

A Fourier-Mukai approach to spectral data for instantons

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Abstract. We study $SU(r)$ instantons on elliptic surfaces with a section and show that they are in one-one correspondence with spectral data consisting of a curve in the dual elliptic surface and a line bundle on that curve. We use relative Fourier-Mukai transforms to analyse their properties and, in the case of the K3 and abelian surface, we show that the moduli space of instantons has a natural Lagrangian fibration structure with respect to the canonical complex symplectic structures.

1. Introduction

The mathematical study of gauge theory was born some 25 years ago, and one of the first main results is just as old, namely the ADHM construction of instantons over the Euclidean 4-space. Since then, several different types of Euclidean instantons have been studied from various points of view: monopoles, calorons, Higgs pairs, doubly-periodic instantons, to name a few. The common feature in the study of such objects is the so-called *Nahm transform*. It relates instantons on \mathbb{R}^4 which are invariant under a subgroup of translations $\Lambda \subset \mathbb{R}^4$ to instantons on the dual $(\mathbb{R}^4)^*$ which are invariant under the dual subgroup of translations Λ^* (see [15], Section 7, for a detailed exposition). The Nahm transform has many interesting properties, but perhaps its greatest virtue is that it often converts the difficult problem of solving nonlinear PDE's into a simpler problem involving only ODE's or just vector spaces and linear maps between them.

More recently, the so-called *Fourier-Mukai transforms* have generalised the Nahm transform, bringing it to the realms of algebraic geometry and derived categories. In this paper, we shall study a particular version of such transforms. It is defined for torsion-free sheaves E over relatively minimal elliptic surfaces $X \xrightarrow{\pi} B$ with a section, transforming them into torsion sheaves on $J_X \xrightarrow{\tilde{\pi}} B$, the relative Jacobian of X , which are supported over the spectral curves studied by Friedman-Morgan-Witten [6], [7] for physical reasons. These constructions are briefly reviewed in Sections 2 and 3, and we observe that there is a correspondence between instantons on a 4-torus, spectral curves with line bundles on it, and instantons on the dual 4-torus.

The bulk of the paper is contained in Section 5, where we compare the μ -stability of E

with the concept of stability introduced by Simpson [16] for its transform. In particular, we show that E is μ -stable and locally-free if and only if its transform is stable in the sense of Simpson.

As an application of these ideas, we shed new light into the moduli space of irreducible $SU(r)$ -instantons over elliptic K3 and abelian surfaces, showing in Sections 6 and 7 that it has the structure of a complex Lagrangian fibration. The case of rank 2 instantons on an elliptic surface is treated in Friedman's book [5]. We extend the results to higher rank and provide some more details about the structure of the fibration. The proofs are greatly simplified using the Fourier-Mukai technology and this enables us to go further in our description of the moduli spaces.

Notation. The elliptic surface X will be polarised by ℓ which can be any polarisation (for the product of elliptic curves it is convenient to choose the sum of the elliptic curves). If F_x denotes the fibre of $X \xrightarrow{\pi} B$ passing through $x \in X$, then $\hat{\pi}^{-1}(\pi(x)) = \hat{F}_{\pi(x)}$ is the dual of F_x . Let J_X denote the dual elliptic surface parametrising flat line bundles on the fibres of π . It also fibres over B and has a canonical section given by the trivial bundles. We shall denote its fibres by \hat{F}_b , for $b \in B$.

Given a sheaf E on X , we write its Chern character as a triple:

$$\text{ch}(E) = (\text{rank}(E), c_1(E), \text{ch}_2(E)),$$

where $\text{ch}_2(E) = \frac{1}{2}c_1(E)^2 - c_2(E)$.

By $\mathcal{P} \rightarrow X \times_B J_X$ we mean the relative Poincaré sheaf parametrising flat line bundles on the smooth fibres. Given $p \in J_X$, let i_p be the inclusion of $F_{\hat{\pi}(p)}$ inside X . We define:

$$E(p) = i_p^*(E \otimes \mathcal{P}_p)$$

where \mathcal{P}_p denotes the torsion sheaf on X corresponding to $p \in J_X$.

Finally, $\mathcal{M}_X(r, k)$ (or just \mathcal{M}) will denote the moduli space of μ -stable locally-free sheaves over X , with Chern classes $(r, 0, -k)$, where $r, k > 1$. Note that this coincides with the moduli space of $SU(r)$ instantons of charge k .

2. Spectral data for instantons

Let \mathcal{E} be a vector bundle over the elliptic surface X , and let A be an anti-self-dual $SU(r)$ connection on \mathcal{E} . Then A induces a holomorphic structure $\bar{\partial}_A$ on \mathcal{E} ; we denote by E the associated holomorphic vector bundle.

Given a holomorphic vector bundle $E \rightarrow X$ as above, we say that E is *generically fibrewise semistable* if its restriction to a generic elliptic fibre is semistable; E is said to be *fibrewise semistable* if its restriction to every elliptic fibre is semistable. We shall also say that E is *regular* if $h^0(F_{\hat{\pi}(p)}, E(p)) \leq 1$ for all $p \in J_X$. Observe that regularity is a generic condition.

Now assume that E is a regular holomorphic vector bundle over X ; we define the *instanton spectral curve* with respect to the projection $X \xrightarrow{\pi} B$ in the following manner:

$$(1) \quad S = \{p \in J_X \mid h^1(F_{\hat{\pi}(p)}, E(p)) \neq 0\}.$$

It is not difficult to see that $S \xrightarrow{\pi} B$ is a branched r -fold covering map.

