0, \frac{1}{2})$ and such that $\mathcal{E}$ splits as a sum of degree zero line bundles at the torus at infinity.

Notice that instanton bundles are generically fibrewise semistable, in the sense that the restriction $\mathcal{E}|_{T_p}$ is semistable for generic $p \in P^1$. Let $\{u_j\}$ be the finite set of points such that $\mathcal{E}|_{T_{u_j}}$ is not semistable. Each such unstable point $u_j$ is assigned a multiplicity $m_j = h^0(T_{u_j}, \mathcal{E}|_{T_{u_j}})$.

An instanton bundle is said to be nilpotent (semisimple) if it is associated with a doubly-periodic instanton having nilpotent (semisimple) asymptotic behaviour.

**Lemma 3.2** If $\mathcal{E}$ is a nilpotent instanton bundle, then $\mathcal{E}|_{T_p}$ must be the non-trivial extension of $\mathcal{O}_T$ by itself for generic $p \in P^1 \setminus \infty$.

**Proof.** It is not difficult to see from (5) that $\mathcal{E}|_{T_p}$ is holomorphically equivalent to the non-trivial extension of $\mathcal{O}_T$ for every $p$ in a sufficiently small deleted neighbourhood of $\infty \in P^1$. Therefore, $h^0(T_p, \mathcal{E}|_{T_p}) = 1$ for every $p$ in some open subset of $P^1$. But $h^0(T_p, \mathcal{E}|_{T_p})$ is an upper semicontinuous function on $T \times P^1$ [8], so $h^0(T_p, \mathcal{E}|_{T_p}) = 1$ for generic $p \in P_1$. The statement now follows easily; see also [1, p. 356].

Fortunately, the subject of holomorphic bundles over elliptic surfaces is very well studied; let us now briefly describe one of the main techniques available.

**4. Fourier–Mukai transform and the graph of an instanton**

Let $\mathcal{F}$ be a sheaf on $T \times P^1$ and consider the following diagram.

\[ \begin{array}{ccc} T \times \hat{T} \times P^1 & \rightarrow & \hat{T} \times P^1 \\ \pi \downarrow & & \downarrow \hat{\pi} \\ T \times P^1 & \rightarrow & \hat{T} \times P^1 \end{array} \]
The Fourier–Mukai transform of $F$ is given by

$$\Psi(F) = R\hat{\pi}_*(\pi^*F \otimes \mathcal{P}),$$

(8)

where $\mathcal{P}$ denotes the pullback of the Poincaré bundle over $\mathbb{T} \times \hat{\mathbb{T}}$. If $F$ is locally free and generically fibrewise semistable, which is the case we are currently interested in, then $\Psi(F)$ is a torsion sheaf of pure dimension one on $\hat{\mathbb{T}} \times \mathbb{P}^1$ [11]. It is supported on a divisor $\mathcal{S}$ called the spectral curve of the sheaf $F$. Generically, one can show that $\mathcal{S}$ is a smooth curve of genus $2k - 1$ [10].

The graph of an instanton. Now recall that the moduli space of $S$-equivalence classes of semistable vector bundles of rank two with trivial determinant over an elliptic curve is just the graph of an instanton. Following Braam and Hurtubise [2], we define the graph $\Gamma(A)$ of the instanton $A$ to be the divisor in $\mathbb{P}^1 \times \hat{\mathbb{P}}^1$ given by

$$\Gamma(A) = \text{graph}(R) + \sum_{j=1}^{q} \{u_j\} \times \hat{\mathbb{P}}^1,$$

(9)

where $q$ is the total number of unstable points (counted without multiplicity). Clearly, $\Gamma(A)$ contains the graph of a rational map $R : \mathbb{P}^1 \to \hat{\mathbb{P}}^1$ whose degree is $k - \sum m_j$ (where $m_j = h^0(T_{u_j}, \mathcal{E}|_{T_{u_j}})$ is the multiplicity associated with the unstable point $u_j$). Moreover, it is not difficult to see that $\Gamma(A) = \mathcal{S}/\mathbb{Z}_2$.

