NONSINGULAR COMPLEX INSTANTONS
ON EUCLIDEAN SPACETIME

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Building on a variation of ’t Hooft’s harmonic function ansatz for $SU(2)$ instantons on $\mathbb{R}^4$, we provide new explicit nonsingular solutions of the Yang–Mills anti-self-duality equations on Euclidean spacetime with gauge group $SL(2, \mathbb{C})$ and $SL(3, \mathbb{R})$.

Keywords: Instantons; periodic instantons.

1. Introduction

Yang–Mills theory has been a rich source of profound mathematical results in the past three decades. It has found applications in a wide variety of research areas, such as differential topology and algebraic geometry [5], representation theory [14], and in the theory of integrable systems [12], to name a few. Usually, one considers the Yang–Mills anti-self-duality (ASD) equations (and its various dimensional reductions) for connections taking values on the Lie algebra of a compact real Lie group. So far, Yang–Mills instantons for complex Lie groups and for noncompact real Lie groups have received little attention, see for instance [3, 4, 15, 16]. However, complex gauge theory has appeared in several recent papers, e.g. [6,9,13]; in particular, there exists an extensive literature on flat connections with values on complex and real noncompact Lie groups, see for instance [1] and the references therein. Furthermore, no general theory has been developed and the proper physical interpretation of such objects is yet to be understood (see [6] for a related discussion).

In this paper we provide explicit nonsingular solutions of the Yang–Mills anti-self-duality equations on Euclidean spacetime with gauge group $SL(2, \mathbb{C})$ and $SL(3, \mathbb{R})$. As it is well-known (see [8,17]), every complex instanton over $\mathbb{C}^4$, regarded as the complexified spacetime, must be singular along a six dimensional submanifold. Thus if we restrict a generic singular instanton on $\mathbb{C}^4$ to the real Euclidean (or Minkowiski) spacetime $\mathbb{R}^4 \subset \mathbb{C}^4$ we can expect to end up with a complex instanton.
which is singular along a 2-dimensional subvariety of $\mathbb{R}^4$. So the existence of smooth anti-self-dual $SL(n, \mathbb{C})$-connections on $\mathbb{R}^4$ is not an obvious fact.

Our starting point is a complex version of the harmonic function ansatz due to ‘t Hooft and outlined in [11]. We show that to each holomorphic function $f : \mathbb{C}^2 \to \mathbb{C}$ one can associate a smooth, anti-self-dual $SL(2, \mathbb{C})$-connection on $\mathbb{R}^4$ of zero action density which is not pure gauge. Motivated by the extensive literature on the explicit construction of $SU(2)$-calorons (see [2, 10] and the references therein), and the partial results so far obtained on the construction of explicit doubly-periodic $SU(2)$-instantons [7], we also give explicit examples of complex calorons and complex doubly-periodic instantons with the same properties.

Complex instantons with zero action density have been found before, cf. [3,4,15]; they were called voidons in [4]. The vanishing of the action density may actually be an advantage from the point of view of the semi-classical approximation, see the discussion in [3]. More recently, $SO(2,1)$-instantons with zero action density were also found in [13], and an interesting connection with the theory of integrable systems was uncovered. Here, integrability once again appears, in the form of the Cauchy–Riemann equations.

2. Complex ’t Hooft Ansatz

We begin by recalling the harmonic function ansatz due to ‘t Hooft and presented by Jackiw, Nohl and Rebbi in [11], slightly adapted to fit our purposes. Let $x_1, x_2, x_3, x_4$ be coordinates of the Euclidean spacetime $\mathbb{R}^4$. Let $\{E_1, E_2, E_3\}$ be a basis for the Lie algebra $su(2)$ satisfying the commutation relations $[E_i, E_j] = \epsilon^{ijk} E_k$; for instance, one may consider $E_j = \frac{1}{2} \sigma_j$, where $\sigma_j$ are the Pauli matrices. Defining

$$\sigma_{ij} = [E_i, E_j]; \quad \sigma_{4i} = E_i,$$

(1)

it is easy to check that

$$\sigma_{ij} = \frac{1}{2} \sum_{k,l} \epsilon_{ijkl} \sigma_{kl} \equiv \ast \sigma_{ij}.$$

(2)

Now consider the $SU(2)$-connection given by the expression $A = \sum_{\mu=1}^4 A_\mu dx_\mu$ with

$$A_\mu = \sum_{\nu=1}^4 \rho_\nu \sigma_{\mu\nu},$$

(3)

where $\rho_\nu : \mathbb{R}^4 \to \mathbb{R}$ are differentiable functions ($\nu = 1, \ldots, 4$). It is not difficult to see that this connection is anti-self-dual provided the following two conditions are satisfied:

$$f_{\mu\nu} \equiv \partial_\mu \rho_\nu - \partial_\nu \rho_\mu = \ast f_{\mu\nu},$$

