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# NONSINGULAR COMPLEX INSTANTONS ON EUCLIDEAN SPACETIME

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Building on a variation of 't Hooft's harmonic function ansatz for $SU(2)$ instantons on $\mathbb{R}^4$, we provide new explicit nonsingular solutions of the Yang–Mills anti-self-duality equations on Euclidean spacetime with gauge group $SL(2,\mathbb{C})$ and $SL(3,\mathbb{R})$.

*Keywords*: Instantons; periodic instantons.

## 1. Introduction

Yang–Mills theory has been a rich source of profound mathematical results in the past three decades. It has found applications in a wide variety of research areas, such as differential topology and algebraic geometry [5], representation theory [14], and in the theory of integrable systems [12], to name a few. Usually, one considers the Yang–Mills anti-self-duality (ASD) equations (and its various dimensional reductions) for connections taking values on the Lie algebra of a compact real Lie group. So far, Yang–Mills instantons for complex Lie groups and for noncompact real Lie groups have received little attention, see for instance [3, 4, 15, 16]. However, complex gauge theory has appeared in several recent papers, e.g. [6, 9, 13]; in particular, there exists an extensive literature on flat connections with values on complex and real noncompact Lie groups, see for instance [1] and the references therein. Furthermore, no general theory has been developed and the proper physical interpretation of such objects is yet to be understood (see [6] for a related discussion).

In this paper we provide explicit nonsingular solutions of the Yang–Mills anti-self-duality equations on Euclidean spacetime with gauge group $SL(2,\mathbb{C})$ and $SL(3,\mathbb{R})$. As it is well-known (see [8,17]), every complex instanton over $\mathbb{C}^4$, regarded as the complexified spacetime, must be singular along a six dimensional submanifold. Thus if we restrict a generic singular instanton on $\mathbb{C}^4$ to the real Euclidean (or Minkowiski) spacetime $\mathbb{R}^4 \subset \mathbb{C}^4$ we can expect to end up with a complex instanton

which is singular along a 2-dimensional subvariety of $\mathbb{R}^4$. So the existence of smooth anti-self-dual $SL(n,\mathbb{C})$-connections on $\mathbb{R}^4$ is not an obvious fact.

Our starting point is a complex version of the harmonic function ansatz due to 't Hooft and outlined in [11]. We show that to each holomorphic function $f:\mathbb{C}^2\to\mathbb{C}$ one can associate a smooth, anti-self-dual $SL(2,\mathbb{C})$-connection on $\mathbb{R}^4$ of zero action density which is not pure gauge. Motivated by the extensive literature on the explicit construction of $SU(2)$-calorons (see [2, 10] and the references therein), and the partial results so far obtained on the construction of explicit doubly-periodic $SU(2)$-instantons [7], we also give explicit examples of complex calorons and complex doubly-periodic instantons with the same properties.

Complex instantons with zero action density have been found before, cf. [3,4,15]; they were called *voidons* in [4]. The vanishing of the action density may actually be an advantage from the point of view of the semi-classical approximation, see the discussion in [3]. More recently, $SO(2,1)$-instantons with zero action density were also found in [13], and an interesting connection with the theory of integrable systems was uncovered. Here, integrability once again appears, in the form of the Cauchy–Riemann equations.

## 2. Complex 't Hooft Ansatz

We begin by recalling the harmonic function ansatz due to 't Hooft and presented by Jackiw, Nohl and Rebbi in [11], slightly adapted to fit our purposes. Let $x_1,x_2,x_3,x_4$ be coordinates of the Euclidean spacetime $\mathbb{R}^4$. Let $\{E_1,E_2,E_3\}$ be a basis for the Lie algebra $\mathfrak{su}(2)$ satisfying the commutation relations $[E_i,E_j]=\epsilon^{ijk}E_k$; for instance, one may consider $E_j=\frac{1}{2i}\sigma_j$, where $\sigma_j$ are the Pauli matrices. Defining

$$\overline{\sigma}_{ij}=[E_i,E_j];\quad \overline{\sigma}_{i4}=E_i, \tag{1}$$

it is easy to check that

$$\overline{\sigma}_{ij}=\frac{1}{2}\sum_{k,l}\varepsilon_{ijkl}\overline{\sigma}_{kl}\equiv *\overline{\sigma}_{ij}. \tag{2}$$

