Nahm Transform and Spectral Curves for Doubly-Periodic Instantons

Marcos Jardim
University of Pennsylvania, Department of Mathematics, Philadelphia, PA 19104-6593, USA

Received: 15 October 1999 / Accepted: 16 October 2001

Abstract: We present the Nahm transform of the doubly-periodic instantons (i.e. instantons on $T^2 \times \mathbb{R}^2$), converting them into certain meromorphic solutions of Hitchin’s equations over an elliptic curve. We then show how to construct a triple consisting of an algebraic curve plus a line bundle with connection over it from a doubly-periodic instanton, and that such data coincides with the Hitchin spectral data associated with the Nahm transformed Higgs bundle.

1. Introduction

In [14], we have shown how certain $SU(2)$ instantons over $\mathbb{R}^4$ which are periodic in two directions, the so-called doubly-periodic instantons, can be constructed from a particular type of singular solutions of Hitchin’s equations (first introduced in [11]) over an elliptic curve. This was done via a procedure known as Nahm transform, which has attracted much attention among physicists recently (see for instance [5, 19, 7] and the references therein).

We now present the inverse construction, showing that all extensible doubly-periodic instantons were obtained in [14].

Recall that given a function $f : T \times \mathbb{C} \to \mathbb{R}$, we say that $f \sim O(|w|^n)$ if:

$$\lim_{w \to \infty} \frac{|f(w)|}{|w|^n} < \infty.$$

We consider anti-self-dual connections $A$ on rank two bundle $E \to T \times \mathbb{C}$ satisfying the following conditions:

1. $|F_A| \sim O(r^{-2})$;
2. there is a holomorphic vector bundle $\mathcal{E} \to T \times \mathbb{P}^1$, the so-called instanton bundle, with trivial determinant such that $\mathcal{E}|_{T \times \mathbb{P}^1(\{\infty\})} \simeq (E, \overline{\nabla}_A)$, where $\overline{\nabla}_A$ is the holomorphic structure on $E$ induced by the instanton connection $A$. 
Such connections are said to be extensible. If $A$ is an extensible instanton connection, then its energy (i.e. $L^2$ norm of the curvature $F_A$) is an integer, the instanton number; furthermore, $A$ splits as a sum of flat connections at the torus added infinity, and such flat connections are called the asymptotic states of $A$ (see Sect. 2).

Let us now outline the contents of this paper. The key feature of Nahm transforms is to try to solve the Dirac equation, and then use its $L^2$-solutions to form a vector bundle over the jacobian torus $\hat{T}$, which parametrises the set of holomorphic flat line bundles over $T \times \mathbb{C}$. Therefore, our first task is to show that the Dirac operator is Fredholm and compute its index.

The bulk of the paper lies in Sects. 4 and 5, where we present the Nahm transform of doubly-periodic instantons and show some of the properties of the transformed objects.

Section 6 is dedicated to prove that the construction here presented is indeed the inverse of the one presented in [14], completing the proof of the main result that motivated these two papers:

**Theorem 1.** The Nahm transform is a bijective correspondence between the following objects:

- gauge equivalence classes of extensible, irreducible $SU(2)$ instanton connections on $E \to T \times \mathbb{C}$ with fixed instanton number $k$ and asymptotic state $\xi_0$; and
- admissible $U(k)$ solutions of Hitchin’s equations over the dual torus $\hat{T}$, such that the Higgs field has at most simple poles at $\pm \xi_0 \in \hat{T}$, with semi-simple residues of rank $\leq 2$ if $\xi_0$ is an element of order 2 in the Jacobian of $T$, and rank $\leq 1$ otherwise.

We also state a higher rank generalization of the above result in Sect. 7.

Finally, we discuss the role played by spectral curves in the correspondence of Theorem 1. More precisely, Hitchin has shown that Higgs pairs are equivalent to a pair consisting of an algebraic curve (the spectral curve) in the total space of the cotangent bundle plus a “line bundle” over it [12]. We conclude this paper by showing how to construct a spectral data, consisting of an algebraic curve plus a line bundle with connection over it, from the instanton (Sect. 8), and proving that it coincides with the Hitchin spectral data for the Nahm transformed Higgs bundle (Sect. 10).

In this way, we complete a circle of ideas analogous to Hitchin’s approach to monopoles [10]:

A similar circle of ideas has also been established for periodic monopoles (that is, solutions of Bogomolny equations on $\mathbb{R}^2 \times S^1$) by Cherkis and Kapustin [5]. Similar correspondences are expected to hold for all translation invariant instantons on $\mathbb{R}^4$.

**Note.** This paper presents the combination of the two previous preprints [15] and [16].
2. Extensibility and Asymptotic Behaviour

We now use the extensibility hypothesis to study the compatibility between the instanton connection $A$ and the extended bundle $E \to T \times \mathbb{P}^1$. More precisely, we first want to show that the holomorphic type of the restriction of the extended bundle to the added divisor $T_\infty = T \times \{\infty\}$ is indeed directly determined by the asymptotic behaviour of the instanton connection $A$. Then we argue that the topology of $E$ is fixed by the action ($L^2$-norm) of $A$.

Before that, we must fix an appropriate trivialisation at infinity.

2.1. Gauge fixing at infinity. Let $B_R$ denote a closed ball in $\mathbb{C}$ of radius $R$, and let $V_R$ be its complement. Also, consider the obvious projection $p : T \times V_R \to T$. We shall need the following technical proposition, which follows from the gauge-fixing result established in [2] (see also the appendix in [13]).

**Proposition 1.** If $|FA| \sim O(r^{-2})$, then, for $R$ sufficiently large, there is a gauge over $T \times V_R$ and a constant flat connection $\alpha_{\text{max}}$ on a topologically trivial rank two bundle over the elliptic curve such that:

$$A - p^* \alpha_{\text{max}} = \alpha \sim O(r^{-1} \log r).$$

2.2. Asymptotic states. By general theory, a constant flat connection on a bundle $S \to T$ determines uniquely a holomorphic structure on this bundle. Moreover, $S$ must split, holomorphically, as the sum of two line bundles, i.e. $S = L_{\xi_0} \oplus L_{-\xi_0}$, uniquely up to $\pm 1$. Here, $\pm \xi_0$ are seen as points in $\hat{T}$, the Jacobian of the elliptic curve $T$.

Therefore, by Proposition 1, to each extensible instanton connection we can associate an unique pair of opposite points $\pm \xi_0 \in \hat{T}$. Such points are called the asymptotic states of $A$.

**Lemma 1.** If an extensible instanton connection $A$ has asymptotic states $\pm \xi_0$, then $E|_{T_\infty} = L_{\xi_0} \oplus L_{-\xi_0}$.

**Proof.** Let $V_\infty \subset \mathbb{P}^1$ be a small neighbourhood centred at $\infty \in \mathbb{P}^1$; let $w$ be a coordinate there. We can regard $E|_{T \times V_\infty}$ as a family of rank 2 bundles over $T$, parametrised by $w$. Furthermore, if $\bar{\delta}$ denotes the holomorphic structure on $E$, let $\bar{\delta}_w$ be the holomorphic structure on the restriction $E|_{T_w}$. Clearly, as operators:

$$\lim_{w \to \infty} \bar{\delta}_w = \bar{\delta}_\infty.$$

However, from condition (2) in the definition of extensibility, we know that $\bar{\delta}_w = \bar{\delta}_{A|_{T_w}}$ away from $\infty$. But Proposition 1 tells us that $\bar{\delta}_{A|_{T_w}}$ approaches $\bar{\delta}_\Gamma$ as $w \to \infty$. Therefore, $\bar{\delta}_\infty = \bar{\delta}_\Gamma$, and the lemma follows. $\square$

2.3. The instanton number. Let us now argue that the topological type of $E$ is determined by the action of the instanton connection:

**Lemma 2.** $c_2(E) = \frac{1}{8\pi^2} \int_{T \times \mathbb{C}} |FA|^2$. 

Proof. Again, let $V$ be a small neighbourhood of $\infty \in \mathbb{P}^1$. Let $\Gamma_{\pm \xi_0}$ be the canonical connection on the bundle $L_{\xi_0} \oplus L_{-\xi_0}$ over an elliptic curve and consider the projection $p : T \times V \to T$.

Now consider a connection $A'$ on the extended bundle $\mathcal{E}$ that coincides with $p^* \Gamma_{\pm \xi_0}$ on $T \times V$. Therefore

$$c_2(\mathcal{E}) = \frac{1}{8\pi^2} \int_{T \times \mathbb{P}^1} \text{Tr}(F_{A'} \wedge F_{A'}) = \frac{1}{8\pi^2} \int_{T \times (\mathbb{P}^1 \setminus \{\infty\})} \text{Tr}(F_{A'} \wedge F_{A'})$$

$$= \frac{1}{8\pi^2} \lim_{R \to \infty} \int_{T \times B_R} \text{Tr}(F_{A'} \wedge F_{A'}). \quad (1)$$

On the other hand, we have from Lemma 1 that $A - A' = \alpha$ is a 1-form in $O(r^{-1} \log(r))$. Define the 1-parameter family of connections $A_t = A' + t \cdot \alpha$, so that the corresponding curvatures:

$$F_{A_t} = t \cdot F_{A'} + (1 - t) \cdot F_{A'}' = \left(t - \frac{t^2}{2}\right) \cdot \alpha \wedge \alpha$$

$$\implies |F_{A_t}| \sim O(r^{-2} \cdot \log^2 r) \quad \forall t \in [0, 1]. \quad (2)$$

So let:

$$i(A) = \frac{1}{8\pi^2} \int_{T \times \mathbb{C}} \text{Tr}(F_A \wedge F_A) = \frac{1}{8\pi^2} \lim_{R \to \infty} \int_{T \times B_R} \text{Tr}(F_A \wedge F_A). \quad (3)$$

Usual Chern–Weil theory tells us that:

$$c_2(\mathcal{E}) - i(A) = \frac{1}{8\pi^2} \lim_{R \to \infty} \left\{ \int_{T \times B_R} (\text{Tr}(F_{A'} \wedge F_{A'}) - \text{Tr}(F_A \wedge F_A)) \right\}$$

$$= \frac{1}{4\pi^2} \lim_{R \to \infty} \left\{ \int_{T \times B_R} d \left( \int_0^1 \text{Tr}(\alpha \wedge F_{A_t}) \right) \right\}$$

$$= \frac{1}{4\pi^2} \lim_{R \to \infty} \left\{ \int_{T \times S^1_R} \left( \int_0^1 \text{Tr}(\alpha \wedge F_{A_t}) \right) \right\} = 0$$

by our estimates in Proposition 1 and Eq. (2). \(\Box\)

We denote the space of extensible connections with fixed instanton number $k$ and asymptotic states $\xi_0$ by $\mathcal{A}(k, \xi_0)$.

### 2.4. Estimating the Dolbeault operator.

Finally, we need one final lemma that will be useful in the following section, where we develop a Fredholm theory for the Dirac operator coupled to an instanton connection $A \in \mathcal{A}(k, \xi_0)$.