To define the second part of the instanton spectral data, recall that $\chi(E(p)) = 0$ for every $p \in J_X$. We define a line bundle \mathcal{L} on S by attaching the vector space $H^1(F_{\hat{\pi}(p)}, E(p))$ to the point $p \in S$. Alternatively, consider the diagram

$$(2) \quad \begin{array}{ccccc} X \times_B J_X & \xleftarrow{\tau_1} & X \times_B S & \xrightarrow{\sigma} & S \\ & & \downarrow \tau_2 & & \\ & & X & & \end{array}$$

where τ_1 is the inclusion map and τ_2, σ are the obvious projections. We define \mathcal{L} as follows:

$$(3) \quad \mathcal{L} = R^1\sigma_*(\tau_2^*E \otimes \tau_1^*\mathcal{P}).$$

Geometrically, it is not difficult to see that \mathcal{L} can be regarded as a *bundle of cokernels* associated to a family of coupled Dirac operators parametrised by S . For the simple case $X = T \times \mathbb{P}^1$, see [10]. This follows from the natural identification between $H^1(F_{\hat{\pi}(p)}, E(p))$ and the cokernel of the Dolbeault operator $\bar{\partial}_A|_{F_{\hat{\pi}(p)}}$.

As we will see in Section 3 below, the above construction is invertible, and the original holomorphic bundle E can be reconstructed from its spectral pair (S, \mathcal{L}) . However, the spectral pair contains only the holomorphic information, so some extra information is needed in order to reconstruct the original instanton connection. One possibility is to construct a connection on $\mathcal{L} \rightarrow S$, as it was done in [10]; see also [11] for a related problem. In this paper we will follow a different path, which will be described in Section 5 below.

3. Spectral data and the relative Fourier-Mukai transform

Given a sheaf E on the relatively minimal elliptic surface $X \rightarrow B$ with a section, we define its relative Fourier-Mukai transform to be the *complex* of sheaves $\Phi(E)$ on J_X given by

$$(4) \quad \Phi(E) = R\hat{\pi}_*(\pi^*E \otimes \mathcal{P})$$

where $X \xleftarrow{\pi} X \times_B J_X \xrightarrow{\hat{\pi}} J_X$ are the projection maps and \mathcal{P} is the relative Poincaré sheaf as before.

We say that E is Φ -WIT $_n$ if $\Phi^i(E) = 0$ unless $i = n$, where $\Phi^i(E)$ denotes the i th homology of the complex representing $\Phi(E)$ which is well defined up to isomorphism.

It can be shown that (4) gives an equivalence between $D(X)$ and $D(J_X)$, the derived categories of bounded complexes of coherent sheaves. Its inverse functor is given by

$$(5) \quad \hat{\Phi}(L) = R\pi_*(\hat{\pi}^*L \otimes \mathcal{P}^\vee)$$

so that $\hat{\Phi}^0(\Phi^1(E)) = E$ and $\Phi^1(\hat{\Phi}^0(L)) = L$.

This is one of a series of Fourier-Mukai transforms for elliptic surfaces parameterized by $SL_2(\mathbb{Z})$. For a description of these and their invertibility, see [1]. The rank and fibre degree of the transform $\Phi(E)$ is related to the rank and fibre degree of E by multiplication by the element of $SL_2(\mathbb{Z})$. The case we shall be interested in this paper is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Let us now recall the details of the definition of stability (due to Mumford and Takemoto) of torsion-free sheaves:

Definition. A torsion-free sheaf E over a polarised surface (X, ℓ) is said to be μ -stable (respectively, μ -semistable) if for all proper subsheaves F of E , $\mu(F) < \mu(E)$ (respectively, $\mu(F) \leq \mu(E)$), where $\mu(\bullet) = c_1(\bullet) \cdot \ell / r(\bullet)$ is called the slope.

Recall that since we are working over surfaces, we can assume that the destabilizing subsheaf is locally-free and that the quotient E/F is torsion-free. Moreover, we also may choose F to be μ -stable.

If E is μ -stable and locally-free, that is if E is a holomorphic vector bundle arising from an irreducible instanton connection via the Hitchin-Kobayashi correspondence, it is easy to see that E is Φ -WIT₁. Hence $\Phi(E) = \Phi^1(E)$ is not a complex of sheaves, but simply a sheaf on J_X . This also applies if E is not necessarily locally-free but is still torsion-free, as we shall see in Section 5.

The following proposition brings together the instanton spectral data described in the previous section with the Fourier-Mukai transform.

Proposition 1. Let \mathcal{E} be a vector bundle with an irreducible $SU(r)$ instanton connection over an elliptic surface X . Let E be the associated μ -stable, regular holomorphic vector bundle with $c_1(E) = 0$. Then $\Phi^1(E)$ is a torsion sheaf supported over the instanton spectral curve $S \subset J_X$. Moreover, the restriction of $\Phi^1(E)$ to its support is naturally isomorphic to \mathcal{L} .