If $A$ is an instanton with asymptotic parameters $(\pm \xi_0, \mu, \alpha)$, then $\Gamma(A)$ contains the point $(\infty, [\pm \xi_0]) \in \mathbb{P}^1 \times \hat{\mathbb{P}}^1$, where $[\pm \xi_0]$ denotes the point in $\mathbb{P}^1$ corresponding to the bundle $\xi_0 \oplus -\xi_0$. As was shown in [1, see p. 366], the residue of $R$ at $p = \infty$ is $\mu$.

Moreover, $\Gamma(A)$ lies in the linear system $|\mathcal{O}(k, 1)|$, where $\mathcal{O}(k, 1) = \mathcal{O}_{\mathbb{P}^1}(k) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. In particular, the set of all possible graphs of charge $k$ instantons with asymptotic state $\pm \xi_0$ and residue $\mu$ is an open subset $\Sigma(k, \pm \xi_0, \mu) \subset \mathbb{P}^{2k-1}$, with $\mathbb{P}^{2k-1}$ being linearly embedded in $\mathbb{P}H^0(\mathbb{P}^1 \times \hat{\mathbb{P}}^1, \mathcal{O}(k, 1))$.

Conversely, a doubly-periodic instanton can be reconstructed from $\sigma \in |\mathcal{O}(k, 1)|$ in the following way. First, regarding $\mathbb{T} \times \mathbb{P}^1$ as a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$, lift $\sigma$ to $\mathbb{T} \times \mathbb{P}^1$, obtaining a curve $\mathcal{S} \subset \mathbb{T} \times \mathbb{P}^1$. Then choose a suitable element $L$ of the Jacobian of $\mathcal{S}$ and extend it to zero to obtain a torsion sheaf on $\mathbb{T} \times \mathbb{P}^1$ supported on $\mathcal{S}$. Finally, apply the inverse Fourier–Mukai functor:

$$\hat{\Psi}(L) = R\pi_*(\check{\pi}^*L \otimes \mathcal{P}^\vee).$$

(10)

As argued in [11, see Theorem 13], $\hat{\Psi}(L)$ turns out to be a locally-free sheaf on $\mathbb{T} \times \mathbb{P}^1$ satisfying all the conditions of Theorem 3, except perhaps $\alpha$-stability.

Thus we can guarantee the existence of certain doubly-periodic instantons simply by checking whether a given divisor in $|\mathcal{O}(k, 1)|$ gives rise to an $\alpha$-stable bundle for a suitable choice of $L$. 

\[\hat{\Psi}(L) = R\pi_*(\check{\pi}^*L \otimes \mathcal{P}^\vee)\]
Similarly, we can guarantee non-existence of certain doubly-periodic instantons by checking whether all divisors in $|\mathcal{O}(k, 1)|$ of a particular type give rise to $\alpha$-unstable bundles. This is the strategy we adopt in the rest of the paper.

**Preliminary observations.** Recall that to check the stability of a given rank-2 bundle over a surface it is enough to check the degree of locally-free subsheaves of rank one. It is then important to characterize all line bundles over $\mathbb{T} \times \mathbb{P}^1$.

**Lemma 4.1** If $L$ is a holomorphic line bundle over $\mathbb{T} \times \mathbb{P}^1$, then $L|_{\mathbb{T}_p}$ is the same for all $p \in \mathbb{P}^1$.

**Proof.** Suppose that the holomorphic line bundle $L \to \mathbb{T} \times \mathbb{P}^1$ has $c_1(L) = a[\mathbb{T}] + b[\mathbb{P}^1]$. It induces a holomorphic map:

$$\mathbb{P}^1 \to \text{Pic}^a(\mathbb{T})$$

$$p \mapsto L|_{\mathbb{T}_p}.$$ 

But the only such map is the constant one.

Thus every line bundle is of the form $Q(b) = p_1^* Q \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(b)$, where $p_1$ and $p_2$ are the obvious projections onto the first and second factors and $Q \in \text{Pic}^b(\mathbb{T})$.

**Lemma 4.2** Let $\mathcal{E} \to \mathbb{T} \times \mathbb{P}^1$ be a holomorphic bundle such that the rational part of its graph is non-constant. Then $\mathcal{E}$ is $\alpha$-stable, for all $\alpha$.