First, note that the bundle $L_{\xi_0} \oplus L_{-\xi_0} \to T$ admits a flat connection with constant coefficients, which we denote by $\partial/\partial \xi_0$. Use the projection $T \times V_R \mathbb{P}^1 \to T$ to pull it back to $T \times V_R$. We show that:

**Lemma 3.** Let $A \in \mathcal{A}(k, \xi_0)$ be any extensible instanton connection. Given $\epsilon > 0$, there is $R$ sufficiently large such that:

$$||\overline{\partial} A - \overline{\partial} \Gamma_{\xi_0}||_{L^2(T \times V_R)} < \epsilon.$$  

**Proof.** Since $\overline{\partial} A - \overline{\partial} \Gamma_{\xi_0}$ is just the $(0, 1)$-part of the 1-form $\alpha = A - \Gamma_{\xi_0}$, the statement is a simple consequence of the gauge-fixing Proposition 1. \(\Box\)
3. Fredholm Theory of the Dirac Operator

We begin by recalling that points in the dual torus $\xi \in \hat{T}$ parametrise the set of flat holomorphic line bundles $L_\xi \to T$. Moreover, such bundles have a natural choice of connection, denoted $i_\xi$, which is consistent with the holomorphic structure; see [14].

In fact, $\hat{T}$ also parametrises the set of flat holomorphic line bundles over $T \times \mathbb{C}$. Using the projection $p_1 : T \times \mathbb{C} \to T$, one obtains the holomorphic line bundle $p_1^*(L_\xi)$ over $T \times \mathbb{C}$, which we shall also denote by $L_\xi$ for simplicity; let $o_\xi$ be the pullback of the flat constant connection on $L_\xi \to T$ described above; clearly, such connection is also flat.

Let $E \to T \times \mathbb{C}$ be a rank 2 bundle provided with an instanton connection $A \in A_{(k,\xi_0)}$. Form the bundle $E \otimes L_\xi$ with the corresponding connection $A_\xi = A \otimes I + I \otimes o_\xi$; since all we have done was to add a flat term to our original instanton, $A_\xi$ is still an instanton on the twisted bundle. We also require $A$ to be irreducible; clearly, its twisted version $A_\xi$ is also irreducible.

Consider now the Dirac operator acting on the bundle $E(\xi) = E \otimes L_\xi$, coupled to the connection $A_\xi$, and its adjoint:

$$D_{A_\xi} : \Gamma(E(\xi) \otimes S^+) \to \Gamma(E(\xi) \otimes S^-),$$

$$D_{A_\xi}^* : \Gamma(E(\xi) \otimes S^-) \to \Gamma(E(\xi) \otimes S^+),$$

where the spaces of sections are provided with norms suitably defined. Since the base manifold is flat and the connection is anti-self-dual, the Weitzenböck formula on $E(\xi) \otimes S^+ \to T \times \mathbb{C}$ is simply:

$$D_{A_\xi}^* D_{A_\xi} = \nabla_{A_\xi} \nabla_{A_\xi}$$

$$\Rightarrow \ ||D_{A_\xi} s||^2 = ||\nabla_{A_\xi} s||^2.$$  (4)

Hence, if $A_\xi$ is irreducible, there are no covariantly constant sections of $E(\xi) \otimes S^+$. This means that the kernel of $D_{A_\xi}$ is trivial. Now, if $D_{A_\xi}$ is a Fredholm operator, then $\ker D_{A_\xi}^*$ (which coincides with $\text{coker} D_{A_\xi}$) is a finite dimensional subspace of $\Gamma(E(\xi) \otimes S^-)$.

In this rather technical but fundamental section, we prove that this is indeed the case:

**Theorem 2.** Given any instanton connection $A \in A_{(k,\xi_0)}$, the Dirac operators:

$$D_{A_\xi}^* : L^2_1(E(\xi) \otimes S^-) \to L^2_1(E(\xi) \otimes S^+)$$

form a smooth family of Fredholm operators parametrised by $\hat{T} \setminus \{\pm \xi_0\}$. Moreover, index $D_{A_\xi}^* = k$, for all $\xi \in \hat{T} \setminus \{\pm \xi_0\}$.

The proof consists of three steps, which we now outline. We first prove that the operators $D_{A_\xi}^* : L^2_1(L_\xi \otimes S^-) \to L^2_1(L_\xi \otimes S^+)$ are invertible for nontrivial $\xi \in \hat{T}$. A
gluing argument then shows that the Dirac operator coupled to a twisted instanton $A_\xi$ is Fredholm if $\xi \neq \xi_0$. To compute its index, we use the Gromov-Lawson Relative Index Theorem [9].

It is important to note here that $D_{A_\xi}$ fails to be Fredholm when $\xi = \pm \xi_0$; the reason will be clear from the proof of the theorem. As we will see, this phenomenon is the source of the singularities that appear in the transformed objects.

3.1. The flat model. Let $L_\xi \to T \times \mathbb{C}$ be the flat line bundle described above, provided with the constant connection $\omega_\xi$. Our starting point to prove Theorem 2 is the following proposition.

**Proposition 2.** For non-trivial $\xi \in \hat{T}$, the coupled Dirac operator

$$D_\xi^* : L^2_1(L_\xi \otimes S^-) \to L^2(L_\xi \otimes S^+)$$

is invertible. Its inverse is denoted by $Q_{\xi}^\infty$.

**Proof.** Let $L_\xi \to T \times \mathbb{C}$ be a flat line bundle as above, provided with the constant connection $\omega_\xi = p^*(-i\xi)$, as described in [14]. Consider the twisted Dirac operator:

$$D_\xi : \Gamma(L_\xi \otimes S^+) \to \Gamma(L_\xi \otimes S^-)$$

and its adjoint $D_\xi^*$.

Since $M = T \times \mathbb{C}$ is a Kähler surface, we have the following decompositions:

$$\begin{align*}
S^+ &= \Lambda^{(0,0)}_M L_\xi \oplus \Lambda^{(0,2)}_M L_\xi \\
S^- &= \Lambda^{(0,1)}_M L_\xi = \Lambda^{(0,1)} T L_\xi \oplus \Lambda^{(0,1)}_C.
\end{align*}$$

With respect to these decompositions, the Dirac operator and its adjoint are given by:

$$D_\xi = \begin{pmatrix}
\frac{1}{2}(\overline{\partial}_\xi^{(z)} - \overline{\partial}_\xi^{(w),*}) \\
\overline{\partial}_\xi^{(w)} - \overline{\partial}_\xi^{(z),*}
\end{pmatrix},
\quad
D_\xi^* = \begin{pmatrix}
-\overline{\partial}_\xi^{(z),*} & -\overline{\partial}_\xi^{(w),*} \\
\overline{\partial}_\xi^{(w)} & \overline{\partial}_\xi^{(z)}
\end{pmatrix},$$

where $\overline{\partial}_\xi^{(z,w)}$ denotes the Dolbeault operator twisted by $\omega_\xi$ along the toroidal $(z)$ and plane $(w)$ complex coordinates, i.e. the components of the covariant derivative. Hence, the coupled Dirac laplacian $\Delta_\xi = D_\xi^* D_\xi$ mapping $\Lambda^{(0,0)}_M L_\xi \oplus \Lambda^{(0,2)}_M L_\xi$ to itself is just:

$$\begin{pmatrix}
\Delta_\xi^{(z)} + \Delta_\xi^{(w)} & 0 \\
0 & \Delta_\xi^{(z)} + \Delta_\xi^{(w)}
\end{pmatrix}.$$
First, we want to solve the homogeneous equation \( \Delta_\xi f = 0 \) for \( f \in \Lambda_M^{(0,0)}(L_\xi) \) and a fixed \( \xi \in \hat{T} \). Now, separate variables, supposing that \( f(z,w) = \psi(z)g(w) \):

\[
\Delta_\xi f = 0 \iff g \Delta_\xi^{(c)} \psi + \psi \Delta_\xi^{(w)} g = 0.
\]

Therefore:

\[
\begin{cases}
(\text{i}) \quad \Delta_\xi^{(c)} \psi = \lambda^2 \psi \\
(\text{ii}) \quad \Delta_\xi^{(w)} g = -\lambda^2 g \Rightarrow (\Delta_\xi^{(w)} + \lambda^2)g = 0,
\end{cases}
\]

where \( \lambda^2 \) are the eigenvalues of the \( \xi \)-twisted laplacian over the torus. They form a discrete, unbounded set \( \{\lambda_n(\xi)\}_{n \in \mathbb{N}} \) of \( \mathbb{R}^+ \), each being a function of the parameter \( \xi \).

Note that since \( H^0(T, L_\xi) = 0 \) for nontrivial \( \xi \in \hat{T} \), we can indeed guarantee that \( \lambda_n(\xi) > 0 \) for all nontrivial \( \xi \). On the other hand, for \( L_\xi = \mathbb{C} \), the laplacian has a 1-dimensional kernel, i.e. one zero eigenvalue.

As usual, we can decompose \( f \) on the eigenstates of \( \Delta_\xi^{(c)} \), i.e.:

\[
f = \sum_n g_n(w)\psi_n(z),
\]

where \( \{\psi_n\} \) is an orthonormal basis for the \( L^2 \) norm on \( \Lambda_M^{(0,0)}(L_\xi) \) of eigenstates with eigenvalues \( \{\lambda_n^2\} \); so, \( \|f\|_{L^2(T \times \mathbb{C})}^2 = \sum_n \|g_n\|_{L^2(T)}^2 \). Moreover:

\[
\Delta_\xi f = \sum_n (\Delta_\xi^{(w)} + \lambda_n^2)g_n \psi_n.
\]

**Proposition 3.** Let \( \rho \in L^2(L_\xi \otimes S^+) \) be compactly supported and suppose that \( \xi \) is nontrivial. Then there is \( f \in L^2(L_\xi \otimes S^+) \) and a constant \( k < \infty \) such that \( \Delta_\xi f = \rho \) and \( \|f\|_{L^2} \leq k \|\rho\|_{L^2} \).

**Proof.** Given (12), solving the equation \( \Delta_\xi f = \rho \) amounts to solve \( (\Delta_\xi^{(w)} + \lambda_n^2)g_n = \rho_n \) for each \( n \), where \( g_n, \rho_n \) are the components of \( g, \rho \) along the eigenspaces of \( \lambda_n^2 \), respectively.

Fix some integer \( n \) and denote by \( F_n \) the fundamental solution of \( (\Delta_\xi^{(w)} + \lambda_n^2)F_n(w) = 0 \). Rescale the plane coordinate \( w' = \lambda_n w \), which transforms the previous equation to \( (\Delta_\xi^{(w)} + 1)F_n(w'_{\lambda_n}) = 0 \). The unique integrable solution for this equation is the Bessel function \( K_0 \) (see below), so that \( F_n(w) = K_0(\lambda_n w) \). Solutions to the non-homogeneous equations will then be given by the convolution:

\[
g_n(w) = \int_{\mathbb{R}^2} F_n(w - x)\rho_n(x)dx\mathrm{d}x
\]

and recall that \( \|g_n\|_{L^2} \leq \|F_n\|_{L^1} \|\rho_n\|_{L^2} \). So, all we need is an estimate for \( \|F_n\|_{L^1} \) which is independent of \( n \).