Proof. Let $j : S \rightarrow J_X$ be the inclusion map. This fits into a commuting diagram of maps introduced in (2):

$$\begin{array}{ccc} X \times_B S & \xrightarrow{\tau_1} & X \times_B J_X \\ \downarrow \sigma & & \downarrow \hat{\pi} \\ S & \xrightarrow{j} & J_X. \end{array}$$

Then

$$\begin{aligned} j_*\mathcal{L} &\cong R(j_*\sigma_*)(\tau_2^*E \otimes \tau_1^*\mathcal{P}) \\ &\cong R\hat{\pi}_*R\tau_{1*}(\tau_2^*E \otimes \tau_1^*\mathcal{P}) \\ &\cong R\hat{\pi}_*(\mathcal{P} \otimes R\tau_{1*}\tau_2^*E). \end{aligned}$$

But $\tau_2 = \pi\tau_1$ (see diagram 2), so $R\tau_{1*}\tau_2^*E \cong \pi^*E \otimes \mathcal{O}_{X \times_B S}$. Hence, we have a natural map $\Phi^1(E) \rightarrow j_*\mathcal{L}$. Since the fibres of $\Phi^1(E)$ and $j_*\mathcal{L}$ are naturally isomorphic, this must be an isomorphism. Hence, $j^*\Phi^1(E) = j^*j_*\mathcal{L} = \mathcal{L}$ as required. \square

4. Instantons on flat 4-tori

Let \mathbb{T} be a flat 4-dimensional torus, with a fixed complex structure. Denote by $\hat{\mathbb{T}}$ the corresponding dual torus, which inherits a flat metric and a complex structure from \mathbb{T} polarized by the associated Kähler class.

We start by considering an irreducible $SU(r)$ bundle $\mathcal{E} \rightarrow \mathbb{T}$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = k$, plus an instanton connection A . Equivalently, we can adopt an algebraic geometric point of view and look at the associated μ -stable holomorphic vector bundle $E \rightarrow \mathbb{T}$ of rank r and the same Chern classes. Stability now implies the irreducibility hypothesis, which in turns implies that $h^0(\mathbb{T}, E) = h^2(\mathbb{T}, E) = 0$, so that $h^1(\mathbb{T}, E) = k$.

The *Nahm transformed bundle* $\hat{E} \rightarrow \hat{\mathbb{T}}$ is then constructed as follows. Let $P \rightarrow \mathbb{T} \times \hat{\mathbb{T}}$ be the Poincaré line bundle, and consider the natural projection maps:

$$(6) \quad \mathbb{T} \xleftarrow{p} \mathbb{T} \times \hat{\mathbb{T}} \xrightarrow{\hat{p}} \hat{\mathbb{T}}.$$

Then $\hat{E} = R^1\hat{p}_*(p^*E \otimes P^\vee)$. The fibre of \hat{E} over $\zeta \in \hat{\mathbb{T}}$ can be canonically identified with $\hat{E}_\zeta = H^1(\mathbb{T}, E \otimes L_\zeta)$, where $L_\zeta \rightarrow \mathbb{T}$ denotes the flat holomorphic line bundle associated with $\zeta \in \hat{\mathbb{T}}$.

Conversely, it can be shown that $E = R^1p_*(\hat{p}^*\hat{E} \otimes P)$, with fibres canonically identified with $E_z = H^1(\hat{\mathbb{T}}, \hat{E} \otimes L_z)$, where $L_z \rightarrow \hat{\mathbb{T}}$ denotes the flat holomorphic bundle associated with $z \in \mathbb{T}$.

Theorem 2. *\hat{E} is a μ -stable holomorphic vector bundle of rank k , flat determinant and $c_2(\hat{E}) = r$. Therefore, since it is invertible, the Nahm transform is a bijective correspondence $\mathcal{M}_{\mathbb{T}}(r, k) \rightarrow \mathcal{M}_{\hat{\mathbb{T}}}(k, r)$.*

For the proof, we refer to [3]. The algebro-geometric version is given in [12]. In particular, in the $k = 1$ case, the theorem tells us that there are no stable holomorphic bundles with this Chern character, thus precluding the existence of instantons with unit charge. For larger k , this result is not really helpful as it only converts the problem of constructing instantons/stable bundles over \mathbb{T} into the same problem over $\hat{\mathbb{T}}$. A better understanding of the moduli space of $SU(r)$ instantons on \mathbb{T} with $k > 1$ requires a new tool, namely the Relative Fourier-Mukai transform introduced in the previous section.

Now assume that $\mathbb{T} = V \times W$, i.e. \mathbb{T} is given by the product of two elliptic curves V and W . We shall denote by \hat{V} and \hat{W} the respective Jacobian curves, so that the dual

torus is given by $\hat{\mathbb{T}} = \hat{V} \times \hat{W}$. This case is particularly interesting because we have two fibration structures leading apparently to two distinct spectral curves, and we would like to understand the relation between them.

Regarding \mathbb{T} as an elliptic surface over V , consider the diagram

$$(7) \quad V \times W \xleftarrow{\pi_V} V \times W \times \hat{W} \xrightarrow{\hat{\pi}_V} V \times \hat{W}$$

where π_V is the projection onto the first and second factors, while $\hat{\pi}_V$ is the projection onto the first and third factors. Let also $\mathbf{P}_W \rightarrow W \times \hat{W}$ be the Poincaré line bundle. Then

$$(8) \quad \mathcal{L} = R^1 \hat{\pi}_{V*} (\pi_V^* E \otimes \mathbf{P}_W)$$

is a torsion sheaf on $V \times \hat{W}$, supported over the instanton spectral curve S .

On the other hand, regard $\hat{\mathbb{T}}$ as an elliptic surface over \hat{W} and consider the Nahm transformed bundle $\hat{E} \rightarrow \hat{\mathbb{T}}$. Based on the diagram

$$(9) \quad \hat{V} \times \hat{W} \xleftarrow{\pi_W} V \times V \times \hat{V} \times \hat{W} \xrightarrow{\hat{\pi}_W} V \times \hat{W}$$

where π_W is the projection onto the second and third factors, while $\hat{\pi}_W$ is the projection onto the first and third factors. Let also $\mathbf{P}_V \rightarrow V \times \hat{V}$ be the Poincaré line bundle. Then

$$(10) \quad \mathcal{N} = R^1 \hat{\pi}_{W*} (\pi_W^* E \otimes \mathbf{P}_V)$$

is a torsion sheaf on $V \times \hat{W}$, supported over the *dual* instanton spectral curve R .