Now consider the $SU(2)$-connection given by the expression $A=\sum_{\mu=1}^4 A_\mu dx_\mu$ with

$$A_\mu=\sum_{\nu=1}^4\rho_\nu\overline{\sigma}_{\mu\nu}, \tag{3}$$

where $\rho_\nu:\mathbb{R}^4\to\mathbb{R}$ are differentiable functions ($\nu=1,\dots,4$). It is not difficult to see that this connection is anti-self-dual provided the following two conditions are satisfied:

$$f_{\mu\nu}\equiv\partial_\mu\rho_\nu-\partial_\nu\rho_\mu=*f_{\mu\nu}, \tag{4}$$

$$\sum_{\nu=1}^4(\partial_\nu\rho_\nu-\rho_\nu^2)=0. \tag{5}$$

If $\rho_\nu = -\log f\nu$ for some non-vanishing function $f : \mathbb{R}^4 \to \mathbb{R}$, we end up with the simpler expression,

$$\frac{1}{f}\nabla^2 f = 0. \tag{6}$$

In summary, given a non-vanishing harmonic function $f : \mathbb{R}^4 \to \mathbb{R}$, the $SU(2)$-connection on $\mathbb{R}^4$ given by the expression $A = \sum_{\mu=1}^4 A_\mu dx_\mu$ with

$$A_\mu = -\sum_{\nu=0}^{4} \overline{\sigma}_{\mu\nu} \frac{\partial}{\partial x_\nu} \ln(f(x)) \tag{7}$$

has anti-self-dual curvature. For instance, if one considers

$$f(x) = 1 + \sum_{i=1}^{k} \frac{\lambda_i^2}{(x - y_i)^2}, \tag{8}$$

where $\lambda_i \in \mathbb{R}$, $y_i \in \mathbb{R}^4$ and $k \geq 0$ is a positive integer, one gets, after an appropriate gauge transformation, $k$ 'basic instantons' with centers $y_i$ and scales $\lambda_i$.

The simplest way of constructing complex instantons would be to consider $f : \mathbb{R}^4 \to \mathbb{C}$, i.e. allowing the superpotential to assume complex values. In other words, consider a pair of nonvanishing harmonic functions $u, v : \mathbb{R}^4 \to \mathbb{R}$ and take $f = u + iv$ into formula (7). This will lead to an $SU(2)$-connection with complex-valued coefficients, which may be regarded as an $SL(2, \mathbb{C})$-connection. Notice that the connection produced in this way may, in principle, have arbitrary topological charge.

Now we assume the connection (3) has complex-valued coefficients, i.e. $\rho_\nu : \mathbb{R}^4 \to \mathbb{C}$. Condition (5) is clearly satisfied if we take $i\rho_1 = \rho_3$ and $i\rho_2 = \rho_4$. It then follows that $\sum \rho_\nu^2 = 0$, hence (5) reduces to

$$\sum_{\nu=1}^{4} \partial_\nu \rho_\nu = 0. \tag{9}$$

It is interesting to remark that if we introduce the matrix $J_\nu^\mu = \partial_\nu \rho_\mu$, then conditions (4) and (9) become $J = J^{\mathrm{t}}$ and $\mathrm{Tr} J = 0$, respectively. *Without loss of generality, we can just focus our attention on the symmetry condition.* A simple computation is sufficient to conclude that this condition is equivalent to the following equations ($j = 1, 2$):

$$i\partial_1 \rho_j = \partial_3 \rho_j, \tag{10}$$

$$i\partial_2 \rho_j = \partial_4 \rho_j, \tag{11}$$

$$\partial_2 \rho_1 = \partial_1 \rho_2. \tag{12}$$

Turning to complex coordinates $u = x_1 + ix_3, v = x_2 + ix_4$, it is not difficult to see that (10) and (11) are exactly the Cauchy–Riemann equations for $\rho_1$ and $\rho_2$.

With all these assumptions, the coefficients of the connection $A = A_\nu dx^\nu$ are given by

$$\begin{aligned} A_1 &= i\rho_2 E_1 - i\rho_1 E_2 + \rho_2 E_3, \\ A_2 &= i\rho_1 E_1 + i\rho_2 E_2 - \rho_1 E_3, \\ A_3 &= iA_1, \quad A_4 = iA_2, \end{aligned} \tag{13}$$

with curvature components

$$\begin{aligned} F_{12} &= i(\partial_1\rho_1 - \partial_2\rho_2 + \rho_1^2 - \rho_2^2)E_1 + 2i(\partial_2\rho_1 + \rho_1\rho_2)E_2 \\ &\quad - (\partial_1\rho_1 + \partial_2\rho_2 + \rho_1^2\rho_2^2)E_3, \\ F_{23} &= -iF_{12}, \\ F_{13} &= 0. \end{aligned} \tag{14}$$

Moreover, the ASD condition holds, i.e.:

$$F_{12} = -F_{34}, \quad F_{23} = -F_{14} \quad \text{and} \quad F_{13} = F_{24}.$$

As usual, we define the topological charge of an instanton by the expression (also considered in [4]):

$$W = -\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}(F \wedge F). \tag{15}$$

We remark that the above expression is $SL(2,\mathbb{C})$ gauge invariant. However, the bilinear form $(A, B) = \mathrm{Tr}(AB)$ is not positive definite in $\mathfrak{sl}(2,\mathbb{C})$, so $W$ may vanish even though $F$ is not pure gauge.