From the expression above, one sees that each \( F_n \) is integrable if the Bessel function \( K_0 \) is: \( \|F_n\|_{L^1} = \lambda_n^{-2}\|K_0\|_{L^1} \). Let \( \lambda = \min\{\lambda_n\}_{n \in \mathbb{N}} \); therefore, \( \|F_n\|_{L^1} \leq \lambda^{-2}\|K_0\|_{L^1} \); putting \( k = \lambda^{-2}\|K_0\|_{L^1} \) we have \( \|g_n\|_{L^2} \leq k\|\rho_n\|_{L^2} \) for each \( n \). This completes the proof. \( \square \)
Consider the Hilbert space $L^2_2(L_ε \otimes S^\pm)$ obtained by the completion of $\Gamma(L_ε \otimes S^\pm)$ with respect to the norm:

$$||s||_{L^2_2} = ||s||_{L^2} + ||\Delta_ε s||_{L^2}.$$  
(14)

The map $\Delta_ε : L^2_2(L_ε \otimes S^-) \rightarrow L^2(L_ε \otimes S^-)$ is then bounded, for clearly $||\Delta_ε s||_{L^2} \leq ||s||_{L^2}$ if $s \in \Gamma(L_ε \otimes S^-)$. Let $G_ε : L^2(L_ε \otimes S^-) \rightarrow L^2_2(L_ε \otimes S^-)$ be the inverse of $\Delta_ε$ given by Proposition 3. Using the inequality of the proposition, one shows that $G_ε$ is also bounded, if $ξ$ is nontrivial:

$$||G_ε s||_{L^2_2} = ||G_ε s||_{L^2} + ||\Delta_ε G_ε s||_{L^2} = ||G_ε s||_{L^2} + ||s||_{L^2} \leq k||s||_{L^2} + ||s||_{L^2} \leq (k + 1) \cdot ||s||_{L^2}.$$  

Moreover, we also conclude that:

$$||G_ε|| < 1 + \frac{C}{\lambda^2}.$$  
(15)

Hence, $G_ε$ is an invertible operator when acting between the above Hilbert spaces, if $ξ$ is non-trivial.

**Remark 1.** We emphasise the necessity of assuming that $ξ$ is nontrivial. If $ξ = \hat{e}$, then the Eq. (10i) admits one zero eigenvalue; on the other hand, the fundamental solution of $\Delta^{(w)} g = 0$ is essentially log $r$, which is not integrable. It is then impossible to get the estimate of Proposition 3, in other words, the operator $\Delta^{(\hat{e} = \hat{e})}$ fails to be invertible. In addition, the parameter $k$ also depends on $ξ$, and $k \rightarrow \infty$ (i.e. $λ \rightarrow 0$) as $ξ \rightarrow 0$.

Now, define the norms:

$$||s||_{L^2_{k+1}} = ||s||_{L^2} + ||D_ε^* s||_{L^2} \text{ if } s \in \Gamma(L_ε \otimes S^-)$$
$$||s||_{L^2_{k+1}} = ||s||_{L^2} + ||D_ε s||_{L^2} \text{ if } s \in \Gamma(L_ε \otimes S^+).$$  
(16)

and consider the Dirac operators as maps between the following Hilbert spaces, obtained by the completion of $\Gamma(L_ε \otimes S^\pm)$ with respect to the above norms:

$$
\begin{align*}
D_ε^* : & L^2_2(L_ε \otimes S^-) \rightarrow L^2(L_ε \otimes S^-) \\
D_ε : & L^2(L_ε \otimes S^+) \rightarrow L^2_2(L_ε \otimes S^+).
\end{align*}
$$  
(17)

Then $D_ε^*$ is clearly bounded. Furthermore, it has an inverse given by $(D_ε^*)^{-1} = D_ε G_ε : L^2(L_ε \otimes S^+) \rightarrow L^2_2(L_ε \otimes S^-)$, which is also bounded:

$$||D_ε^*^{-1} s||_{L^2_2} = ||(D_ε^*)^{-1} s||_{L^2} + ||D_ε^* (D_ε^*)^{-1} s||_{L^2} = ||D_ε G_ε s||_{L^2} + ||s||_{L^2} = ||D_ε G_ε s||_{L^2} \leq (k + 1) \cdot ||s||_{L^2}.$$  

So, $D_ε^*$ is also Fredholm when acting as in (17), and our proof is complete. To further reference, we shall denote $Q_ε^\infty = (D_ε^*)^{-1}$; note, moreover, that this is a bounded, elliptic, pseudo-differential operator of order $-1$. □
We are left with one point to establish: the integrability of the fundamental solution of \((\Delta + 1)F = 0\) in the plane. Indeed, first note that since the operator \(\Delta + 1\) has polar symmetry, then the fundamental solution \(F\) also has. After imposing this symmetry, we obtain the following ODE, for \(r > 0\):

\[(\Delta + 1)F(r) = 0 \Rightarrow F'' + \frac{1}{r}F' - F = 0.\]

This is a Bessel equation with parameter \(\nu = 0\). Its solutions are linear combinations of the Bessel functions of imaginary argument \(I_0\) and \(K_0\) (see [1], chapter 11). Below are possible integral representations for these functions (see [8]):

\[
K_0(r) = \int_1^\infty e^{-rt}(t^2 - 1)^{-\frac{1}{2}}dt,
\]

\[
I_0(r) = \int_{-1}^1 \cosh(rt)(t^2 - 1)^{-\frac{1}{2}}dt.
\]

It is easy to see that \(I_0(r)\) increases exponentially with \(r\); it is also finite for \(r = 0\). For the purpose of finding a Green’s function for the operator \(\Delta + 1\), this solution can be eliminated.

With the help of a table of integrals, one finds out that \(K_0\) is integrable; indeed, by [8]:

\[
\int_{\mathbb{R}^2} K_0(r)d^2\text{vol} = \int_0^\infty \int_{0}^{2\pi} K_0(r)rdrd\theta = 2\pi \int_0^\infty rK_0(r)dr = 2\pi.
\]

This means that \(\|K_0\|_{L^1} = 2\pi\).

**Proposition 4.** The solution \(f\) of the flat laplacian problem \(\Delta f = \rho\) of Proposition (3) decays exponentially if \(\xi\) is nontrivial, in the sense that there is a real constant \(\lambda > 0\) such that:

\[
\lim_{r \to \infty} e^{\lambda r} \|f\| < \infty.
\]

**Proof.** As \(r \to \infty\), the Bessel function \(K_0\) admits the following asymptotic expansion ([20], p. 202):

\[
K_0(r) \sim \left(\frac{\pi}{2}\right)^\frac{1}{2} e^{-r} \sqrt{r} \left[1 - \frac{1}{8r} + \frac{9}{128r^2} + \cdots\right].
\]

(18)

Now since each \(\rho_n\) has compact support, it follows from (13) that each \(g_n\) will also decay exponentially:

\[
g_n(w) \sim \left(\frac{\pi}{2}\right)^\frac{1}{2} \int_{\Omega} \frac{e^{-\lambda_n|w-x|}}{\sqrt{\lambda_n|w-x|}} \left[1 - \frac{1}{8\lambda_n|w-x|} + \cdots\right] \rho_n(x)dxd\tau,
\]

where \(\Omega\) is the support of \(\rho\). As \(|w| \to \infty\), then also \(|w-x| \sim |w|\) for all \(x \in \Omega\). Therefore,

\[
g_n(w) \sim \left(\frac{\pi}{2}\right)^\frac{1}{2} \frac{e^{-\lambda_n|w|}}{\sqrt{\lambda_n|w|}} \left[1 - \frac{1}{8\lambda_n|w|} + \cdots\right]\int_{\Omega} \rho_n(x)dxd\tau, \quad \text{as } |w| \to \infty.
\]

Choosing \(0 < \lambda < \min\{\lambda_n\}_{n \in \mathbb{N}}\), the statement follows from the eigenspace decomposition of \(f\) (11) and (12). \(\square\)
In particular, note that \((f/w)\) also belongs to \(L^2(L_\xi \otimes S^+)\). Define \(\widetilde{L}^2(L_\xi \otimes S^+)\) as the space of all \(\psi \in \Gamma(L_\xi \otimes S^+)\) such that \(\psi/w\) is square-integrable. The proposition just proved implies that the flat model laplacian acting as follows:
\[
\triangle_\xi : \widetilde{L}^2(L_\xi \otimes S^+) \to L^2(L_\xi \otimes S^+)
\]
is an invertible operator. Since \(\triangle_\xi = D_\xi D_\xi^\ast\), we conclude that:
\[
D_\xi^\ast : \widetilde{L}^2(L_\xi \otimes S^-) \to L^2(L_\xi \otimes S^+)
\]  
(19)
is also invertible.

### 3.2. Completing the proof of Theorem 2.

Let \(K\) denote a closed ball in \(\mathbb{C}\) of sufficiently large radius \(R\); its complement is \(D_R\) defined as above. To show that \(D_\ast\) is Fredholm, first note that the usual elliptic theory for compact manifolds guarantees the existence of a parametrix for \(D_\ast\) inside this compact core \(T \times K\); this is a bounded, elliptic, pseudo-differential operator:
\[
Q^K_{A_\xi} : L^2(E(\xi) \otimes S^+|_{T \times K}) \to L^2(E(\xi) \otimes S^-|_{T \times K})
\]
of order \(-1\).

On the other hand, it follows from Lemma 3 that:
\[
||D_\ast^\xi - (D_\ast^\xi \oplus D_\ast^\xi \oplus D_\ast^\xi \oplus D_\ast^\xi)||^2_{L^2(T \times \mathbb{D}_R)} < 2\epsilon,
\]
where \(\epsilon\) can be made arbitrarily small. Thus, \(D_\ast^\xi|_{T \times D_R}\) is also invertible for sufficiently large \(R \gg 0\), if \(\xi \neq \pm \xi_0\). Denote this inverse by \(Q^\infty_{A_\xi}\); this is also a bounded, elliptic, pseudo-differential operator of order \(-1\).

Now choose \(\beta_1, \beta_2 : \mathbb{C} \to \mathbb{R}\) respectively supported over \(K\) and \(D_R\) and satisfying \(\beta_1^2 + \beta_2^2 = 1\) everywhere. We can patch together our two parametrix \(Q^K_{A_\xi}\) and \(Q^\infty_{A_\xi}\) in the following way:
\[
P_{A_\xi} g = \beta_1 Q^K_{A_\xi}(\beta_1 g) + \beta_2 Q^\infty_{A_\xi}(\beta_2 g).
\]  
(20)
This is the same as restricting the section \(g\) to \(T \times K\) (respectively, \(T \times D_R\)), apply \(Q^K_{A_\xi}\) (\(Q^\infty_{A_\xi}\)) and restricting the result again to \(T \times K\) \((T \times D_R)\). Note that \(P_{A_\xi}\) acts as follows:
\[
P_{A_\xi} : L^2(E(\xi) \otimes S^+) \to L^2(E(\xi) \otimes S^-).
\]

We want to show that this is a parametrix for \(D_\ast^\xi\). In fact, take \(g \in L^2(E(\xi) \otimes S^+)\); then:
\[
D_\ast^\xi P_{A_\xi} g = D_\ast^\xi [\beta_1 Q^K_{A_\xi}(\beta_1 g)] + D_\ast^\xi [\beta_2 Q^\infty_{A_\xi}(\beta_2 g)]
\]
\[
= [\beta_1 D_\ast^\xi Q^K_{A_\xi}(\beta_1 g) + \beta_2 D_\ast^\xi Q^\infty_{A_\xi}(\beta_2 g)] + d\beta_1 Q^K_{A_\xi}(\beta_1 g) + d\beta_2 Q^\infty_{A_\xi}(\beta_2 g).
\]  
(21)
where "\(\cdot\)" means Clifford multiplication.
Since $Q^K_{\mathcal{A}_t}$ is a parametrix for $D^*_{\mathcal{A}_t}$ inside $T \times K$, the first term (inside brackets) equals the identity plus a compact operator $S^K$ acting on $\beta g$. Similarly, in the second term, $Q^\infty_{\mathcal{A}_t}$ is the inverse of the Dirac operator outside $K$. Together, the first two terms form the identity operator plus $S^K$. Hence:

$$(D^*_{\mathcal{A}_t} P_{\mathcal{A}_t} - I)g = S^K g + S^\infty g,$$

where $S^\infty : L^2(E(\xi) \otimes S^+) \to L^2(E(\xi) \otimes S^+)$ is the operator over the brackets in (21). Since $Q^K_{\mathcal{A}_t}$ and $Q^\infty_{\mathcal{A}_t}$ are bounded operators, so is $S^\infty$; we argue that this is also a compact operator.