Since the bundles E and \hat{E} are related via Nahm transform, it seems natural to ask how are the two sets of spectral data (S, \mathcal{L}) and (R, \mathcal{N}) are related to one another.

Regarding \mathbb{T} as an elliptic surface over V , we have just seen that the spectral data (S, \mathcal{L}) is encoded into the torsion sheaf $\Phi^1(E)$ on $V \times \hat{W}$. On the other hand, regarding $\hat{\mathbb{T}}$ as an elliptic surface over \hat{W} let us denote the Fourier-Mukai transform of \hat{E} by $\Psi(\hat{E})$. It is a torsion sheaf on $V \times \hat{W}$ encoding the dual spectral data (R, \mathcal{N}) . Let also $\hat{\Psi}$ denote the inverse of Ψ .

We are finally in a position to prove our first main result:

Theorem 3. *The spectral pairs (S, \mathcal{L}) and (R, \mathcal{N}) are equivalent, in the sense that \mathcal{L} and \mathcal{N} can be canonically identified as sheaves on J_X .*

Proof. If we define the functor $\mathcal{F} : D(\mathbb{T}) \rightarrow D(\hat{\mathbb{T}})$ as follows:

$$\mathcal{F}(E) = R\hat{p}_*(p^*E \otimes P^\vee)$$

where p and \hat{p} are the projection maps in (6), then $\hat{E} = \mathcal{F}^1(E)$. The functor \mathcal{F} is Mukai's original Fourier transform introduced in [13]. Using this notation, it is enough to show that $\hat{\Psi} \circ \Phi = \mathcal{F}$.

To see this it suffices to show that $\hat{\Psi}(\Phi(\mathcal{O}_x)) = \hat{\Psi}(\mathcal{P}_x) = P_x$ over $\hat{\mathbb{T}}$. But $\mathcal{P}_x \otimes \mathcal{P}_y^\vee$ is supported at the intersection of translates of V and \hat{W} . Hence, \mathcal{P}_x is Ψ -WIT₀ and $\hat{\Psi}(\mathcal{P}_x)$ is a flat line bundle. On the other hand, the properties of Ψ and Φ imply that translating x to y twists $\hat{\Psi}(\mathcal{P}_x)$ by P_{y-x} and so by normalizing the Poincaré bundles appropriately, we have $\hat{\Psi}(\mathcal{P}_x) = P_x$. \square

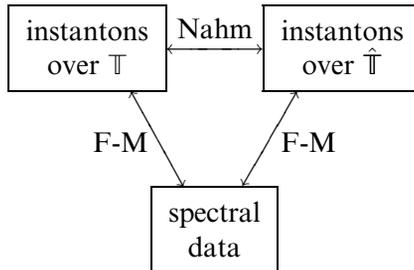
Similarly, one could regard \mathbb{T} as an elliptic surface over W and $\hat{\mathbb{T}}$ as an elliptic surface over \hat{V} . The analogue of Theorem 3 would again hold, so that the corresponding spectral data would also coincide as torsion sheaves in $\hat{V} \times W$.

More generally, one has the following commuting diagram of derived categories:

$$(11) \quad \begin{array}{ccc} D(V \times W) & \xrightarrow{\Phi} & D(V \times \hat{W}) \\ \downarrow \Upsilon & \searrow \mathcal{F} & \downarrow \Psi \\ D(\hat{V} \times W) & \xrightarrow{\Xi} & D(\hat{V} \times \hat{W}). \end{array}$$

Of course, the functor Υ is given by the Fourier-Mukai transform regarding \mathbb{T} as an elliptic surface over W , while the functor Ξ is the Fourier-Mukai transform regarding $\hat{\mathbb{T}}$ as an elliptic surface over \hat{V} .

Remark. One concludes from Theorem 3 that there is an equivalence between the following three objects: instantons on a flat 4-torus (i.e. μ -stable bundles over \mathbb{T}), spectral data, and instantons on the dual 4-torus (i.e. μ -stable bundles over $\hat{\mathbb{T}}$). Such “circle of ideas” has been previously established for monopoles [9], doubly-periodic instantons [10] and periodic monopoles [2]. Indeed, one expects that a similar scheme will hold for all translation invariant instantons on \mathbb{R}^4 .



5. Preservation of stability

We now aim at establishing the link between the stability of E and the stability of $L = \Phi^1(E)$. For torsion sheaves, we have a concept of stability due to Simpson [16]:

Definition. A torsion sheaf L on a polarised variety (X, ℓ) is said to be p -stable (respectively, p -semistable) if it is of pure dimension (i.e. the supports of all subsheaves have the same dimension) and for all proper subsheaves M of L we have $p(M, n) < p(L, n)$ (respectively, $p(M, n) \leq p(L, n)$) for all sufficiently large n , where $p(L, n)$ denotes the reduced Hilbert polynomial of $L \otimes \mathcal{O}(n\ell)$.

In particular, a torsion sheaf which is supported on an irreducible curve and whose restriction has rank one is automatically p -stable with respect to any polarisation.

Now let $X \rightarrow B$ be a (relatively minimal) elliptic surface with a section σ ; let E be a torsion-free sheaf on X with $\text{ch}(E) = (r, 0, -k)$.