It is easy to check that if $A$ is anti-self-dual, then $F \wedge F = F_{12}^2 + F_{13}^2 + F_{23}^2$; in particular, our connections satisfy

$$F \wedge F = F_{12}^2 + (-iF_{12})^2 = 0. \tag{16}$$

In other words, all connections constructed in this section have zero energy density.

## 3. Instantons with Holomorphic Conditions

Let $u = x_1 + ix_3, v = x_2 + ix_4$ be complex coordinates, so that

$$\begin{aligned} du &= \frac{1}{2}(dx^1 + idx^3), \quad dv = \frac{1}{2}(dx^2 + idx^4), \\ \partial_u &= \partial_1 - i\partial_3, \quad \partial_v = \partial_2 - i\partial_4, \end{aligned} \tag{17}$$

$$\begin{aligned} d\bar{u} &= \frac{1}{2}(dx^1 - idx^3), \quad d\bar{v} = \frac{1}{2}(dx^2 - idx^4), \\ \partial_{\bar{u}} &= \partial_1 + i\partial_3, \quad \partial_{\bar{v}} = \partial_2 + i\partial_4. \end{aligned} \tag{18}$$

One can check that the following equality holds

$$A = A_\nu dx^\nu = (A_1 - iA_3)du + (A_2 - iA_4)dv + (A_1 + iA_3)d\bar{u} + (A_2 + iA_4)d\bar{v}.$$

We fix the notation $A_u, A_v, A_{\bar{u}}, A_{\bar{v}}$ for the corresponding components of $A$ in the complex coordinates above. Using the notation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\nu, A_\mu], \quad \mu, \nu = u, v, \bar{u}, \bar{v},$$

we have the ASD conditions for $A$ expressed as

$$F_{u\bar{v}} = 0, \quad F_{v\bar{u}} = 0, \quad F_{u\bar{u}} = F_{v\bar{v}}, \tag{19}$$

while the energy density in the ASD case is given by

$$\mathrm{Tr}(F_{12}^2 + F_{23}^2 + F_{13}^2) = \frac{1}{4}\mathrm{Tr}(F_{uv}F_{\bar{u}\bar{v}} - F_{u\bar{u}}^2). \tag{20}$$

Now we suppose that $A_{\bar{u}} = A_{\bar{v}} = 0$, hence the ASD conditions (19) simplify to

$$\partial_{\bar{v}} A_u = 0, \quad \partial_{\bar{u}} A_v = 0, \quad \partial_{\bar{u}} A_u = \partial_{\bar{v}} A_v. \tag{21}$$

The curvature components of such a connection are then given by

$$F_{uv} = \partial_u A_v - \partial_v A_u + [A_u, A_v], \tag{22}$$

$$F_{\bar{u}\bar{v}} = 0, \tag{23}$$

$$F_{u\bar{u}} = -\partial_{\bar{u}} A_u, \tag{24}$$

with energy density

$$\mathrm{Tr}(F \wedge F) = -\frac{1}{4}\mathrm{Tr}\{(\partial_{\bar{u}} A_u)^2\}. \tag{25}$$

From the condition $A_{\bar{u}} = A_{\bar{v}} = 0$ and (21), we have that:

$$\partial_{\bar{v}} A_1 = 0, \quad \partial_{\bar{u}} A_2 = 0, \quad \partial_{\bar{v}} A_2 = \partial_{\bar{u}} A_1 \tag{26}$$

and

$$iA_1 = A_3, \quad iA_2 = A_4. \tag{27}$$

Summing up, we conclude that if $A = A_\nu dx^\nu$ is a connection satisfying (26) and (27), then $A$ is a ASD. Furthermore, if $\partial_{\bar{u}} A_1 = 0$, then the energy density (hence the topological charge) of $A$ vanishes. One can easily notice from (13) that the complex 't Hooft ansatz obtained in the previous section is a particular case of the holomorphic instanton constructed above. It is also worth noting that, by the weaker conditions exposed in this section, (12) can be neglected without loss of anti-self-duality.