In fact, let $\tilde{\partial}K$ denote the closure of the support of $d\beta_1$ and $d\beta_2$ (which is an annulus around the boundary of $K$). Consider the diagram:

$$L^2(E(\xi) \otimes S^+) \xrightarrow{s} L^2(E(\xi) \otimes S^+|_{T^2 \times \partial K}) \xrightarrow{i} L^2(E(\xi) \otimes S^+|_{T^2 \times \partial K}).$$

Now, let $\Upsilon \subset L^2(E(\xi) \otimes S^+)$ be a bounded set; since $s$ is a bounded operator, $s(\Upsilon)$ is also bounded. By the Rellich lemma, the map $i$ is a compact inclusion; note that $\partial K$ is a compact subset of the plane. Hence, $i(s(\Upsilon))$ is a relatively compact subset of $L^2(E(\xi) \otimes S^+|_{T^2 \times \partial K})$, and clearly also a relatively compact subset of $L^2(E(\xi) \otimes S^+)$. This means that:

$$S^\infty = i \circ s : L^2(E(\xi) \otimes S^+) \to L^2(E(\xi) \otimes S^+)$$

is a compact operator, as have we claimed. We conclude that:

$$D^*_{\mathcal{A}_t} P_{\mathcal{A}_t} - I = [\text{compact operator}]$$

so that (20) is indeed a parametrix for $D^*_{\mathcal{A}_t}$ if $\xi \neq \pm \xi_0$.

Finally, to compute the index of $D^*_{\mathcal{A}_t}$, we use the Relative Index Theorem of Gromov & Lawson [9] (see also the appendix in [13]). One can show that:

**Lemma 4.** If $A \in \mathcal{A}_{(\xi, \xi_0)}$, then $\text{index}[D^*_{\mathcal{A}_t}] = k$, for all $\xi \neq \pm \xi_0$.

### 3.3. The Green’s operator

Clearly, the Dirac laplacian, with the norms as in (14):

$$\Delta_{\mathcal{A}_t} : L^2(E \otimes L_\xi \otimes S^+) \to L^2(E \otimes L_\xi \otimes S^+)$$

$$\Delta_{\mathcal{A}_t} = D^*_{\mathcal{A}_t} D_{\mathcal{A}_t}$$

is also a Fredholm operator. In particular, by general Fredholm theory, there is a bounded operator $G_{\mathcal{A}_t}$, called the Green’s operator, such that:

$$\Delta_{\mathcal{A}_t} G_{\mathcal{A}_t} = I - H_\xi,$$

where $H_\xi$ is the finite rank orthogonal projection operator:

$$H_\xi : L^2(E \otimes L_\xi \otimes S^+) \to \ker(\Delta_{\mathcal{A}_t}).$$
3.4. Harmonic spinors and cohomology. To conclude this chapter, we want to interpret the harmonic spinors $\psi \in \ker D^*_A$ as some holomorphic object defined in terms of the compactified bundle $E \to T \times \mathbb{P}^1$. Indeed, we aim to establish the following identification:

**Proposition 5.** If $\Lambda$ has nontrivial asymptotic state $\xi_0 \in \hat{T}$ and $k > 0$, then there is an isomorphism $H^1(T \times \mathbb{P}^1, E) \cong \ker D^*_A$.

Note that $\ker D^*_A \subset L^2_1(E \otimes S^-)$, with the norm defined in (6). First, we must show that $H^1(T \times \mathbb{P}^1, E)$ has the correct dimension.

**Vanishing theorem.** Since $\chi(E) = -k$, it is enough to show that the cohomologies of orders 0 and 2 vanish in order to conclude that $h^1(T \times \mathbb{P}^1, \mathcal{O}(E)) = k$.

A holomorphic bundle $E \to T \times \mathbb{P}^1$ is said to be **generically fibrewise semistable** if the restriction $E|_{T_w}$ is semistable \(^1\) for generic $w \in \mathbb{P}^1$ (here, $T_w = T \times \{w\}$). Similarly, $E$ is said to be **fibrewise semistable (regular)** if the restriction $E|_{T_w}$ is semistable (regular) for all $w \in \mathbb{P}^1$. Notice that every instanton bundle is generically fibrewise semistable, since $E|_{T_w}$ is semistable, which is a generic condition.

This observation leads to the desired vanishing result:

**Lemma 5.** If $E$ is an irreducible instanton bundle and $k > 0$, then:

$$h^0(T \times \mathbb{P}^1, E(\xi)) = h^2(T \times \mathbb{P}^1, E(\xi)) = 0, \forall \xi \in \hat{T}.$$

Let $L_{\xi} \to T$ be a flat line bundle as described in [14]; denote:

$$E(\xi) = E \otimes p_1^*L_{\xi} \quad \text{and} \quad \tilde{E}(\xi) = E \otimes p_1^*L_{\xi} \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1).$$

Note that we can regard $p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ as the line bundle corresponding to the divisor $T_{\infty}$.

It follows from the lemma that:

$$h^1(T \times \mathbb{P}^1, E(\xi)) = h^1(T \times \mathbb{P}^1, \tilde{E}(\xi)) = k$$

for every $\xi \in \hat{T}$.

**Proof.** Take $w \in \mathbb{P}^1$ such that $E(\xi)|_{T_w} = L_{\xi_1} \oplus L_{\xi_2}$ for some non-trivial $\xi_1, \xi_2 \in \hat{T}$ and let $V \subset \mathbb{P}^1$ be an open neighbourhood of $w$ such that every point of $V$ satisfies the same condition; the existence of such an open set is guaranteed by the fact that $E$ is generically fibrewise semistable.

Suppose there is a holomorphic section $s \in H^0(M, E(\xi))$; it gives rise to a holomorphic section $s_w$ of $E(\xi)|_{T_w} \to T_w$. On the other hand, we have that $h^0(T, E(\xi)|_{T \times \{w\}}) = 0$, hence $s_w \equiv 0$. Moreover, $s_w \equiv 0$ for all $w \in V$, so that $s$ must vanish identically on the open set $T \times V$, hence vanish everywhere and $h^0(E(\xi)) = 0$. The vanishing of $h^0(E(\xi))$ is proved in the very same way by noting $\tilde{E}(\xi)|_{T_w} \equiv \tilde{E}(\xi)|_{T_w}$ since $p_2^*\mathcal{O}_{\mathbb{P}^1}(1)|_{T_w} = \mathcal{O}$. The vanishing of the $h^2$’s follows from Serre duality and a similar argument for the bundle $E(\xi) \otimes K_{\mathbb{P}^1}$. More precisely, Serre duality implies that:

$$H^2(T \times \mathbb{P}^1, E(\xi)) = H^0(T \times \mathbb{P}^1, E(\xi)^\vee \otimes K_{T \times \mathbb{P}^1})^* = H^0(T \times \mathbb{P}^1, E(\xi)^\vee \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(-2))^*.$$

\(^1\) Recall that every semistable, rank 2 vector bundle over an elliptic with trivial determinant either splits as a sum of flat line bundles or it is the unique nontrivial extension of a flat line bundle of order 2 by itself. Such bundle is regular if it is not the sum of trivial line bundles.
On the other hand, it is easy to see that:

\[ \mathcal{E}(\xi)_{\mid T_w} = (\mathcal{E}(\xi)^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)(-2))_{\mid T_w} \]

so that we can apply the same argument as above to show that

\[ h^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)^\vee \otimes K_{T \times \mathbb{P}^1}) = 0. \]

\( \square \)

**Proof of Proposition 5.** Let \( \{w_i\} \subset \mathbb{P}^1 \) be such that \( H^0(T_{w_i}, \mathcal{E}|_{T_{w_i}}) \) does not vanish. As we argued above, there are only finitely many such points; in fact, it can be shown that there are at most \( k \) such points (see Lemma 2 of [14]). Suppose that \#\( \{w_i\} = p \leq k \); note also that \( \infty \notin \{w_i\} \) if \( \xi_0 \) is nontrivial.

Denote by \( B \) the divisor in \( T \times \mathbb{P}^1 \) consisting of the elliptic curves lying over these points, i.e. \( B = \sum_i T \times \{w_i\} \). Also, denote \( \mathcal{E}(p) = \mathcal{E} \otimes \mathcal{O}_{T \times \mathbb{P}^1}(B) \).

Consider the exact sequence of sheaves:

\[ 0 \to \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E}(p)) \to \mathcal{O}(\mathcal{E}(p)|_B) \to 0 \]

which induces the following sequence of cohomology:

\[ 0 \to H^0(B, \mathcal{E}(p)|_B) \to H^1(T \times \mathbb{P}^1, \mathcal{E}) \to \underbrace{H^1(T \times \mathbb{P}^1, \mathcal{E}(p))}_{\text{dim} = k} \to H^1(B, \mathcal{E}(p)|_B) \to 0 \]

(23)

and note that \( p \leq h^0(B, \mathcal{E}(p)|_B) = h^1(B, \mathcal{E}(p)|_B) \leq 2k \). It follows from (23) that \( h^0(B, \mathcal{E}(p)|_B) = h^1(B, \mathcal{E}(p)|_B) = k \), so that the map \( H^0(B, \mathcal{E}(p)|_B) \to H^1(T \times \mathbb{P}^1, \mathcal{E}) \) is an isomorphism.

This means that each element in \( H^1(T \times \mathbb{P}^1, \mathcal{E}) \) can be represented by a \((0,1)\)-form \( \theta \) supported on tubular neighbourhoods of the fibres \( T \times \{w_i\} \). Pulling \( \theta \) back to \( T \times \mathbb{C} \), we obtain a compactly supported \((0,1)\)-form, which we also denote by \( \theta \), since \( \xi_0 \) is nontrivial.

We want to fashion a solution \( \psi \) of \( D_A^* \psi = 0 \) out of \( \theta \), and within the same cohomology class. In other words, we want to find a section \( s \in L^2(A^0_{\mathcal{E}}) \) such that \( D_A^*(\theta + \overline{\partial}_A s) = 0 \). Since \( D_A^* = \overline{\partial}_A - \partial_A \), this is the same as solving the equation:

\[ \overline{\partial}_A \partial_A s = \Delta_A s = -\overline{\partial}_A \theta \]

for a compactly supported \( \theta \).

In the Fredholm theory for the Dirac operator developed above, we constructed the Green’s operator \( G_A \) of the Dirac laplacian \( \Delta_A \). Thus, we can write \( s = -G_A \overline{\partial}_A \theta \) and \( \psi = \theta - \overline{\partial}_A G_A \overline{\partial}_A \theta = P \theta \), where \( P \) denotes the \( L^2 \) projection \( L^2(E \otimes S^-) \to \ker \overline{\partial}_A \).