Proposition 4. *Suppose that the restriction of E to some smooth fibre F_s is semistable. Then E is Φ -WIT₁ and $L = \Phi^1(E)$ has Chern character $(0, k\hat{f} + r\hat{\sigma}, 0)$, where $\hat{\sigma}$ is the class given by the zero section of J_X . In particular, $p(L, n) = n$.*

Proof. First note that if E is torsion-free then $\Phi^0(E) = \hat{\pi}_*(\pi^*E \otimes \mathcal{P})$ is also torsion-free. Hence its support consists either empty or consists of the whole relative jacobian surface J_X . Moreover, if E is not Φ -WIT₁ then the support of $\Phi^0(E)$ is contained in the support of $\Phi^1(E)$ since $\chi(E|_{F_x}) = 0$ for all $x \in X$.

However, $\Phi^1(E)$ cannot be supported on the whole surface since it does not contain the whole fibre \hat{F}_s where $E|_{F_s}$ is semistable. Thus $\Phi^0(E) = 0$ and E is Φ -WIT₁.

Now $\Phi^1(\mathcal{O}_X)$ is the trivial line bundle supported on the zero section of J_X and $\Phi^0(\mathcal{O}_x)$ is a flat line bundle supported on (a divisor in) \hat{f} . Hence, since $\text{ch}(E) = r \cdot \text{ch}(\mathcal{O}_X) + k \cdot \text{ch}(\mathcal{O}_x)$, we conclude that

$$\text{ch}(\Phi^1(E)) = r \cdot \text{ch}(\Phi^1(\mathcal{O}_X)) + k \cdot \text{ch}(\Phi^0(\mathcal{O}_x)) = (0, k\hat{f} + r\hat{\sigma}, 0)$$

as required. \square

Note that the rank and fibre degrees of L are given by $r(L) = c_1(E) \cdot f = 0$ and $c_1(L) \cdot \hat{f} = r$ from the definition of the relative transform.

Let S denote the support of $\Phi^1(E)$; the statement above does not imply that S is a divisor in J_X , since it might contain some 0-dimensional components (i.e. isolated points).

Lemma 5. *If E is generically fibrewise semistable, then $S = \text{supp } \Phi^1(E)$ has no 0-dimensional components. Moreover, $L = \Phi^1(E)$ is of pure dimension one.*

Proof. Suppose that $q \in S$ is an isolated point. It cannot belong to a fibre \hat{F}_x such that $E|_{F_x}$ is semistable, since the restriction to every nearby fibre is also semistable and the restriction of E to the fibres varies holomorphically.

So $E|_{F_q}$ is unstable. But S contains all such fibres, since $h^1(F_{\hat{\pi}(p)}, E(p)) > 0$ for all $p \in \hat{\pi}^{-1}(\pi(q))$.

To obtain the second statement, we must show that L has no proper subsheaves supported on points. For a contradiction, suppose it does, and let F be a subsheaf of L with 0-dimensional support. Then $\hat{\Phi}^0(F)$ is a torsion subsheaf of $\hat{\Phi}^0(L) = E$, thus contradicting the hypothesis that E is torsion-free. \square

Thus, we will call S the *spectral curve* associated to the generically fibrewise semistable torsion-free sheaf E , since it generalizes our previous definition for regular locally-

free sheaves. As before, let us denote by \mathcal{L} the restriction of $\Phi^1(E)$ to its support; in general, it is a coherent sheaf on S . Note that $\text{deg}(\mathcal{L}) = 0$.

Remark. If the restriction of E to some smooth fibre has no sections, then semi-continuity implies that E is generically fibrewise semistable, and hence Φ -WIT₁. It is somewhat surprising that the spectral data can actually be defined in an interesting, meaningful way under such mild conditions.

Summing up, we also conclude:

Corollary 6. *If E is a torsion-free and Φ -WIT₁, then $\Phi(E) = \Phi^1(E)$ is a torsion sheaf of pure dimension one.*

Conversely, we also have:

Lemma 7. *If L is a $\hat{\Phi}$ -WIT₀ torsion sheaf of pure dimension one on J_X with $\text{ch}(L) = (0, k\hat{f} + r\hat{\sigma}, 0)$, then $\hat{\Phi}^0(L)$ is torsion-free.*

Proof. Suppose that the support of L decomposes as $\Sigma + F$, where F is the sum of fibres. When this happens, we have a subsheaf K of L supported on F with degree 0. Assume that F is the maximal such effective subdivisor of S . Let $Q = L/K$. Then Q is $\hat{\Phi}$ -WIT₀ and K is $\hat{\Phi}$ -WIT₁. The resulting sequence after transforming with $\hat{\Phi}$ is then

$$(12) \quad 0 \rightarrow E \rightarrow E^{**} \rightarrow \mathcal{O}_Z \rightarrow 0$$

where $E = \hat{\Phi}^0(L)$, for some zero dimensional subscheme Z . \square

Remark. Notice that applying Φ to the sequence (12) shows that if E is torsion-free but not locally-free, then the support of L contains fibres.

The following lemma due to Bridgeland [1] characterizes $\hat{\Phi}$ -WIT₀ sheaves on J_X :

Lemma 8. *A sheaf L on J_X is $\hat{\Phi}$ -WIT₀ if and only if*

$$\text{Hom}(L, \mathcal{P}_x) = 0 \quad \forall x \in X.$$

5.1. Suitable polarizations. Now let ℓ be a polarisation of the elliptic surface X , and let $\hat{\ell}$ be the induced polarisation on J_X . If ℓ is arbitrary, it is not difficult to see that a μ -unstable torsion-free sheaf on X can have a p -stable transform.