(4)

$$\sum_{\nu=1}^4 (\partial_\nu \rho_\nu - \rho_\nu^2) = 0.$$

(5)
If \( \rho_\nu = -\log f_\nu \) for some non-vanishing function \( f : \mathbb{R}^4 \to \mathbb{R} \), we end up with the simpler expression,

\[
\frac{1}{f} \nabla^2 f = 0.
\]

(6)

In summary, given a non-vanishing harmonic function \( f : \mathbb{R}^4 \to \mathbb{R} \), the \( SU(2) \)-connection on \( \mathbb{R}^4 \) given by the expression

\[
A_\mu = -\sum_{\nu=0}^{4} \sigma_{\mu\nu} \frac{\partial}{\partial x_\nu} \ln(f(x))
\]

(7)

has anti-self-dual curvature. For instance, if one considers

\[
f(x) = 1 + \sum_{i=1}^{k} \frac{\lambda_i^2}{(x - y_i)^2},
\]

(8)

where \( \lambda_i \in \mathbb{R} \), \( y_i \in \mathbb{R}^4 \) and \( k \geq 0 \) is a positive integer, one gets, after an appropriate gauge transformation, \( k \) ‘basic instantons’ with centers \( y_i \) and scales \( \lambda_i \).

The simplest way of constructing complex instantons would be to consider \( f : \mathbb{R}^4 \to \mathbb{C} \), i.e. allowing the superpotential to assume complex values. In other words, consider a pair of nonvanishing harmonic functions \( u, v : \mathbb{R}^4 \to \mathbb{R} \) and take \( f = u + iv \) into formula (7). This will lead to an \( SU(2) \)-connection with complex-valued coefficients, which may be regarded as an \( SL(2, \mathbb{C}) \)-connection. Notice that the connection produced in this way may, in principle, have arbitrary topological charge.

Now we assume the connection (3) has complex-valued coefficients, i.e. \( \rho_\nu : \mathbb{R}^4 \to \mathbb{C} \). Condition (5) is clearly satisfied if we take \( i\rho_1 = \rho_3 \) and \( i\rho_2 = \rho_4 \). It then follows that \( \sum_{\nu=1}^{4} \rho_\nu^2 = 0 \), hence (5) reduces to

\[
\sum_{\nu=1}^{4} \partial_\nu \rho_\nu = 0.
\]

(9)

It is interesting to remark that if we introduce the matrix \( J_\mu^\nu = \partial_\nu \rho_\mu \), then conditions (4) and (9) become \( J = J^* \) and Tr\( J = 0 \), respectively. Without loss of generality, we can just focus our attention on the symmetry condition. A simple computation is sufficient to conclude that this condition is equivalent to the following equations \( (j = 1, 2) \):

\[
i\partial_1 \rho_j = \partial_3 \rho_j,
\]

(10)

\[
i\partial_2 \rho_j = \partial_4 \rho_j,
\]

(11)

\[
\partial_2 \rho_1 = \partial_1 \rho_2.
\]

(12)

Turning to complex coordinates \( u = x_1 + ix_3, v = x_2 + ix_4 \), it is not difficult to see that (10) and (11) are exactly the Cauchy–Riemann equations for \( \rho_1 \) and \( \rho_2 \).
With all these assumptions, the coefficients of the connection \( A = A_\nu dx^\nu \) are given by

\[
A_1 = i\rho_2 E_1 - i\rho_1 E_2 + \rho_2 E_3,
A_2 = i\rho_1 E_1 + i\rho_2 E_2 - \rho_1 E_3, \\
A_3 = iA_1, \\ A_4 = iA_2,
\]

(13)

with curvature components

\[
F_{12} = i(\partial_1 \rho_1 - \partial_2 \rho_2 + \rho_1^2 - \rho_2^2) E_1 + 2i(\partial_2 \rho_1 + \rho_1 \rho_2) E_2 \\
\quad - (\partial_1 \rho_1 + \partial_2 \rho_2 + \rho_1^2 \rho_2^2) E_3, \\
F_{23} = -iF_{12}, \\
F_{13} = 0.
\]

(14)

Moreover, the ASD condition holds, i.e.: 

\[
F_{12} = -F_{34}, \quad F_{23} = -F_{14} \quad \text{and} \quad F_{13} = F_{24}.
\]

As usual, we define the topological charge of an instanton by the expression (also considered in [4]):

\[
W = -\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{Tr}(F \wedge F).
\]

(15)

We remark that the above expression is \( SL(2,\mathbb{C}) \) gauge invariant. However, the bilinear form \( (A, B) = \text{Tr}(AB) \) is not positive definite in \( \mathfrak{sl}(2,\mathbb{C}) \), so \( W \) may vanish even though \( F \) is not pure gauge.