We must verify that \( \psi \in L^2(E \otimes S^-) \); it is enough to show that \( \overline{\partial}_A G_A \overline{\partial}_A \theta \) is square-integrable for any compactly supported \((0,1)\)-form \( \theta \). First note that \( \gamma = \overline{\partial}_A \theta \) also has compact support, thus \( s = G_A \gamma \in L^2(A^0_{\mathcal{E}}) \). Therefore, we have:

\[ ||\overline{\partial}_A s||_{L^2}^2 = \langle \overline{\partial}_A s, \overline{\partial}_A s \rangle = \langle \overline{\partial}_A s, (\overline{\partial}_A G_A) \gamma \rangle \]

\[ = \langle (\overline{\partial}_A G_A)^* \overline{\partial}_A s, \gamma \rangle \]

which is finite, since \( \gamma \) is compactly supported. Note the integration by parts made from the first to the second line is justified by the same fact. Therefore, \( \psi \) is indeed a square-integrable solution of \( D_A^* \psi = 0 \).
Finally, to see that the map defined above is injective (hence an isomorphism), let $\theta'$ be another $(0,1)$-form supported around $B$ and within the same cohomology class as $\theta$, so that $\theta - \theta' = \bar{\partial}_A \alpha$. Thus:

\[
\psi - \psi' = (\theta - \bar{\partial}_A G_A \bar{\partial}_A \theta) - (\theta' - \bar{\partial}_A G_A \bar{\partial}_A \theta')
\]
\[
= (\theta - \theta') - \bar{\partial}_A G_A \bar{\partial}_A (\theta - \theta')
\]
\[
= \bar{\partial}_A \alpha - \bar{\partial}_A G_A \bar{\partial}_A \bar{\partial}_A \alpha = \bar{\partial}_A \alpha - \bar{\partial}_A \alpha = 0.
\]

This completes the proof. \qed

4. Nahm Transform of Doubly-Periodic Instantons

Recall that our starting point is a rank two vector bundle $E \to T \times \mathbb{C}$ provided with an instanton connection $A \in \mathcal{A}(k,\xi_0)$, where the instanton number $k$ and the asymptotic state $\xi_0$ are from now on fixed.

Over the punctured Jacobian torus $\hat{T} \setminus \{\pm \xi_0\}$, consider the trivial Hilbert bundle $\hat{H} \to \hat{T} \setminus \{\pm \xi_0\}$ whose fibres are $\hat{H}_\xi = L^2_1(E(\xi) \otimes S^-)$. Taking the $L^2_1$-norm on the fibres, $\hat{H}$ becomes an hermitian bundle. Moreover, call $\hat{d}$ the trivial covariant derivative on $\hat{H}$; such derivative is clearly unitary, hence one can define a holomorphic structure over $\hat{H}$.

Now consider the finite-dimensional sub-bundle $V \hookrightarrow \hat{H}$ over $\hat{T} \setminus \{\pm \xi_0\}$ whose fibres are given by $V_\xi = \ker D^*_A$. Let $i : V \to \hat{H}$ be the natural inclusion and $P : \hat{H} \to V$ the fibrewise orthogonal $L^2_1$ projection; more precisely, $P_\xi = I - D_A \hat{G}_A D^*_A$ for each $\xi \in \hat{T} \setminus \{\pm \xi_0\}$, where $G_A$ denotes the Green’s operator for (22), $I$ is the identity operator. We can define a connection on $V$ via the projection formula:

\[
\nabla_B = P \circ \hat{d} \circ i,
\]

where $B$ is the associated connection form.

Clearly, $V$ inherits the hermitian metric from $\hat{H}$, and $B$ is also unitary with respect to this induced metric. Hence, we can provide $V$ with the holomorphic structure coming from the unitary connection $B$.

Alternatively, $V$ also admits an interpretation in terms of monads, see [6]. The Dirac operator can be unfolded into a family of elliptic complexes parametrised by $\hat{T} \setminus \{\pm \xi_0\}$, namely:

\[
0 \to L^2_1(\Lambda^0 E(\xi)) \xrightarrow{\bar{\partial}_A} L^2_1(\Lambda^{0,1} E(\xi)) \xrightarrow{-\bar{\partial}_A} L^2_1(\Lambda^{0,2} E(\xi)) \to 0
\]

which, of course, are also Fredholm. Moreover, the cohomologies of order 0 and 2 must vanish, by Proposition 5. As in [6], such a holomorphic family defines a holomorphic vector bundle $V \to (\hat{T} \setminus \{\pm \xi_0\})$, with fibres $V_\xi = \ker(D^*_A)$, plus an unitary connection, induced by orthogonal projection, which is compatible with the given holomorphic structure. Such a connection will be denoted by $B$. We will invoke this construction repeatedly throughout this work.
The curvature \( F_B \) of \( B \) is simply given by:

\[
F_B = \nabla_B \nabla_B = P \hat{d}(P \hat{d}).
\]

Explicit formulas for the matrix elements on an arbitrary local trivialisation of \( V \to (\hat{T} \setminus \{\pm \xi_0\}) \) will be useful later on. For instance, pick up an orthonormal frame \[\{\psi_i\}_{n=1}^k\]
over an open set \( U \subset \hat{T} \setminus \{\pm \xi_0\} \). Then, we have that:

\[
(B)_{ij} = \langle \psi_j, \nabla_B \psi_i \rangle = \langle \psi_j, \hat{d} \psi_j \rangle,
\]

\[
(F_B)_{ij} = \langle \psi_j, F_B \psi_i \rangle = \langle \psi_j, P \hat{d}(P \hat{d} \psi_i) \rangle = \langle \psi_j, \hat{d}(P \hat{d} \psi_i) \rangle.
\]

(27)

**Higgs field.** We now define the Higgs field \( \Phi \in \text{End}(V) \otimes K \). Let \( w \) be the complex coordinate of the plane, and \( \psi \in \text{ker} \ D^* A(\xi) \), i.e. for each \( \xi \in \hat{T} \setminus \{\pm \xi_0\} \), \( \psi[\xi] \in \ker D^*_A \).

For a fixed \( \xi' \), the Higgs field will act on \( \psi[\xi'] \) by multiplying this section by the plane coordinate \( w \) and then projecting it back to \( \ker D^* A(\xi) \):

\[
(\Phi(\psi))[\xi'] = 2\sqrt{2}\pi P_\xi' (w \psi[\xi']) d\xi.
\]

(28)

Its conjugate is clearly given by \( (\Phi^*(\psi))[\xi'] = 2\sqrt{2}\pi P_\xi' (\bar{w} \psi[\xi']) d\bar{\xi} \).

There is a subtle analytical point here. The spinors \( \psi \) belong to \( L^2(E(\xi) \otimes S^-) \) but is not necessarily the case that \( w \psi \) also belong to \( L^2(E(\xi) \otimes S^-) \). However, we have the following lemma:

**Lemma 6.** If \( \psi \in \ker D^*_A \) and \( A \) has nontrivial asymptotic state, then \( w \psi \in L^2(E \otimes S^-) \).

**Proof.** The key result here is Proposition 4, and the observation that follows it, in particular the invertibility of the operator (19).

Let \( K \subset T \times \mathbb{C} \) be a compact subset such that \( D^*_A \) is sufficiently close to the flat Dirac operator \( D^*_{\pm \xi_0} \) outside \( K \). Thus, restricted to the complement of \( K \), \( D^*_A \) is invertible acting from \( \tilde{L}^2 \to L^2 \).

Now if \( \psi \in \ker D^*_A \), then \( D^*_A (w \psi) = dw \cdot \psi \in L^2(E \otimes S^+|_{T \times \mathbb{C} \setminus K}) \) and the proposition follows. \( \square \)

Note that the dependence of \( (B, \Phi) \) on the original instanton \( A \) is contained on the \( L^2 \)-projection operator \( P \), i.e. on the \( k \) solutions of \( D^*_A \psi = 0 \). It is easy to see that the finite dimensional space spanned by these \( \psi \) is gauge invariant; moreover, the multiplication by \( w \) also commutes with gauge transformations \( \hat{g} \in \text{Aut}(V) \). Therefore, we have that:

**Proposition 6.** If \( A \) and \( A' \) are gauge equivalent irreducible instantons, then the corresponding pairs \( (B, \Phi) \) and \( (B', \Phi') \) are also gauge equivalent.

A pair \( (B, \Phi) \) is called a Higgs pair on the bundle \( V \to \hat{T} \setminus \{\pm \xi_0\} \) if it satisfies Hitchin’s self-duality equations:

\[
\begin{align*}
(i) & \quad F_B + [\Phi, \Phi^*] = 0 \\
(ii) & \quad \bar{\partial}_B \Phi = 0.
\end{align*}
\]

(29)
Recall from [14] that the unitary connection of the Poincaré line bundle $P \to T \times \hat{T}$ and its corresponding curvature are given by:

$$\omega(z, \xi) = i\pi \cdot \sum_{\mu=1}^{2} (\xi_{\mu} dz_{\mu} - z_{\mu} d\xi_{\mu}) \quad \text{and} \quad \Omega(z, \xi) = 2i\pi \cdot \sum_{\mu=1}^{2} d\xi_{\mu} \wedge dz_{\mu}.$$ 

From Braam & Baal [4], we know that if $s \in \Gamma(E(\xi) \otimes S^{-})$, then:

$$D_{A_{k}}^{*} (\hat{d}s) = [D_{A_{k}}, \hat{d}]s = -\Omega \cdot s,$$

where “$\cdot$” means Clifford multiplication. The local formula for the curvature (27) may now be cast on a more convenient form:

$$(F_{B})_{ij} = \langle \psi_{j}, \hat{d}(P \hat{d}\psi_{i}) \rangle = \langle \psi_{j}, \hat{d}(D_{A_{k}} G_{A_{k}} D_{A_{k}}^{*} \hat{d}\psi_{i}) \rangle$$

$$= \langle -D_{A_{k}}^{*} \hat{d}\psi_{j}, G_{A_{k}} (D_{A_{k}}^{*} \hat{d}\psi_{i}) \rangle = \langle \Omega, \psi_{j}, G_{A_{k}} (\Omega \cdot \psi_{i}) \rangle.$$ 

Since the Clifford multiplication commutes with the Green’s operator, we end up with:

$$(F_{B})_{ij} = -\langle (\Omega \wedge \Omega) \cdot \psi_{i}, G_{A_{k}} \psi_{i} \rangle$$

$$= 8\pi^{2} \langle (dz_{1} \wedge dz_{2}) \cdot \psi_{j}, G_{A_{k}} \psi_{i} \rangle d\xi_{1} \wedge d\xi_{2} \quad (30)$$

Note moreover that the inner product is taken in $L^2(E(\xi) \otimes S^{-})$, integrating out the $(z, w)$ coordinates.

**Theorem 3.** If $A \in \mathcal{A}(k, \xi_{0})$, then the associated pair $(B, \Phi)$ on the dual bundle $V \to \hat{T} \setminus \{\pm \xi_{0}\}$ constructed above satisfies the Hitchin’s equations (29).

**Proof.** Choose an open set $U \in \hat{T} \setminus \{\pm \xi_{0}\}$ and pick up a local orthonormal trivialisation of $V \to \hat{T} \setminus \{\pm \xi_{0}\}$ over $U$, such that the corresponding local frame $\{\psi_{i}\}_{i=1}^{n}$ is parallel at \( \xi \). Recall that $\psi_{i}(\xi) \in \ker D_{A_{k}}^{*}$.