Indeed, let X be an elliptic surface whose fibres are all smooth, and let E be the locally-free sheaf given by the following extension:

$$0 \rightarrow \mathcal{O}(-\sigma + df) \rightarrow E \rightarrow \mathcal{O}(\sigma - df) \rightarrow 0.$$

Clearly, $c_1(E) = 0$ and $c_2(E) = -\sigma^2 + 2d$. For d sufficiently large, $c_2(E) > 0$ and $\ell \bullet (-\sigma + df) = -(\ell \bullet \sigma) + d(\ell \bullet f) > 0$, so E is μ -unstable with respect to ℓ .

On the other hand, notice that the restriction of E to each fibre is an extension of

$Q \in \text{Pic}^d(T)$ by its dual. Thus, the bundles obtained by the above extension are generically regular. This means $\Phi^1(E)$ is supported on a smooth, irreducible curve, so that $\Phi^1(E)$ is necessarily p -stable with respect to $\hat{\ell}$.

Therefore, we can only expect the Fourier-Mukai transform to preserve stability if we restrict the choice of polarisation on X in some convenient way.

Definition. Let c be a positive integer, and consider the set

$$\Xi(c) = \{\xi \in \text{Div}(X) \mid -4c \leq \xi^2 < 0 \text{ and } \xi \bmod 2 = 0\}.$$

Let W^ξ be the intersection of the hyperplane ξ^\perp with the ample cone of X . A polarisation ℓ is said to be c -suitable if $\ell \notin W^\xi$ and $\text{sign}(\ell \bullet \xi) = \text{sign}(f \bullet \xi)$ for all $\xi \in \Xi(c)$.

It is easy to see that suitable polarizations exist for every c . The following important result is due to Friedman, Morgan and Witten [7]:

Theorem 9. *Let E be a torsion-free sheaf on X with $ch(E) = (r, 0, -k)$. If E is μ -semistable with respect to a k -suitable polarisation ℓ , then E is generically fibrewise semistable.*

5.2. Preservation of stability. It follows from Theorem 9 that if E is a torsion-free sheaf on X with $ch(E) = (r, 0, -k)$, which is μ -semistable with respect to a k -suitable polarisation ℓ , then $L = \Phi^1(E)$ is a torsion sheaf of pure dimension one. Let us now consider its stability (in the sense of Simpson).

Proposition 10. *Let E be a torsion-free sheaf on X which is μ -semistable with respect to a k -suitable polarisation ℓ . Then $L = \Phi^1(E)$ is p -semistable with respect to $\hat{\ell}$.*

Proof. Suppose that

$$0 \rightarrow M \rightarrow L \rightarrow N \rightarrow 0$$

is a destabilizing sequence for L . We can assume M is semistable and N has pure dimension 1. Now $p(M, n) = n + \alpha$, for some $\alpha > 0$. Then $\text{Hom}(M, \mathcal{P}_x) = 0$ for all $x \in X$, since $p(\mathcal{P}_x, n) = n$. Thus M is $\hat{\Phi}$ -WIT₀ by Lemma 8. Arguing as in Proposition 4, we have $c_1(\hat{\Phi}^0(M)) = \alpha f$ hence $\mu(\hat{\Phi}^0(M)) > 0$. So $\hat{\Phi}^0(M)$ will destabilize $E = \hat{\Phi}^0(L)$ unless $r(\hat{\Phi}^0(M)) = r(E)$. For this to be the case we must have $c_1(M) \cdot \hat{f} = c_1(L) \cdot \hat{f}$. Then N is supported on a sum of fibres. But since N must be $\hat{\Phi}$ -WIT₀ we see that it has non-negative degree on these fibres and so $\alpha = 0$, thus contradicting the assumption on M . \square

If we examine the proof more carefully, we can also see that the assumption that E is μ -stable implies that L is p -stable unless its support decomposes as $D + D'$, where D is the sum of fibres. When this happens, we have a subsheaf K of L supported on D with degree 0. Assume that D is the maximal such effective subdivisor of S . Let $Q = L/K$. Then Q is $\hat{\Phi}$ -WIT₀ and K is $\hat{\Phi}$ -WIT₁. The resulting sequence after transforming with $\hat{\Phi}$ is then $E \rightarrow E^{**} \rightarrow \mathcal{O}_Z$ for some zero dimensional subscheme Z . Conversely, applying Φ to this sequence shows that the support of L contains fibres. We have therefore established:

Proposition 11. *Suppose E is μ -semistable with respect to a k -suitable polarisation ℓ with $c_1(E) = 0$. If E is μ -stable and locally-free then L is p -stable. Moreover, E is locally-free if and only if L is destabilized by a sheaf supported on a fibre.*

We can now consider the opposite question: if we assume L is p -semistable what can we say about its transform?

Proposition 12. *Suppose L is a p -stable sheaf on J_X with Chern character $(0, k\hat{f} + r\hat{\sigma}, 0)$, where $r, k > 1$. Then L is $\hat{\Phi}$ -WIT₀ and $\hat{\Phi}^0(L)$ is locally-free, μ -stable with respect to a k -suitable polarisation and such that $ch(\hat{\Phi}^0(L)) = (r, 0, -k)$.*

Proof. The first statement follows from Lemma 8, which requires us to show that $\text{Hom}(L, \mathcal{P}_x) = 0$ for all x . However, $p(\mathcal{P}_x, n) = n$ and any map $L \rightarrow \mathcal{P}_x$ would contradict the stability of L .