**Proof.** Suppose $Q(b)$ is a subsheaf of $\mathcal{E}$. Then $\deg Q$ must be negative, otherwise there would be no maps $Q(b)|_{\mathbb{T}_p} \to \mathcal{E}|_{\mathbb{T}_p}$ for generic $p \in \mathbb{P}^1$. However, no such subsheaf satisfies $Q(b)|_{\mathbb{T}_\infty} \subset \pm \delta_0$, thus $\mathcal{E}$ is automatically $\alpha$-stable.
In particular, given a (non-constant) rational map \( R : \mathbb{P}^1 \to \mathbb{P}^1 \) of degree \( k \), there is a semisimple doubly-periodic instanton of charge \( k \) whose graph coincides with the graph of \( R \).

Furthermore, we can also conclude from Lemma 4.2 that the only non-stable bundles are among those whose graph is of the following form:

\[
P^1 \times [\pm \xi_0] + \sum_{j=1}^{q} \{u_j\} \times \mathbb{P}^1 \tag{11}
\]

called totally degenerate graphs. Such bundles can be obtained as extensions:

\[
0 \to -\xi(b) \to E \to \xi(-b) \otimes \mathcal{I}_k \to 0,
\]

where \( \xi \in \hat{T} \) and \( \mathcal{I}_k \) is the ideal sheaf of \( k \) points in \( T \times \mathbb{P}^1 \) (none of them lying in the \( T_\infty \)). It is easy to see that \( E \) is not \( \alpha \)-stable if and only if \( b \geq 0 \).

In addition, note that every non-stable bundle must contain unstable points. However, the converse is not true. For example, consider a graph consisting of the graph of a (non-constant) rational map of degree \( k - 1 \) plus a copy of \( \mathbb{P}^1 \); the associated bundle is \( \alpha \)-stable by Lemma 4.2, and contains exactly one unstable point of multiplicity one.

**Lemma 4.3** Suppose \( A \) is a charge-\( k \) instanton with asymptotic state \( \pm \xi_0 \) and whose graph is totally degenerate. Then there are line bundles \( F \) and \( F' \) over \( T \times \mathbb{P}^1 \) such that \( F|_{T_\infty} = -\xi_0 \) and \( F'|_{T_\infty} = \xi_0 \), which map non-trivially into \( \mathcal{E}_A \).

If \( \xi_0 = -\xi_0 \), then \( F = F' \).

**Proof.** If \( A \) is semisimple, then \( \mathcal{E}_A|_{T_p} = \xi_0 \oplus -\xi_0 \) for generic \( p \in \mathbb{P}^1 \). Otherwise, \( \xi_0 = 0 \) and \( A \) is nilpotent, so that \( \mathcal{E}_A|_{T_p} \) is the non-trivial extension of \( \mathcal{O}_T \) by itself for generic \( p \in \mathbb{P}^1 \).

If \( \xi_0 \neq -\xi_0 \), then \( h^0(T_p, \mathcal{E}_A(\xi_0)|_{T_p}) = 1 \) for generic \( p \in \mathbb{P}^1 \). Since \( p_{2*} \mathcal{E}_A(\xi_0) \) is a torsion-free sheaf on \( \mathbb{P}^1 \), we conclude that \( p_{2*} \mathcal{E}_A(\xi_0) = \mathcal{O}_\mathbb{P}^1(b) \). By the projection formula,

\[
p_{2*}(\mathcal{E}_A(\xi_0) \otimes p_*^2 \mathcal{O}_\mathbb{P}^1(-b)) = p_{2*} \mathcal{E}_A(\xi_0) \otimes \mathcal{O}_\mathbb{P}^1(-b) = \mathcal{O}_\mathbb{P}^1.
\]

Thus \( h^0(T \times \mathbb{P}^1, \mathcal{E}_A(\xi_0) \otimes p_*^2 \mathcal{O}_\mathbb{P}^1(-b)) = 1 \), and so there is a map \(-\xi_0(b) \to \mathcal{E}_A\); we define \( F = -\xi_0(b) \).

Similarly, we get that \( p_{2*} \mathcal{E}_A(-\xi_0) = \mathcal{O}_\mathbb{P}^1(b') \), and there is a map \( F' \to \mathcal{E}_A \), where \( F' = \xi_0(b') \).

If \( \xi_0 = -\xi_0 \), one must consider two cases: either \( A \) is semisimple, or \( \xi_0 = 0 \) and \( A \) is nilpotent. The same argument as above will produce the desired line bundle.