First, we shall look at the second equation of (29), and recall that $\hat{T} \setminus \{\pm \xi_{0}\}$ was given the flat Euclidean metric induced from the quotient. Once a local trivialisation is chosen, the endomorphism $\Phi$ can then be put in matrix form, with matrix elements given by:

$$a_{ij}(\xi) = \langle \psi_{j}(\xi), \Phi[\psi_{i}](\xi) \rangle,$$

where $\langle , \rangle$ is the inner product on $L^2(E(\xi) \otimes S^{-})$, integrating out the $(z, w)$ coordinates. Clearly, $\Phi$ is a holomorphic endomorphism if its matrix elements in a holomorphic trivialisation are holomorphic functions. However:

$$\Phi[\psi_{i}](\xi) = P_{\xi}(w\psi_{i}(\xi))d\overline{\xi} = (I - D_{A_{k}} G_{A_{k}} D_{A_{k}}^{*})(w\psi_{i}(\xi))d\overline{\xi}$$

so that:

$$a_{ij}(\xi) = 2\sqrt{2}\pi \left\{ \langle \psi_{j}(\xi), w\psi_{i}(\xi) \rangle - \langle \psi_{j}(\xi), D_{A_{k}} G_{A_{k}} D_{A_{k}}^{*}(w\psi_{i}(\xi)) \rangle \right\}$$

$$= 2\sqrt{2}\pi \left\{ \langle \psi_{j}(\xi), w\psi_{i}(\xi) \rangle - \langle D_{A_{k}}^{*} \psi_{j}(\xi), G_{A_{k}} D_{A_{k}}^{*}(w\psi_{i}(\xi)) \rangle \right\}$$

$$= 2\sqrt{2}\pi \langle \psi_{j}(\xi), w\psi_{i}(\xi) \rangle.$$
Therefore:
\[
\frac{\partial a_{ij}}{\partial \xi} (\xi) = 2\sqrt{2}\pi \left\{ \langle \partial_B \psi_j, w \psi_i \rangle + \langle \psi_j, \overline{\partial}_B (w \psi_i) \rangle \right\} \\
= 2\sqrt{2}\pi \langle \psi_j, \left( \frac{\partial w}{\partial \xi} \right) \psi_i + \overline{\partial}_B \psi_i \rangle = 0
\]
as \psi_i \ is parallel at \ \xi. \ Since this can be done for all \ \xi \in \hat{T} \setminus \{ \pm \xi_0 \}, \ the \ second \ equation \ is \ satisfied.

Now, we move back to (29(i)). Let us first compute the matrix elements \ ([\Phi, \Phi^*])_{ij}.

Note that:
\[
\begin{align*}
(i) & \quad [D_{A_i}^*, \overline{\Phi}] \psi_i(\xi) = D_{A_i}^* (\overline{\Phi} \psi_i(\xi)) = -d \overline{w} \cdot \psi_i(\xi) \\
(ii) & \quad [D_{A_i}^*, w] \psi_i(\xi) = D_{A_i}^* (w \psi_i(\xi)) = 0,
\end{align*}
\]
where we used the fact that \(D_{A_i} = \overline{D}_{A_i}^{-} - \overline{D}_{A_i}^{+} \).

Recall that for 1-forms \([\Phi, \Phi^*] = \Phi \Phi^* + \Phi^* \Phi \). We compute each term separately:
\[
\begin{align*}
\Phi^* \Phi(\psi_i) & = 8\pi^2 P[\overline{\Phi} P(w \psi_i)]d\xi \wedge d\overline{\xi} \\
& = 8\pi^2 \left[ \overline{\Phi} P(w \psi_i) - D_{A_i} \overline{G}_{A_i} D_{A_i}^* \overline{P}(w \psi_i) \right] d\xi \wedge d\overline{\xi} \\
& = 8\pi^2 \left[ \overline{w} w \psi_i - \overline{w} D_{A_i} \overline{G}_{A_i} D_{A_i}^* (w \psi_i) \right] d\xi \wedge d\overline{\xi} \\
& = 8\pi^2 \left[ \overline{w} w \psi_i - \overline{w} D_{A_i} \overline{G}_{A_i} D_{A_i}^* (w \psi_i) \right] d\xi \wedge d\overline{\xi}.
\end{align*}
\]

The two first terms of \(\Phi^* \Phi\) and \(\Phi^* \Phi\) cancel each other and the third terms will cancel out when we take the inner product with \(\psi_j\). Moreover, the second term of \(\Phi^* \Phi\) is zero by (31(ii)). So we are left with:

\[
([\Phi, \Phi^*])_{ij} = 8\pi^2 \langle \psi_j, [\Phi, \Phi^*] \psi_i \rangle \\
= 8\pi^2 \langle \psi_j, w D_{A_i} \overline{G}_{A_i} D_{A_i}^* (\overline{\Phi} \psi_i) \rangle d\xi \wedge d\overline{\xi} \\
= 8\pi^2 \langle D_{A_i}^* (\overline{\Phi} \psi_j), \overline{G}_{A_i} D_{A_i}^* (\overline{\Phi} \psi_i) \rangle d\xi \wedge d\overline{\xi} \\
= -8\pi^2 \langle d \psi_j, G_{\xi} \psi_i \rangle d\xi \wedge d\overline{\xi} \\
= -4\pi^2 i \langle (d w_1 \wedge d w_2) \cdot \psi_j, G_{\xi} \psi_i \rangle d\xi \wedge d\overline{\xi},
\]

where we have once more used the fact that the Clifford multiplication commutes with the Green’s operator. Summing the final expression above with (30), one gets:
\[(F_B)_{ij} + ([\Phi, \Phi^*])_{ij} = -4\pi^2 i \langle (d z_1 \wedge d z_2 + d w_1 \wedge d w_2) \cdot \psi_j, G_{\xi} \psi_i \rangle d\xi \wedge d\overline{\xi} = 0\]
for the first term of the inner product is zero since it consists of a self-dual form (the Kähler form) acting on a negative spinor. \(\square\)
Clearly, the above result has two weak points: it tells nothing about the behaviour of the Higgs field around the singular points \( \pm \xi_0 \); and it fails to show that the Higgs pairs so obtained are admissible in the sense of [14]. In fact, establishing the first point requires the use of algebraic-geometric methods, and will be taken up in Sect. 5 below. The second point will be clarified in Sect. 6.

5. Holomorphic Version

The vanishing results of Sect. 3.4 put us in position to define the transformed bundle \( \mathcal{V} \rightarrow \hat{T} \). Indeed, consider the following elliptic complex:

\[
0 \rightarrow L^2(\Lambda^0 \mathcal{E}(\xi)) \xrightarrow{\overline{\partial}_A \xi} L^2(\Lambda^1 \mathcal{E}(\xi)) \xrightarrow{-\overline{\partial}_A \xi} L^2(\Lambda^2 \mathcal{E}(\xi)) \rightarrow 0.
\]  

(32)

According to Proposition 5, \( H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) \) is the only nontrivial cohomology of this complex. It then follows that the family of vector spaces given by \( \mathcal{V}_\xi = H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) \) forms a holomorphic vector bundle of rank \( k \) over \( \hat{T} \); denote such holomorphic structure by \( \partial \mathcal{V} \). Note that \( \mathcal{V}_\xi \) is defined even if \( \xi = \pm \xi_0 \). Furthermore, by Proposition 5, \( \mathcal{V}_\xi \big|_{\hat{T} \setminus \{\pm \xi_0\}} \) coincides holomorphically with the dual bundle \( V \) defined on the previous section, i.e.:

\[
(V, \overline{\partial}_V) \big|_{\hat{T} \setminus \{\pm \xi_0\}} \simeq (V, \partial_B).
\]

Moreover, \( V \) comes equipped with a hermitian metric \( h' \), which we want to compare with \( h \), the hermitian metric on \( V \) induced from the monad (26). The key point is a fact we noted before in Lemma 3: given an 1-form \( a \) on \( T \times \mathbb{P}^1 \), its \( L^2 \)-norm with respect to the round metric is always larger than its \( L^2 \)-norm with respect to the flat metric on \( T \times (\mathbb{P}^1 \setminus \{\infty\}) \):

\[
||a||_{L^2_R} > ||a||_{L^2_F}.
\]

Thus, comparing the monads (26) and (32), one sees that \( h \) is bounded above by \( h' \). In particular, the metric \( h \) is bounded at \( \pm \xi_0 \).

We can regard \( \mathcal{V} \) as an index bundle for the family of Dirac operators over \( T \times \mathbb{P}^1 \) parametrised by \( \xi \in \hat{T} \). Hence, its degree can be computed by the Atiyah-Singer index theorem for families. Consider now the bundle \( G = p_{12}^* \mathcal{E} \otimes p_{13}^* \mathcal{P} \) over \( T \times \mathbb{P}^1 \times \hat{T} \), and note that \( G \big|_{T \times \mathbb{P}^1 \times \{\xi\}} = \mathcal{E}(\xi) \). Then we have:

\[
ch(V) = -ch(G) \cdot td(T \times \mathbb{P}^1) / [T \times \mathbb{P}^1] = - \left( 2 + 2c_1(\mathcal{P}) + c_1(\mathcal{P})^2 - c_2(\mathcal{E}) \right) \left( 1 + \frac{1}{2}c_1(\mathbb{P}^1) \right) / [T \times \mathbb{P}^1] = k - \frac{1}{2}c_1(\mathcal{P})^2c_1(\mathbb{P}^1) / [T \times \mathbb{P}^1] = k - 2\hat{t},
\]

where the “−” sign in the first line is needed since \( V \) is formed by the null spaces of the adjoint Dirac operator.

Summing up:

Lemma 7. The dual bundle \( (V, \overline{\partial}_B) \rightarrow \hat{T} \setminus \{\pm \xi_0\} \) admits a holomorphic extension \( V \rightarrow \hat{T} \) of degree \(-2\). Moreover, its hermitian metric \( h \) is bounded above at the punctures \( \pm \xi_0 \).
The determinant line bundle of $\mathcal{V}$ is not fixed, however. In fact, let $t_x : T \times \mathbb{P}^1 \to T \times \mathbb{P}^1$ be the translation of the torus by $x \in T$, acting trivially on $\mathbb{P}^1$, and let $\mathcal{E}' = t_x^{*}\mathcal{E}$. If $\mathcal{V}'$ is the dual bundle associated with $\mathcal{E}'$ then $\mathcal{V}' = \mathcal{V} \otimes L_x$. Indeed:

$$
\mathcal{V}'_{\xi} = H^1(T \times \mathbb{P}^1, \mathcal{E}'(\xi)) = H^1\left(T \times \mathbb{P}^1, p_{12}^{*}(t_x^{*}\mathcal{E}) \otimes p_{13}^{*}\mathcal{P}|_{T \times \mathbb{P}^1 \times \{\xi}\}}\right)
$$

$$
= H^1\left(T \times \mathbb{P}^1, t_x^{*}(p_{12}^{*}\mathcal{E} \otimes p_{13}^{*}\mathcal{P}) \otimes p_{13}^{*}\mathcal{L}_x|_{T \times \mathbb{P}^1 \times \{\xi}\}}\right)
$$

$$
= H^1\left(T \times \mathbb{P}^1, p_{12}^{*}\mathcal{E} \otimes p_{13}^{*}\mathcal{P}|_{T \times \mathbb{P}^1 \times \{\xi}\}}\right) \otimes (L_x)_{\xi}
$$

$$
\Rightarrow \mathcal{V}'_{\xi} = \mathcal{V}_\xi \otimes (L_x)_{\xi}
$$
as a canonical isomorphism for each $\xi \in \hat{T}$. Thus $\mathcal{V}' = \mathcal{V} \otimes L_x$.