## 4. Charge Zero $SL(n, \mathbb{R})$ Instantons for $n \geq 3$

We now assume a reality condition on $A$ by supposing that $A_{\bar{u}} = \bar{A}_u$ and $A_{\bar{v}} = \bar{A}_v$, i.e. the matrices $A_i$ have real valued coefficients. We further assume that the complex coefficients of $A$ are holomorphic, i.e. $\partial_{\bar{u}} A_u = \partial_{\bar{v}} A_u = \partial_{\bar{u}} A_v = \partial_{\bar{v}} A_v = 0$; we also have $\partial_u A_{\bar{u}} = \partial_u A_{\bar{v}} = \partial_u A_{\bar{v}} = \partial_v A_{\bar{v}} = 0$ since the former is the complex

conjugate of the latter. In this situation, from (19), the ASD condition simplifies to a purely algebraic condition:

$$[A_u, A_{\bar{u}}] = [A_v, A_{\bar{v}}] \tag{28}$$

$$[A_u, A_{\bar{v}}] = 0 \tag{29}$$

$$[A_v, A_{\bar{u}}] = 0. \tag{30}$$

One can also check that (29) is equivalent to (30).

Now we suppose

$$A_u = \rho(x)M, \quad A_v = \mu(x)M, \tag{31}$$

for $\rho, \mu$ holomorphic functions. Equations (28) and (29) became

$$\rho\mu[M, \bar{M}] = 0 \tag{32}$$

$$|\rho|^2[M, \bar{M}] = |\mu|^2[M, \bar{M}]. \tag{33}$$

As one can see the interesting case is $[M, \bar{M}] = 0$. Setting $M = a + ib$ with $a, b$ real matrices, then $[M, \bar{M}] = 0$ iff $[a, b] = 0$. Unfortunately, if $a, b \in \mathfrak{sl}(2, \mathbb{R})$ then $a$ and $b$ are multiple of each other, and the corresponding instanton would turn out to be abelian. However, for $n \geq 3$, there exist pairs of traceless commuting matrices that are not multiple of one another, see the examples below.

The energy density of this connection will be

$$\mathrm{Tr}(F \wedge F) = -(|\partial_v \rho - \partial_u \mu|^2 \mathrm{Tr} M\bar{M}) du \wedge dv \wedge d\bar{u} \wedge d\bar{v}. \tag{34}$$

Concluding that the charge will be zero if the right-hand side vanishes, i.e. when either $\partial_v \rho = \partial_u \mu$ or $\mathrm{Tr} M\bar{M} = \mathrm{Tr}(a^2 + b^2) = 0$.

## 5. Explicit Examples

A good way for constructing examples is to take some $\rho : \mathbb{C}^2 \to \mathbb{C}$ holomorphic in $z_1, z_2$ and make $\rho_j = \frac{\partial \rho}{\partial z_j}$. This is enough for conditions (10)–(12). Here we let $E_j = \frac{1}{2i}\sigma_j$ where $\sigma_j$ are the Pauli matrices. We use in this section the notation $z_1 = x_1 + ix_3, z_2 = x_2 + ix_4$.

**Example 1.** Taking $\rho = z_1 z_2$ we have $\rho_1 = z_2$ and $\rho_2 = z_1$. Remembering that the curvature is completely defined with its $F_{12}$ component as seen in (14) and by (13) we have

$$A_1 = -iA_3 = \frac{1}{2}\begin{pmatrix} -iz_1 & z_1 + iz_2 \\ z_1 - iz_2 & iz_1 \end{pmatrix},$$

$$A_2 = -iA_4 = \frac{1}{2}\begin{pmatrix} iz_2 & z_2 - iz_1 \\ z_2 + iz_1 & -iz_2 \end{pmatrix},$$

$$F_{12} = i(z_2^2 - z_1^2)E_1 + 2i(1 + z_1 z_2)E_2 - (z_2^2 + z_1^2)E_3.$$

This is a nonsingular charge zero instanton with nonvanishing curvature. The vanishing energy density (16) and the nonzero curvature contrasts with the $SU(2)$ case where the energy density vanishes only when the curvature vanishes.

**Example 2 (Complex Caloron).** Setting $\rho = z_1 e^{z_2}$, we have $\rho_1 = e^{z_2}$ and $\rho_2 = z_1 e^{z_2}$. By (13) and (14) the connection and curvature will be

$$A_1 = -iA_3 = \frac{e^{z_2}}{2}\begin{pmatrix} -iz_1 & z_1 + i \\ z_1 - i & iz_1 \end{pmatrix},$$

$$A_2 = -iA_4 = \frac{e^{z_2}}{2}\begin{pmatrix} i & 1 - iz_1 \\ 1 + iz_1 & -i \end{pmatrix},$$

$$F_{12} = ie^{z_2}(e^{z_2} - z_1 - z_1^2 e^{z_2})E_1 + 2ie^{z_2}(1 + z_1 e^{z_2})E_2 - z_1 e^{z_2}(1 + z_1 e^{3z_2})E_3.$$

This solution is nonsingular and periodic in the $x_4$ direction, i.e. it is a complex caloron. It also has zero energy density.