Notice that it follows from Lemma 3.2 that nilpotent instantons have totally degenerate graphs.
The moduli space of doubly-periodic instantons. We can summarize the remarks above with the results of [1] in the following statement.

**Theorem 4.4** The moduli space $\mathcal{M}(k, \pm\xi_0, \alpha, \mu)$ of doubly-periodic instantons with fixed instanton number $k$ and asymptotic parameters $(\pm\xi_0, \alpha, \mu)$, if non-empty, is a smooth, complete, hyperkähler manifold of real dimension $8k - 4$. Furthermore, there is a surjective map $\Pi: \mathcal{M}(k, \pm\xi_0, \alpha, \mu) \to \Sigma_{(k, \pm\xi_0, \mu)}$, whose fibres $\Pi^{-1}(\mathcal{S})$ are given by suitable subspaces of the Jacobian of $\mathcal{S}$.

Since both the base and the generic fibre of $\mathcal{M}(k, \pm\xi_0, \alpha, \mu)$ happen to have the same dimension, it seems reasonable to expect that the fibres of $\Pi$ are actually Lagrangian with respect to the natural complex symplectic structure on $\mathcal{M}(k, \pm\xi_0, \alpha, \mu)$. See [11] for a similar statement concerning the moduli of instanton over elliptic K3 and elliptic abelian surfaces; the proof for the present case would follow the same algebraic geometric argument.

5. Instantons with unit charge

In the simplest case of charge-one doubly-periodic instantons, the moduli space $\mathcal{M}_{(1, \pm\xi_0, \alpha, \mu)}$ can be fully described, for all possible values of the asymptotic parameters $(\pm\xi_0, \alpha, \mu)$.

**Proof of Theorem 2.3.** Given an instanton $A$, let $\mathcal{E}_A\to\mathbb{T}\times\mathbb{P}^1$ denote the corresponding holomorphic bundle. First, we will prove the non-existence part of the theorem by contradiction, considering three possibilities:

**Case I:** $\xi_0 \neq -\xi_0$ and $\mu = 0$. Let $A$ be a charge one instanton with $\xi_0 \neq -\xi_0$ and $\mu = 0$. Then $\Gamma(A)$ is totally degenerate; let $\mathcal{F}$ and $\mathcal{F}'$ be the line bundles produced by Lemma 4.3. We argue that either $\deg \mathcal{F} = b > 0$ or $\deg \mathcal{F}' = b' > 0$, which contradicts the $\alpha$-stability of $\mathcal{E}_A$.

Since $\mathcal{E}_A = \mathcal{E}_A^\vee$, relative Serre duality\footnote{That is, $h^0(\mathbb{P}^1, \mathcal{E}_A(\xi_0)|_{\mathbb{P}^1}) = h^1(\mathbb{T}_p, \mathcal{E}_A(-\xi_0)|_{\mathbb{T}_p})$ for all $p \in \mathbb{P}^1 \setminus \infty$.} implies that

$$p_{2*}(\mathcal{E}_A \otimes \mathcal{F}^\vee) = (R^1 p_{2*}(\mathcal{E}_A \otimes \mathcal{F}^\vee)) = -c_2(\mathcal{E} \otimes \mathcal{F}^\vee) = -1.$$

By the index theorem for families, we conclude that

$$c_1(p_{2*}(\mathcal{E}_A \otimes \mathcal{F})) = -c_1(R^1 p_{2*}(\mathcal{E}_A \otimes \mathcal{F}^\vee)) = -c_2(\mathcal{E} \otimes \mathcal{F}^\vee) = -1.$$

But $\mathcal{E}_A \otimes \mathcal{F} = \mathcal{E}_A(\pm\xi_0) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(b)$, so $h^0(\mathbb{T}_p, \mathcal{E}_A \otimes \mathcal{F}|_{\mathbb{T}_p}) = 1$ for all $p \in \mathbb{P}^1$, thus

$$p_{2*} \mathcal{E}_A(-\xi_0) \otimes \mathcal{O}_{\mathbb{P}^1}(b) = \mathcal{E}_A(-\xi_0) \otimes \mathcal{O}_{\mathbb{P}^1}(b) = \mathcal{E}_A(-\xi_0) \otimes \mathcal{O}_{\mathbb{P}^1}(b).$$

Hence $b' + b = -1$, which implies that either $b > 0$ or $b' > 0$, as desired.