It is easy to check that if \( A \) is anti-self-dual, then \( F \wedge F = F_{12}^2 + F_{13}^2 + F_{23}^2 \); in particular, our connections satisfy

\[
F \wedge F = F_{12}^2 + (-iF_{12})^2 = 0.
\]

(16)

In other words, all connections constructed in this section have zero energy density.

3. Instantons with Holomorphic Conditions

Let \( u = x_1 + ix_3, v = x_2 + ix_4 \) be complex coordinates, so that

\[
du = \frac{1}{2}(dx^1 + idx^3), \\ dv = \frac{1}{2}(dx^2 + idx^4), \\
\partial_u = \partial_1 - i\partial_3, \\ \partial_v = \partial_2 - i\partial_4,
\]

(17)

\[
d\bar{u} = \frac{1}{2}(dx^1 - idx^3), \\ d\bar{v} = \frac{1}{2}(dx^2 - idx^4), \\
\partial_u = \partial_1 + i\partial_3, \\ \partial_v = \partial_2 + i\partial_4.
\]

(18)

One can check that the following equality holds

\[
A = A_\nu dx^\nu = (A_1 - iA_3)du + (A_2 - iA_4)dv + (A_1 + iA_3)d\bar{u} + (A_2 + iA_4)d\bar{v}.
\]
We fix the notation $A_u, A_v, A_{\bar{u}}, A_{\bar{v}}$ for the corresponding components of $A$ in the complex coordinates above. Using the notation

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad \mu, \nu = u, v, \bar{u}, \bar{v},$$

we have the ASD conditions for $A$ expressed as

$$F_{u \bar{v}} = 0, \quad F_{v \bar{u}} = 0, \quad F_{u \bar{u}} = F_{v \bar{v}},$$

while the energy density in the ASD case is given by

$$\text{Tr}(F^2_{12} + F^2_{23} + F^2_{13}) = \frac{1}{4} \text{Tr}(F_{uv} F_{\bar{u} \bar{v}} - F^2_{u \bar{u}}).$$

Now we suppose that $A_{\bar{u}} = A_{\bar{v}} = 0$, hence the ASD conditions (19) simplify to

$$\partial_\bar{v} A_u = 0, \quad \partial_\bar{u} A_v = 0, \quad \partial_\bar{u} A_u = \partial_\bar{v} A_v. \quad (21)$$

The curvature components of such a connection are then given by

$$F_{uv} = \partial_u A_v - \partial_v A_u + [A_u, A_v], \quad (22)$$

$$F_{\bar{u} \bar{v}} = 0, \quad (23)$$

$$F_{u \bar{u}} = -\partial_\bar{u} A_u, \quad (24)$$

with energy density

$$\text{Tr}(F \wedge F) = -\frac{1}{4} \text{Tr}((\partial_\bar{u} A_u)^2). \quad (25)$$

From the condition $A_{\bar{u}} = A_{\bar{v}} = 0$ and (21), we have that:

$$\partial_\bar{v} A_1 = 0, \quad \partial_\bar{u} A_2 = 0, \quad \partial_\bar{u} A_1 = \partial_\bar{v} A_2. \quad (26)$$

and

$$iA_1 = A_3, \quad iA_2 = A_4. \quad (27)$$

Summing up, we conclude that if $A = \tilde{A}_u dx^u$ is a connection satisfying (26) and (27), then $A$ is a ASD. Furthermore, if $\partial_\bar{u} A_1 = 0$, then the energy density (hence the topological charge) of $A$ vanishes. One can easily notice from (13) that the complex 't Hooft ansatz obtained in the previous section is a particular case of the holomorphic instanton constructed above. It is also worth noting that, by the weaker conditions exposed in this section, (12) can be neglected without loss of anti-self-duality.