Now suppose that $\hat{\Phi}^0(L)$ is not μ -stable and let A be the destabilizing subsheaf, so that $\mu(A) \geq \mu(E) = 0$. Moreover, we may assume that A is μ -stable, thus A is Φ -WIT₁.

We may also assume that the quotient $B = E/F$ is μ -stable. Then both A and B are Φ -WIT₁. Since their transforms must have zero rank, A and B both have zero fibre degree and so $p(\Phi^1(A), n) \geq n$ and this contradicts the stability of L .

The fact that $\hat{\Phi}^0(L)$ is locally-free follows from the last part of Proposition 11. \square

A similar argument shows that if L is p -semistable and $\hat{\Phi}$ -WIT₀ then $\hat{\Phi}^0(L)$ must be μ -semistable. Note, however, that the semistability of L does not imply that L is $\hat{\Phi}$ -WIT₀. Examining the proof above we see that this happens precisely when L is destabilized by mapping to a sheaf supported on a fibre. In particular, such sheaves are S-equivalent to sheaves which are $\hat{\Phi}$ -WIT₀ but whose transforms are not locally-free.

We summarize the results of this section in the following theorem.

Theorem 13. *Suppose E is a torsion-free sheaf with rank r , $c_1(E) = 0$ and $c_2(E) = k$ on a relatively minimal elliptic surface X over B . Let ω be a k -suitable polarisation on X . Then:*

(1) *E is a μ -stable locally free sheaf if and only if its transform is a p -stable torsion sheaf supported on a divisor in $|k\hat{f} + r\hat{\sigma}|$, where $\hat{\sigma}$ is a section of J_X and \hat{f} is a fibre class.*

(2) *E is μ -stable properly torsion-free if and only if its transform is a p -semistable torsion sheaf supported on a reducible divisor in $|k\hat{f} + r\hat{\sigma}|$ which is destabilized only by a sheaf supported on a fibre.*

(3) *E is properly μ -semistable if and only if its transform is a p -semistable torsion sheaf supported on a reducible divisor in $|k\hat{f} + r\hat{\sigma}|$.*

One can also give criteria for the Gieseker stability of E in terms of destabilizing properties of L , but these seem to be less useful.

Remark. Recently, Hernández Ruipérez and Muñoz Porras [8] and Yoshioka [17] have independently obtained stability results close to those presented in this section.

Remark. It follows from the first item in Theorem 13 that the Fourier-Mukai functor Φ induces a bijective map from $\mathcal{M}_X(r, k)$ onto $\mathcal{S}_{J_X}(0, k\hat{f} + r\hat{\sigma}, 0)$, the Simpson moduli space of p -stable torsion sheaves on J_X with the given Chern classes. As we will observe in Section 7 below, this map is also a hyperkähler isometry when X is an elliptic K3 or abelian surface (i.e. when X is hyperkähler).

We can see that the geometry of the spectral data is easily linked to the sheaf theoretic properties of the original sheaf. This should make it very easy in practice to use the spectral data to analyse properties of the specimen sheaf. We shall see an example of this in the subsequent sections where we use the spectral data to explore the fibration structure on the moduli spaces.

6. The fibration structure

From now on we assume that X is either an elliptic K3 surface with a section or a product of elliptic curves. We have seen that there is a natural map Π from the moduli space of μ -stable torsion-free sheaves $\mathcal{M}_X^{\text{TF}}(r, k)$ to the set \mathcal{B} of spectral curves. In the case of the K3 surface the base \mathcal{B} is just the linear system $|k\hat{f} + r\hat{\sigma}|$ while for the abelian surface it can be expressed as the total space of the projectivized bundle $\mathbb{P}\mathcal{F}'(\mathcal{O}(k\hat{f} + r\hat{\sigma}))$, where \mathcal{F}' is the Mukai transform $D(\hat{V} \times W) \rightarrow D(V \times \hat{W})$ (this can be factored as $\Phi\hat{Y} = \hat{\Psi}\Xi$ from (11)). The fibre of Π over S is given by suitable subspaces of $\text{Jac}_{g(S)-1}(S)$. The hypothesis on X implies that for \mathcal{M} to be non-empty we must have both r and k at least 2.

Theorem 14. *The map $\mathcal{M}_X(r, k) \xrightarrow{\Pi} \mathcal{B}$ is a surjective map of varieties.*

Proof. We must first show that Π is a well defined map of varieties. To see this we use the argument given by Friedman and Morgan in [4], Chapter VII, Thm. 1.14. The key idea is to observe that Π coincides (locally in the étale or complex topology) with the projection map from the universal sheaf corresponding to the relative Picard scheme of degree $g - 1$ line bundles on families of genus g curves. The universal sheaf only exists locally in these topologies but this is enough to show that Π is holomorphic.

Given a spectral curve S , observe that its structure sheaf \mathcal{O}_S is $\hat{\Phi}$ -WIT₀ since in the short exact sequence $0 \rightarrow \Lambda^{-1} \rightarrow \mathcal{O} \rightarrow \mathcal{O}_S \rightarrow 0$, Λ^{-1} is $\hat{\Phi}$ -WIT₁ and \mathcal{O} is $\hat{\Phi}$ -WIT₀.