**Example 3 (Complex doubly-periodic instanton).** Taking $\rho = e^{z_1} e^{z_2}$, we have $\rho_1 = e^{z_1} e^{z_2}$ and $\rho_2 = e^{z_1} e^{z_2}$. Again by (13) and (14) we arrive at

$$A_1 = -iA_3 = \frac{e^{z_1+z_2}}{2}\begin{pmatrix} -i & 1 + i \\ 1 - i & i \end{pmatrix},$$

$$A_2 = -iA_4 = \frac{e^{z_1+z_2}}{2}\begin{pmatrix} i & 1 - i \\ 1 + i & -i \end{pmatrix},$$

$$F_{12} = 2i(e^{z_1} e^{z_2} + e^{2z_1} e^{2z_2})(E_2 - E_3).$$

This solution is nonsingular and periodic in the $x_2$ and $x_4$ directions, so it is a complex doubly-periodic instanton. It also has zero energy density. As far as we know, this is the first explicit example of a doubly-periodic instanton.

Now we give an example of complex solution that is not from the above complex 't Hooft ansatz.

**Example 4.** Setting $\rho_1 = z_1 e^{z_2}$ and $\rho_2 = e^{z_1} z_2$ in the construction of Sec. 3, one obtains:

$$A_1 = \frac{1}{2}\begin{pmatrix} -ie^{z_1} z_2 & e^{z_1} z_2 + iz_1 e^{z_2} \\ e^{z_1} z_2 - iz_1 e^{z_2} & ie^{z_1} z_2 \end{pmatrix}$$

$$A_2 = \frac{1}{2}\begin{pmatrix} iz_1 e^{z_2} & z_1 e^{z_2} - ie^{z_1} z_2 \\ z_1 e^{z_2} + ie^{z_1} z_2 & -iz_1 e^{z_2} \end{pmatrix}.$$

Therefore,

$$\begin{aligned} F_{12} = {} & i(e^{z_2} - e^{z_1} + z_1^2 e^{2z_2} - z_2^2 e^{2z_1})E_1 + i(z_2 e^{z_1} + z_1 e^{z_2} + 2z_1^2 z_2^2 e^{2z_1} e^{2z_2})E_2 \\ & - (e^{z_1} + e^{z_2} + z_1^2 z_2^2 e^{2(z_1+z_2)})E_3. \end{aligned}$$

Note that Eq. (13) do not hold, showing that this example does not arise from the complex 't Hooft ansatz.

Finally, we conclude with an explicit example of a $SL(3,\mathbb{R})$ instanton with non-zero curvature but zero action density.

**Example 5 ($SL(3,\mathbb{R})$ instanton).** We consider in Eq. (31) $\rho = 2e^{z_2}, \mu = 2z_1$ with $M = a + ib$,

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{35}$$

It follows that

$$\begin{aligned} A_1 &= e^{x_2}(\cos x_4 a - \sin x_4 b), \\ A_2 &= x_1 a - x_3 b, \\ A_3 &= e^{x_2}(\sin x_4 a + \cos x_4 b), \\ A_4 &= x_3 a + x_1 b, \end{aligned}$$

with vanishing energy density and the $F_{uv}$ component of the complex curvature is

$$F_{uv} = F_{12} - F_{34} + i(F_{23} - F_{14}) = 4(1 - 2e^{z_2})M.$$

So the curvature in real coordinates cannot be zero, concluding that it is not a pure gauge connection and contrasting again with the $SU(n)$ case.

## 6. Conclusion

In this paper we presented two ways of constructing explicit, nonsingular complex instantons on the Euclidean $\mathbb{R}^4$ with zero action density. Further examples could be obtained by replacing the conditions $A_{\bar{u}} = A_{\bar{v}} = 0$ and $A_{\bar{u}} = \bar{A}_u$, $A_{\bar{v}} = \bar{A}_v$, imposed respectively in Sec. 3 and in Sec. 4, by another simplifying assumption, showing that our method is not completely exhausted. We hope that these examples will be useful as motivation for the development of a general deformation theory of complex instantons, and lead to better understanding of its physical interpretation and of the structure of its moduli spaces.

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