4. Charge Zero $SL(n, \mathbb{R})$ Instantons for $n \geq 3$

We now assume a reality condition on $A$ by supposing that $A_{\bar{u}} = \bar{A}_u$ and $A_{\bar{v}} = \bar{A}_v$, i.e. the matrices $A_i$ have real valued coefficients. We further assume that the complex coefficients of $A$ are holomorphic, i.e. $\partial_\bar{u} A_u = \partial_\bar{u} A_u = \partial_\bar{v} A_v = \partial_\bar{v} A_v = 0$; we also have $\partial_\bar{u} A_{\bar{u}} = \partial_\bar{u} A_{\bar{v}} = \partial_\bar{v} A_{\bar{u}} = \partial_\bar{v} A_{\bar{v}} = 0$ since the former is the complex
conjugate of the latter. In this situation, from (19), the ASD condition simplifies to a purely algebraic condition:

\[ [A_u, A_\bar{u}] = [A_v, A_\bar{v}] \quad (28) \]
\[ [A_u, A_\bar{v}] = 0 \quad (29) \]
\[ [A_v, A_\bar{u}] = 0. \quad (30) \]

One can also check that (29) is equivalent to (30).

Now we suppose

\[ A_u = \rho(x)M, \quad A_v = \mu(x)M, \quad (31) \]

for \( \rho, \mu \) holomorphic functions. Equations (28) and (29) became

\[ \rho \bar{\mu}[M, \bar{M}] = 0 \quad (32) \]
\[ |\rho|^2[M, \bar{M}] = |\mu|^2[M, \bar{M}] \quad (33) \]

As one can see the interesting case is \([M, \bar{M}] = 0\). Setting \( M = a + ib \) with \( a, b \) real matrices, then \([M, \bar{M}] = 0\) iff \([a, b] = 0\). Unfortunately, if \( a, b \in \mathfrak{sl}(2, \mathbb{R})\) then \( a \) and \( b \) are multiple of each other, and the corresponding instanton would turn out to be abelian. However, for \( n \geq 3 \), there exist pairs of traceless commuting matrices that are not multiple of one another, see the examples below.

The energy density of this connection will be

\[ \text{Tr}(F \wedge F) = -((\partial_v \rho - \partial_u \mu)[M, \bar{M}] du \wedge dv \wedge d\bar{u} \wedge d\bar{v}) \quad (34) \]

Concluding that the charge will be zero if the right-hand side vanishes, i.e. when either \( \partial_v \rho = \partial_u \mu \) or \( \text{Tr} M \bar{M} = \text{Tr}(a^2 + b^2) = 0 \).

5. Explicit Examples

A good way for constructing examples is to take some \( \rho : \mathbb{C}^2 \to \mathbb{C} \) holomorphic in \( z_1, z_2 \) and make \( \rho_j = \frac{\partial \rho}{\partial z_j} \). This is enough for conditions (10)–(12). Here we let \( E_j = \frac{1}{2\pi} \sigma_j \) where \( \sigma_j \) are the Pauli matrices. We use in this section the notation \( z_1 = x_1 + ix_3, z_2 = x_2 + ix_4 \).

**Example 1.** Taking \( \rho = z_1 z_2 \) we have \( \rho_1 = z_2 \) and \( \rho_2 = z_1 \). Remembering that the curvature is completely defined with its \( F_{12} \) component as seen in (14) and by (13) we have

\[ A_1 = -iA_3 = \frac{1}{2} \left( \begin{array}{cc} -iz_1 & z_1 + iz_2 \\ z_1 - iz_2 & iz_1 \end{array} \right), \]
\[ A_2 = -iA_4 = \frac{1}{2} \left( \begin{array}{cc} iz_2 & z_2 - iz_1 \\ z_2 + iz_1 & -iz_2 \end{array} \right), \]

\[ F_{12} = i(z_2^2 - z_1^2)E_1 + 2i(1 + z_1 z_2)E_2 - (z_2^2 + z_1^2)E_3. \]
This is a nonsingular charge zero instanton with nonvanishing curvature. The vanishing energy density (16) and the nonzero curvature contrasts with the \( SU(2) \) case where the energy density vanishes only when the curvature vanishes.

**Example 2 (Complex Caloron).** Setting \( \rho = z_1 e^{z_2} \), we have \( \rho_1 = e^{z_2} \) and \( \rho_2 = z_1 e^{z_2} \). By (13) and (14) the connection and curvature will be

\[
A_1 = -iA_3 = \frac{e^{z_2}}{2} \begin{pmatrix} -iz_1 & z_1 + i \\ z_1 & 1 \\ \end{pmatrix},
\]

\[
A_2 = -iA_4 = \frac{e^{z_2}}{2} \begin{pmatrix} i & 1 - iz_1 \\ 1 + iz_1 & -i \\ \end{pmatrix},
\]

\[
F_{12} = ie^{z_2} (e^{z_2} - z_1 - z_1^2 e^{z_2}) E_1 + 2ie^{z_2} (1 + z_1 e^{z_2}) E_2 - z_1 e^{z_2} (1 + z_1 e^{z_2}) E_3.
\]

This solution is nonsingular and periodic in the \( x_4 \) direction, i.e. it is a complex caloron. It also has zero energy density.