Alternatively, this case can also be ruled out via the Nahm transform; see Theorem 2.5.

**Case II:** $\xi_0 = -\xi_0$ and $\mu \neq 0$. Let $A$ be a charge-one instanton with $\xi_0 = -\xi_0$ and $\mu \neq 0$. Since $\xi_0 = -\xi_0$, we have that $h^0(\mathbb{T}_\infty, \mathcal{E}_A(\xi_0)|_{\mathbb{T}_\infty}) = 2$. If $\mu \neq 0$, then $\Gamma(A)$ consists only of the graph of a degree-one rational function $\mathbb{P}^1 \to \mathbb{P}^1$. This implies that $h^0(\mathbb{T}_p, \mathcal{E}_A(\xi_0)|_{\mathbb{T}_p}) = 0$ for all $p \in \mathbb{P}^1 \setminus \infty$.\footnote{That is, $h^0(\mathbb{T}_p, \mathcal{E}_A(\xi_0)|_{\mathbb{T}_p}) = h^1(\mathbb{T}_p, \mathcal{E}_A(-\xi_0)|_{\mathbb{T}_p})$ for all $p \in \mathbb{P}^1$.}
Proof. By the results of [5], with the flat metric.

Case III: $\xi_0 = -\xi_0$ and $\mu = 0$. Let $A$ be a charge-one instanton with $\xi_0 = -\xi_0$ and $\mu = 0$. Thus the rational part of $\Gamma(A)$ is constant, hence either $E_A|_{T_p} = \xi_0 \oplus -\xi_0$ (if $A$ is semisimple) or $E_A|_{T_p}$ is the non-trivial extension of $O_T$ by itself (if $A$ is nilpotent) for all but one unstable point $u \in P^1 \setminus \infty$, where $E_A|_{T_u} = O \oplus Q'$ for some $Q \in \text{Pic}^1(T)$.

Case III(a). If $A$ is semisimple, we see that $h^0(T_p, E_A(\xi_0)|_{T_p}) = 2$ for all $p \in P^1 \setminus \{u\}$, while $h^0(T_p, E_A(\xi_0)|_{T_p}) = 1$. But this contradicts the fact that $h^0(T_p, E_A(\xi_0)|_{T_p})$ is an upper semicontinuous function $P^1 \to \mathbb{Z}$; see [8, section III.12].

Case III(b). If $A$ is nilpotent, $p_{2*}E_A$ is a torsion-free sheaf of rank one, thus $p_{2*}E_A = O_{P^1}(b)$. As in the proof of Lemma 4.3, we conclude that there is a map $p_{2*}O_{P^1}(b) \to E_A$. We argue that $b = 0$. Since $\alpha \text{-deg}(p_{2*}O_{P^1}(b)) = b$, this implies that $E_A$ is not $\alpha$-stable, and hence contradicts the existence of a nilpotent instanton of unit charge.

Indeed, by the index theorem for families, $c_1(R^1 p_{2*}E_A) = c_2(E_A) = 1$. Since $h^1(T_p, E_A|_{T_p}) = 1$ for all $p \in P^1 \setminus \infty$ and $h^1(T_{\infty}, E_A|_{T_{\infty}}) = 2$, we conclude that $R^1 p_{2*}E_A = O_{P^1} \oplus O_{\infty}$, where $O_{\infty}$ is the skyscraper sheaf supported at $\infty \in P^1$. Using again relative Serre duality and the fact that $E_A = E_A$, we conclude that $p_{2*}E_A = O_{P^1}$, as desired.

This completes the non-existence part of the theorem. To guarantee existence when $\xi_0 \neq -\xi_0$ and $\mu \neq 0$, let $R : P^1 \to \hat{P}^1$ be a rational map of degree one such that $R(\infty) = [\pm \xi_0]$ and $R'(\infty) = \mu$. Lift its graph to $\hat{T} \times \hat{P}^1$ to obtain a (smooth) curve $S \subset \hat{T} \times \hat{P}^1$, and let $\mathcal{E} = \hat{\Psi}(\mathcal{O}_S)$. As we have seen in Lemma 4.2, $\mathcal{E}$ is $\alpha$-stable, for all $\alpha$. Thus Theorem 3.1 guarantees the existence of a doubly-periodic instanton with $\xi_0 \neq -\xi_0$ and $\mu \neq 0$ for all $\alpha$.