We aim to construct a $g(S) - 1$ degree line bundle L over the curve S , which is stable as a torsion sheaf on J_X . That is, we need to choose $g(S) - 1$ points on S and guarantee that the restriction of L to any proper component S_i of S satisfies $\text{deg}(L|_{S_i}) > 0$. We do this by choosing $g(S_i)$ points on each component S_i . This can always be done because if $g(S_i) > 0$ then S_i intersects $S - S_i$ at least twice since k and r are both at least 2 and so $g(S - S_i) < g(S) - g(S_i)$. Without loss of generality, we assume that the set we have just chosen $Z \subset S$ consists of distinct points away from the singularities of S . Therefore,

$$\text{Ext}^1(\mathcal{O}_Z, \mathcal{O}_S) = \bigoplus_{z \in Z} \text{Ext}^1(\mathcal{O}_z, \mathcal{O}_S) = \bigoplus_{z \in Z} T_z S.$$

We pick a class in $\text{Ext}^1(\mathcal{O}_Z, \mathcal{O}_S)$ which is non-zero on each factor. This defines a torsion sheaf L on J_X given by this extension class. Then L is stable and locally free on its support by the choice of Z .

By Proposition 12, L is $\hat{\Phi}$ -WIT₀ and $E = \hat{\Phi}^0(L)$ is μ -stable vector bundle such that $\Pi(E) = S$. \square

From this proof we can also see that any $E \in \mathcal{M}$ can be written as an extension

$$0 \rightarrow \hat{\Phi}^0(\mathcal{O}_S) \rightarrow E \rightarrow \hat{\Phi}^0(\mathcal{O}_Z) \rightarrow 0.$$

We shall make use of this in the next section. Note that such representation is not unique.

Such a fibration structure will exist for the moduli spaces over any elliptic surface with a section but this surjectivity result may not hold. For the rank 2 case see [5], Thm. 37, or [4], Thm. 1.14. It is also the case that the fibres will not exist in the middle dimension which is the situation we wish to consider in the next section.

7. The hyperkähler structure

When X is K3 or a torus, it is well known that the moduli spaces of sheaves are hyperkähler. To see this, note that by a result of Mukai’s [14], for each complex structure I on X we obtain a complex symplectic structure Ω_I on \mathcal{M}_X . Then, by Yau’s proof of the Calabi conjecture, one obtains the full hyperkähler structure on \mathcal{M}_X .

In fact, the complex symplectic structures arise in a very natural way:

$$T_{[E]}\mathcal{M} \times T_{[E]}\mathcal{M} \cong \text{Ext}^1(E, E) \times \text{Ext}^1(E, E) \xrightarrow{\cup} \text{Ext}^2(E, E) \cong \mathbb{C},$$

where the E is an \mathcal{O}_X -module with respect to I . But we can express this even more simply by observing that $\text{Ext}^i(E, E) = \text{Hom}_{D(X)}(E, E[i])$. Then the cup product \cup is just composition of maps in the derived category. Since Φ is a functor, it must preserve these and so we see that Φ induces a complex symplectomorphism:

$$\mathcal{M}_X(r, k) \rightarrow \mathcal{S}_{J_X}(0, k\hat{f} + r\hat{\sigma}, 0).$$

Since this happens for each complex structure, the moduli map induced by Φ is actually a hyperkähler isometry.

We aim to show now that the fibration structure we have defined in the last section on \mathcal{M} has Lagrangian fibres with respect to this complex symplectic structure.

Proposition 15. *If t_1 and t_2 are two tangent vectors to the fibre of Π then $\Omega_I(t_1, t_2) = 0$.*

Proof. By continuity it suffices to prove this when t_i are defined over a point E given by an extension:

$$0 \rightarrow \hat{\Phi}(\mathcal{O}_S) \rightarrow E \rightarrow \hat{\Phi}(\mathcal{O}_Z) \rightarrow 0$$

where Z consists of discrete points and S is smooth. Deformations of E arising from the fibre of Π are determined by deformations of Z along S . Then $\text{Ext}^2(E, E) \cong \text{Ext}^2(L, L)$ is generated by a non-zero vector in $\text{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$ and the fibre Π_S over S has tangent space given by $\bigoplus_{z \in Z} \langle \lambda_z \rangle$, where λ_z generates the tangent space to S at z . But $\lambda_z \cup \lambda_z = 0$ in $\text{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$ for each z and so $\Omega_I(t_1, t_2) = 0$ as required. \square

In the case of a product abelian surface we have natural hyperkähler actions of the torus and its dual via translations and twisting by flat line bundles. These act naturally on the fibration structure and the resulting quotient is also a Lagrangian fibration.

We can say a little more in the particular case of the product of two elliptic curves. For $\mathbb{T} = V \times W$, observe that $\mathcal{M}_{\mathbb{T}}(r, k)$ has two such fibration structures:

$$(13) \quad \begin{array}{ccc} & \mathcal{M}_{\mathbb{T}}(r, k) & \\ & \swarrow \quad \searrow & \\ \mathcal{B} & & \mathcal{B} \end{array}$$

where $\hat{\mathcal{B}} = \mathbb{P}\hat{\mathcal{F}}(\mathcal{O}(kf + r\sigma))$. Note that the fibres are Lagrangian with respect to the same complex symplectic structure and the bases are biholomorphic since the underlying vector bundles can be canonically identified by pulling back along the isomorphism $V \times \hat{W} \rightarrow \hat{V} \times W$.

In summary we have proved:

Theorem 16. *If X is an elliptic K3 surface with a section or a product of elliptic curves, then the moduli space of instantons admits a fibration structure over a compact base (which is a projective space or a projective bundle over an abelian surface, respectively) and the fibres are Lagrangian with respect to the natural complex symplectic structure.*

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