**Example 3 (Complex doubly-periodic instanton).** Taking \( \rho = e^{z_1} e^{z_2} \), we have \( \rho_1 = e^{z_1} e^{z_2} \) and \( \rho_2 = e^{z_1} e^{z_2} \). Again by (13) and (14) we arrive at

\[
A_1 = -iA_3 = \frac{e^{z_1 + z_2}}{2} \begin{pmatrix} -i & 1 + i \\ 1 - i & i \\ \end{pmatrix},
\]

\[
A_2 = -iA_4 = \frac{e^{z_1 + z_2}}{2} \begin{pmatrix} i & 1 - i \\ 1 + i & -i \\ \end{pmatrix},
\]

\[
F_{12} = 2i(e^{z_1} e^{z_2} + e^{z_1} e^{z_2}) (E_2 - E_3).
\]

This solution is nonsingular and periodic in the \( x_2 \) and \( x_4 \) directions, so it is a complex doubly-periodic instanton. It also has zero energy density. As far as we know, this is the first explicit example of a doubly-periodic instanton.

Now we give an example of complex solution that is not from the above complex \'t Hooft ansatz.

**Example 4.** Setting \( \rho_1 = z_1 e^{z_2} \) and \( \rho_2 = e^{z_1} z_2 \) in the construction of Sec. 3, one obtains:

\[
A_1 = \frac{1}{2} \begin{pmatrix} -ie^{z_1} z_2 & e^{z_1} z_2 + iz_1 e^{z_2} \\ e^{z_1} z_2 - iz_1 e^{z_2} & ie^{z_1} z_2 \\ \end{pmatrix},
\]

\[
A_2 = \frac{1}{2} \begin{pmatrix} iz_1 e^{z_2} & z_1 e^{z_2} - ie^{z_1} z_2 \\ z_1 e^{z_2} + ie^{z_1} z_2 & -iz_1 e^{z_2} \\ \end{pmatrix}.
\]

Therefore,

\[
F_{12} = i(e^{z_2} - e^{z_1} + z_1^2 e^{2z_2} - z_1^2 e^{2z_1}) E_1 + i(z_2 e^{z_1} + z_1 e^{z_2} + 2z_1^2 z_2 e^{2z_1} e^{2z_2}) E_2
\]

\[-(e^{z_1} + e^{z_2} + z_1^2 e^{2(z_1 + z_2)}) E_3.
\]

Note that Eq. (13) do not hold, showing that this example does not arise from the complex \'t Hooft ansatz.
Finally, we conclude with an explicit example of a $SL(3, \mathbb{R})$ instanton with non-zero curvature but zero action density.

**Example 5 ($SL(3, \mathbb{R})$ instanton).** We consider in Eq. (31) $\rho = 2e^{z_2}, \mu = 2z_1$ with $M = a + ib$,

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

(35)

It follows that

\[
\begin{align*}
A_1 &= e^{z_2} (\cos x_4 a - \sin x_4 b), \\
A_2 &= x_1 a - x_3 b, \\
A_3 &= e^{z_2} (\sin x_4 a + \cos x_4 b), \\
A_4 &= x_3 a + x_1 b,
\end{align*}
\]

with vanishing energy density and the $F_{uv}$ component of the complex curvature is

\[
F_{uv} = F_{12} - F_{34} + i(F_{23} - F_{14}) = 4(1 - 2e^{z_2})M.
\]

So the curvature in real coordinates cannot be zero, concluding that it is not a pure gauge connection and contrasting again with the $SU(n)$ case.

**6. Conclusion**

In this paper we presented two ways of constructing explicit, nonsingular complex instantons on the Euclidean $\mathbb{R}^4$ with zero action density. Further examples could be obtained by replacing the conditions $A_u = A_v = 0$ and $A_u = \tilde{A}_u$, $A_v = \tilde{A}_v$, imposed respectively in Sec. 3 and in Sec. 4, by another simplifying assumption, showing that our method is not completely exhausted. We hope that these examples will be useful as motivation for the development of a general deformation theory of complex instantons, and lead to better understanding of its physical interpretation and of the structure of its moduli spaces.

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