Note also that if $B$ is an admissible connection, $\mathcal{V}$ admits no splitting $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{L}$ compatible with $B$ for any flat line bundle $\mathcal{L}$.

**Defining the Higgs field.** The next step is to give a holomorphic description of the Higgs field $\Phi$.

Recall that $h^0(T \times \mathbb{P}^1, p_{2*}\mathcal{O}_\mathbb{P}^1(1)) = 2$, and regarding $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, we can fix two holomorphic sections $s_0, s_\infty \in H^0(\mathbb{P}^1, \mathcal{O}_\mathbb{P}^1(1))$ such that $s_0$ vanishes at $0 \in \mathbb{C}$ and $s_\infty$ vanishes at the point added at infinity. In homogeneous coordinates $((w_1, w_2) \in \mathbb{C}^2 | w_2 \neq 0)$ and $((w_1, w_2) \in \mathbb{C}^2 | w_1 \neq 0)$, we have that, respectively $(w = w_1/w_2)$:

$$
s_0(w) = w, \quad s_0(w) = 1,
$$

$$
s_\infty(w) = 1, \quad s_\infty(w) = \frac{1}{w}.
$$

Let us first consider an alternative definition of the transformed Higgs field. For each $\xi \in \hat{T}$, we define the map:

$$
H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) \times H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) \xrightarrow{\Psi_{\xi}} H^1(T \times \mathbb{P}^1, \tilde{\mathcal{E}}(\xi))
$$

$$
(\alpha, \beta) \mapsto \alpha \otimes s_0 - \beta \otimes s_\infty.
$$

If $(\alpha, \beta) \in \ker\Psi_{\xi}$, we define an endomorphism $\varphi$ of $H^1(T \times \mathbb{P}^1, \tilde{\mathcal{E}}(\xi))$ at the point $\xi \in \hat{T}$ as follows:

$$
\varphi_{\xi}(\alpha) = \beta.
$$

We check that $\varphi$ actually coincides with the Higgs field $\Phi$ we defined in the previous section, up to a multiplicative constant. Note that:

$$
\alpha \otimes s_0 - \beta \otimes s_\infty = 0 \iff \beta = \alpha (\otimes s_0)(\otimes s_\infty)^{-1}.
$$

Moreover, recall that, for any trivialisation of $\mathcal{O}_\mathbb{P}^1(1)$ with local coordinate $w$ on $\mathbb{P}^1$, the quotient $\frac{s_0(w)}{s_\infty(w)} = w$. The claim now follows from the proof of Proposition 5; we denote $\Phi[\xi] = 2\sqrt{2\pi} \cdot \varphi_{\xi}$.

**Proposition 7.** The eigenvalues of the Higgs field $\Phi$ have at most simple poles at $\pm \xi_0$. Moreover, the residues of $\Phi$ are semi-simple and have rank $\leq 2$ if $\xi_0$ is an element of order 2 in the Jacobian of $T$, and rank $\leq 1$ otherwise.
Proof. Suppose \( \alpha(\xi) \) is an eigenvector of \( \Phi_\xi \) with eigenvalue \( \epsilon'(\xi) = 1/\epsilon(\xi) \), i.e.
\[
\Phi_\xi(\alpha(\xi)) = \epsilon'(\xi) \cdot \alpha(\xi) .
\]
Thus,
\[
\alpha(\xi) \otimes s_0 - \epsilon'(\xi) \cdot \alpha(\xi) \otimes s_\infty = 0 . \quad \Rightarrow \quad \alpha(\xi) \otimes (\epsilon(\xi) \cdot s_0 - s_\infty) = 0
\]
Therefore, denoting \( s_\epsilon(\xi) = \epsilon(\xi) \cdot s_0 - s_\infty \), we have that \( \alpha(\xi) \in \ker(\otimes s_\epsilon(\xi)) \).

On the other hand, consider the sheaf sequence:
\[
0 \rightarrow \mathcal{E}(\xi) \otimes s_\epsilon(\xi) \rightarrow \tilde{\mathcal{E}}(\xi) \rightarrow \tilde{\mathcal{E}}(\xi)|_{\mathcal{T}(\xi)} \rightarrow 0 ,
\]
since the section \( s_\epsilon(\xi) \) vanishes at \( \epsilon'(\xi) \). It induces the cohomology sequence:
\[
0 \rightarrow H^0(\mathcal{T}(\xi), \tilde{\mathcal{E}}(\xi)|_{\mathcal{T}(\xi)}) \rightarrow H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi) \otimes s_\epsilon(\xi)) \rightarrow \ldots
\]
so that \( \ker(\otimes s_\epsilon(\xi)) = H^0(\mathcal{T}(\xi), \tilde{\mathcal{E}}(\xi)|_{\mathcal{T}(\xi)}) \) which is non-empty if and only if \( \mathcal{E}(\xi)|_{\mathcal{T}(\xi)} = L_\xi \oplus L_{-\xi} \) or \( F_2 \otimes L_\xi \).

Hence, as \( \xi \) approaches \( \infty \), since \( \tilde{\mathcal{E}}|_{\mathcal{T}(\xi)} = L_\xi \oplus L_{-\xi} \). Moreover, \( s_\epsilon(\xi) \rightarrow s_\infty \), so that:
\[
\lim_{\xi \rightarrow \pm \xi_0} \alpha(\xi) \in \ker(\otimes s_\infty) = H^0(T_\infty, \tilde{\mathcal{E}}(\xi)|_{T_\infty}).
\]

Therefore, we conclude that, if \( \xi_0 \neq -\xi_0 \), then one of the eigenvalues of \( \Phi \) has a simple pole at \( \pm \xi_0 \) since \( h^0(T_\infty, \mathcal{E}(\pm \xi_0)|_{T_\infty}) = 1 \); similarly, if \( \xi_0 = -\xi_0 \), then two of the eigenvalues of \( \Phi \) have a simple poles at \( \xi_0 \).

Note in particular that the images of the residues of \( \Phi \) at \( \pm \xi_0 \) are precisely given by:
\[
H^0(T_\infty, \tilde{\mathcal{E}}(\pm \xi_0)|_{T_\infty}) \subset H^1(T \times \mathbb{P}^1, \mathcal{E}(\pm \xi_0)). \quad \Box
\]

This proposition almost concludes the main task of this paper, namely to construct the inverse of the Nahm transform of [14]. It only remains to be shown that the Nahm transformed Higgs pair is admissible. We must then show how to match the \( SU(2) \) bundle \( \hat{E} \rightarrow T \times \mathbb{C} \) with doubly-periodic instanton \( \hat{A} \) constructed from the transformed Higgs pair \( (B, \Phi) \) as in [14] with the original objects \( A \) and \( E \rightarrow T \times \mathbb{C} \) we started with in the present paper. These tasks are taken up in the following section.

6. Proof of Inversion

So far, we have established that the Nahm transform of a doubly-periodic instanton is the same kind of singular Higgs pair as those we started with in the first part of this series [14].

We must now show that the transform presented here is actually the inverse of the construction of instantons of [14]. More precisely, we show that if we start with a doubly-periodic instanton \( A \), apply the Nahm transform to obtain a Higgs pair \( (B, \Phi) \), then the corresponding doubly-periodic instanton constructed as in [14] is gauge equivalent to the original object.

First, consider the six-dimensional manifold \( T \times \mathbb{C} \times (\hat{T} \setminus \{ \pm \xi_0 \}) \). To shorten notation, we denote \( M_\xi = T \times \mathbb{C} \times \{ \xi \} \) and \( \hat{T}_{(z,w)} = \{ z \} \times \{ w \} \times (\hat{T} \setminus \{ \pm \xi_0 \}) \).
Now take the bundle $\mathcal{G} = p_2^*E \otimes p_1^*P$ over $T \times \mathbb{C} \times (\hat{T} \setminus \{\pm \xi_0\})$; note that $\mathcal{G}|_{M_\xi} = E(\xi)$ and $\mathcal{G}|_{\hat{T}(z, w)} = E_{(z, w)} \otimes L_z$, where $E_{(z, w)}$ denotes a trivial rank 2 bundle over $\hat{T} \setminus \{\pm \xi_0\}$ with the fibres canonically identified with the vector space $E_{(z, w)}$.

$\mathcal{G}$ is clearly holomorphic; we denote by $\overline{\partial}_M$ the action of the associated Dolbeault operator along the $T \times P_1$ direction, and by $\overline{\partial}_{\hat{T}}$ its action along the $\hat{T}$ direction. In particular, $\overline{\partial}_M|_{M_\xi} \equiv \partial A_\xi$.

Let $C_{p,q} = \Lambda^p_0 T \times \mathbb{C} \times (\hat{T} \setminus \{\pm \xi_0\}) \otimes \Lambda^q_0 (\hat{T} \setminus \{\pm \xi_0\})$; in other words, $C_{p,q}$ consists of the $(p + q)$-forms over $T \times \mathbb{C} \times (\hat{T} \setminus \{\pm \xi_0\})$ with values in $\mathcal{G}$ spanned by forms of the shape:

$$s(z, w, \xi)dz_{i_1}d\overline{w}_{i_2}d\xi_{j_1}d\overline{\xi}_{j_2},$$

$i_1, i_2, j_1, j_2 \in \{0, 1\}$ and $i_1 + i_2 = p$, $j_1 + j_2 = q$.

Analytically, we want to regard $C_{p,q}$ as the completion of the set of smooth forms of the shape above with respect to a Sobolev norm described as follows:

$$\left| |s|_{T \times \mathbb{C} \times \{\xi\}} \right| \in L^2_q(\Lambda^{2-q}E(\xi)) \quad \text{for each } \xi \in \hat{T} \setminus \{\pm \xi_0\},$$

$$\left| |s|_{(z, w) \in T \setminus \{\pm \xi_0\}} \right| \in L^2_q(\Lambda^{2-q}L_z) \quad \text{for each } (z, w) \in T \times \mathbb{C}.$$

Now, define the maps:

$$C_{p,0} \xrightarrow{\delta_1} C_{p,1} \xrightarrow{\delta_2} C_{p,2} \quad \text{for } (s_1, s_2) \in \Lambda^0_{T \times \mathbb{C}}(\mathcal{G}) \otimes \left(\Lambda^0_{\hat{T}}(\mathcal{G}) \oplus \Lambda^1_{\hat{T}}(\mathcal{G})\right) \equiv C(p, 1).$$

The inversion result will follow from the analysis of the spectral sequences associated to the following double complex (for the general theory of spectral sequences and double complexes, we refer to [3]):

$$\begin{array}{cccc}
C^{0,2} \xrightarrow{\delta_1} C^{1,2} \xrightarrow{\delta_2} C^{2,2} \\
\uparrow \delta_2 \quad \uparrow \delta_2 \quad \uparrow \delta_2 \\
\xrightarrow{\delta_1} C^{1,1} \xrightarrow{\delta_1} C^{2,0} \\
\xrightarrow{\delta_1} C^{0,1} \xrightarrow{\delta_1} C^{2,0} \\
\end{array}$$

The idea is to compute the total cohomology of the spectral sequence in the two possible different ways and compare the filtrations of the total cohomology.