In particular, we conclude that there are no charge-one instantons with nilpotent asymptotic behaviour.

It is interesting to note that the instanton bundles associated with charge-one instantons are exactly the universal bundles of rank two constructed by Friedman, Morgan and Witten [5]. More precisely, these are rank-2 bundles $V \to T \times P^1$ with $\text{det}(V) = O_{T \times P^1}$ and $c_2(V) = 1$ such that $V|_{T_p}$ is regular (that is, $h^0(T_p, V|_{T_p}) \leq 1$) for all $p \in P^1$. Such bundles have one special point, namely there is a unique $e \in P^1 \setminus \infty$ such that $V|_{T_e}$ is the non-trivial extension of $O_T$ by itself. In other words, $e$ is the only point for which $h^0(T_p, V|_{T_p}) = 1$. This special point may be interpreted as the centre of the associated doubly-periodic instanton.

\textbf{Theorem 5.1} Moreover, the moduli space $\mathcal{M}_{1, (1, \pm \xi_0, \alpha, \mu)}$ of such instantons is isometric to $T^2 \times \mathbb{R}^2$ with the flat metric.

\textbf{Proof.} By the results of [1], $\mathcal{M}_{1, (1, \pm \xi_0, \alpha, \mu)}$ is a complete hyperkähler manifold of real dimension 4. Since $T^2 \times \mathbb{R}^2$ acts isometrically on $\mathcal{M}_{1, (1, \pm \xi_0, \alpha, \mu)}$ via translations, the last assertion follows easily.

The theorem implies that there exists a ‘basic instanton’ of unit charge, from which all others are obtained by translation. Ford et al. have obtained closed formulae for the charge-one instanton restricted to the plane [4], while González-Arroyo and Montero have studied it numerically [7]. However, the precise analytic formula for the charge-one instanton is still unknown.
Therefore, it is also reasonable to interpret the $8k - 4$ real parameters needed to describe the moduli space of charge-$k$ instantons as $4k$ parameters describing the positions of $k$ basic instantons in $T^2 \times \mathbb{R}^2$, plus $4(k - 1)$ parameters describing their relative positions.

6. Instantons of higher charge

While there are no charge-one instantons over $T^4$, instantons over $T^4$ with higher charge are known to exist [4]. Our goal in this section is to argue that the situation for doubly-periodic instantons is more nuanced: existence depends on the asymptotic behaviour.

It is hard to obtain a complete description of the obstructions to existence posed by asymptotic behaviour, as we did for unit charge in the previous section. We conclude this paper with the proof of our second main result.

Proof of Theorem 2.4. This follows as the existence argument in the proof of Theorem 2.3; let $R : \mathbb{P}^1 \to \mathbb{P}^1$ be a non-constant rational map of degree $k$ such that $R(\infty) = [\pm \xi_0]$ and $R'(\infty) = \mu$. Lift its graph to $\hat{T} \times \mathbb{P}^1$ to obtain a (smooth) curve $S \subset \hat{T} \times \mathbb{P}^1$, and let $E = \Psi(O_S)$. As we have seen in Lemma 4.2, $E$ is $\alpha$-stable, for all $\alpha$.

In particular, notice that the theorem guarantees the existence of instantons with $k \geq 2$, $\xi_0 = -\xi_0$ and $\mu \neq 0$, for all $\alpha$, in contrast to the $k = 1$ case.

Furthermore, semisimple instantons with $k \geq 2$, $\xi_0 = -\xi_0$ and $\mu = 0$ can be produced in the same way, since there are non-constant rational maps of degree at least 2 with derivative vanishing at $\infty \in \mathbb{P}^1$. However, we shall leave open the question of whether there exist nilpotent instantons of charge $k \geq 2$.

Finally, it is worth noting that Theorems 2.3 and 2.4 can also be interpreted as existence/non-existence results for the type of singular solutions of Hitchin’s equations on the torus obtained via the Nahm transform (see [10]).

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References