**Lemma 8.** By first taking the cohomology of the rows, we obtain

$$E^p_{2,q} \rightarrow 0 \quad H^2(C(e, 0)) \rightarrow 0$$

$$q \uparrow \quad 0 \quad H^1(C(e, 0)) \rightarrow 0$$

$$\rightarrow 0 \quad H^0(C(e, 0)) \rightarrow 0.$$
where $H^i(C(e, 0))$ are the cohomology groups of the complex that yields the monad description of the construction of doubly-periodic instantons in [14] (see Proposition 3 there).

Proof. First, note that the rows coincide with the complex (26).

Moreover, we can regard elements in $C^{p,q}$ as $q$-forms over $\hat{T}$ with values in $L_{2-p}(\Lambda^{0,p}G)$. To see this, fix some $\xi' \in \hat{T}$; by (36), $s(z, w, \xi') \in \Lambda^{0,p}G|_{M_{\xi'}}$. So, by varying $\xi'$ we get the interpretation above.

This said, it is clear that the first and second columns of $E_1^{p,q}$ must vanish, since $A$ is irreducible. In the middle column, we get $q$-forms over $\hat{T}$ with values in $\ker(\partial M - \overline{\partial}_M)$, which for a fixed $\xi'$ restricts to $\ker(D^*_M - \overline{\partial}_M)$.

Therefore, after taking the cohomologies of the rows, we are left with:

\[
\begin{array}{ccc}
0 & L^p(\Lambda^{1,1}V) & 0 \\
C_1^{p,q} & \uparrow (\overline{\partial}_B + \Phi) & \\
0 & L^p_1(\Lambda^{1,0}V \oplus \Lambda^{0,1}V) & 0 \\
q \uparrow & (\Phi + \overline{\partial}_B) & \\
0 & L^p_2(\Lambda^0V) & 0.
\end{array}
\]

But this is just the complex that yields the monad description of the construction of doubly-periodic instantons in [14]. The lemma follows after taking the cohomology of the remaining column. \qed

Total cohomology and admissibility. Note that, as we pointed out in the beginning of this section, we still do not know if the Higgs pair $(B, \Phi)$ arising from the instanton $(E, A)$ is admissible or not, i.e. the hypercohomology spaces $\mathbb{H}^0$ and $\mathbb{H}^2$ might be nontrivial. The next lemma deals with this problem.

Lemma 9. The only nontrivial cohomology of the total complex is $H^2(C(p, q))$, which is naturally isomorphic to the fibre $E(e, 0)$.

In particular, this shows that the Higgs pairs $(B, \Phi)$ obtained via Nahm transform on instanton connection $A \in A(k, \xi_0)$ are indeed admissible, see [14].

Proof. First note that we can regard an element in $C^{p,q}$ as a $(0, p)$-form over $\hat{T} \times \mathbb{C}$ with values in $\Lambda^{q1, q2}(\hat{G})$. Since $\hat{G}|_{\hat{T} \times \mathbb{C}} = E(e, w) \otimes L_e$, $\ker M$ and $ker \overline{\partial}_M$ are nontrivial only if $z = e$, the identity element in the group law of $T$. Hence, it is enough to work on a tubular neighbourhood of $\{e\} \times \mathbb{P}^1(\hat{T} \setminus \{\pm \xi_0\})$.

More precisely, we define another double complex (germ $C$)$^{p,q}$, consisting of forms defined on arbitrary neighbourhoods of $\{e\} \times \mathbb{P}^1(\hat{T} \setminus \{\pm \xi_0\})$. Then we have a restriction map $C^{p,q} \to (\text{germ } C)^{p,q}$ commuting with $\overline{\partial}_M, \delta_1$ and $\delta_2$. Such a map also induces an isomorphism between the total cohomologies of $C^{p,q}$ and $(\text{germ } C)^{p,q}$. So we can work with $(\text{germ } C)^{p,q}$ to prove the lemma.
Let $V_e$ be some neighbourhood of $e \in T$. By the Poincaré lemma applied to $\overline{\mathcal{T}}$, we get:

$\Lambda^2_{V_e}(\mathcal{G}) \rightarrow 0$

$(\text{germ } \mathbf{C})_{1}^{p,q} \Lambda^1_{V_e}(\mathcal{G}) \rightarrow 0$

$q \uparrow \Lambda^0_{V_e}(\mathcal{G}) \rightarrow 0$

where $V_e$ denotes a tubular neighbourhood of $N_e = \{e\} \times \mathbb{P}^1 \times (\overline{T} \setminus \{\pm \xi_0\})$.

As in [6] (see pp. 91–92), the complex in the first row is, after restriction, mapped into a Koszul complex over $N_e$:

$\mathcal{O}_{N_e}(\mathcal{G}) \xrightarrow{(w \xi)} \mathcal{O}_{N_e}(\mathcal{G}) \oplus \mathcal{O}_{N_e}(\mathcal{G}) \xrightarrow{(-\xi, z)} \mathcal{O}_{N_e}(\mathcal{G})$

so that:

$(\text{germ } \mathbf{C})_{2}^{p,q} E(e,0) \rightarrow 0$

$q \uparrow 0 \rightarrow 0$

It then follows from Lemmas 8 and 9 that there is a natural isomorphism of vector spaces $\mathcal{I}_{I}: H^1(\mathcal{C}(e,0)) \equiv \tilde{E}(e,0) \rightarrow E(e,0)$, which in principle may depend on the choice of complex structure $I$ on $T \times \mathbb{C}$.

**Matching** ($\tilde{\mathbf{E}}, \tilde{\mathbf{A}}$) **with the original data.** Since the choice of identity element in $T$ and of origin in $\mathbb{C}$ is arbitrary, we can extend $\mathcal{I}_{I}$ to a bundle isomorphism $E \rightarrow \tilde{E}$. More precisely, let $t_{(u,v)}: T \times \mathbb{C} \rightarrow T \times \mathbb{C}$ be the translation map $(z, w) \rightarrow (z + u, w + v)$. Clearly, the connection $t_{(u,v)}^{*}A$ on the pullback bundle $t_{(u,v)}^{*}E$ is also irreducible and $t_{(u,v)}^{*}E(e,0) \equiv E(u,v)$. Computing the total cohomology of the double complex (38) associated to the bundle $t_{(u,v)}^{*}\mathcal{G}$ (where $t_{(u,v)}^{*}$ acts trivially on $\tilde{T}$ coordinate), Lemmas 8 and 9 lead to an isomorphism of vector spaces $H^1(\mathcal{C}(u, v)) \equiv \tilde{E}(u,v) \rightarrow E(u,v)$.

It is clear from the naturality of the constructions that these fibre isomorphisms fit together to define a holomorphic bundle isomorphism $\mathcal{I}_{I}: E \rightarrow \tilde{E}$. In particular, $\mathcal{I}_{I}$ takes the Dolbeault operator $\overline{\partial}_{A}$ of the holomorphic bundle $E \rightarrow T \times \mathbb{C}$ to the Dolbeault operator $\overline{\partial}_{\tilde{A}}$ of the holomorphic bundle $\tilde{E} \rightarrow T \times \mathbb{C}$. It also follows from this observation that the holomorphic extensions $\mathcal{E}$ and $\tilde{\mathcal{E}}$ must be isomorphic as holomorphic vector bundles.

However, such fact still does not guarantee that the connections $A$ and $\tilde{A}$ are gauge-equivalent. This is accomplished if we can show that $\mathcal{I}_{I}$ is actually independent of the choice of complex structure in $T \times \mathbb{C}$. Therefore, the proof of the main theorem 1 is completed by the following proposition:
Proposition 8. The bundle map $\mathcal{I}_I : \tilde{E} \to E$ is independent of the choice of complex structure on $T \times \mathbb{C}$.

Proof. Again, it is sufficient to consider only the fibre over $(e, 0)$. As in [6], the idea is to present an explicit description of $\mathcal{I}_I : \tilde{E}(e, 0) \to E(e, 0)$, and then show that it is Euclidean invariant.

Let $\alpha \in H^1(C(e, 0)) \subset C^{1,1}$. To find $\mathcal{I}_I([\alpha])$ we have to find $\beta \in C^{0,2}$ such that $\bar{\partial}_M \beta = \delta_2 \alpha$. A solution to this equation is provided by the Hodge theory for the $\bar{\partial}_M$ operator:

$$\beta = G_M(\bar{\partial}_M \delta_2 \alpha),$$

where $G_M$ denotes the Green’s operator for $\bar{\partial}_M \bar{\partial}_M$, which can be regarded fibrewise as the family of Green’s operators $G_{\xi} = G_M|_{\tilde{T}_\xi}$ parametrised by $\xi \in (\tilde{T} \setminus \{\pm \xi_0\})$.

In principle, $\beta$ depends on the complex structure $I$ via the operators $\bar{\partial}_M$ and $G_M$. However, by the Weitzenböck formula applied to the bundle $G$, we have:

$$\bar{\partial}_M \bar{\partial}_M = \nabla^* \nabla \nabla^* \nabla,$$

Here, $\nabla_M$ is the covariant derivative in the $T \times \mathbb{C}$ direction on $G$. With this interpretation, $G_M = (\nabla^* \nabla_M)^{-1}$ is seen to be independent of the complex structure $I$; in fact, it is Euclidean invariant.

Now $\beta$ as an element of $C^{1,1}$ has the form $\beta(z, w; \xi) d\xi d\overline{\xi}$, so that the restriction $r(e, 0)(\beta) = \beta|_{\tilde{T}(e, 0)}$ is a $(1, 1)$-form over $\tilde{T} \setminus \{\pm \xi_0\}$ with values in $E(e, 0)$. Take its cohomology class in $H^2(\tilde{T} \setminus \{\pm \xi_0\}, \mathbb{C} \otimes E(e, 0))$, so that:

$$\mathcal{I}_I([\alpha]) = \int_{\tilde{T}(e, 0)} r(e, 0)(\beta)$$

which is the desired explicit description. □

Together with the work done in [14], we have thus proven Theorem 1.

7. Instantons of Higher Rank

One easily realizes that there is nothing really special about rank two bundles; the whole proof could be generalised to higher rank. Indeed, the only point in restricting to the rank two case is to reduce the number of possible vector bundles over an elliptic curve, and avoid a tedious case-by-case study throughout the various stages of the proof.

Before we can state the generalisation of the main theorem 1, we must review our definitions of asymptotic state and irreducibility.

The restriction of the instanton bundle $\mathcal{E} \to T \times \mathbb{P}^1$ to the added divisor $T_\infty$ is a flat $SU(n)$ bundle, i.e.

$$\mathcal{E}|_{T_\infty} = L_{\xi_1} \oplus \cdots \oplus L_{\xi_k}$$

such that $\bigotimes_{i=1}^k L_{\xi_i} = \mathcal{O}_T$.

In other words, $\mathcal{E}|_{T_\infty}$ is determined by a set of points $(\xi_1, \ldots, \xi_j) \in J(T)$ with multiplicities $(m_1, \ldots, m_j)$, and such that $\sum_{i=1}^j m_i \xi_i = 0$. We call such data the generalised asymptotic state.
Moreover, we will say that \((E, A)\) is 1-irreducible if there is no flat line bundle \(E \to T \times \mathbb{C}\) such that \(E\) admits a splitting \(E' \oplus L\) which is compatible with the connection \(A\).

**Theorem 4.** There is a bijective correspondence between the following objects:

- Gauge equivalence classes of 1-irreducible \(SU(n)\)-instantons over \(T \times \mathbb{C}\) with fixed instanton number \(k\) and generalised asymptotic state \((\xi_1, \ldots, \xi_j)\) with multiplicities \((m_1, \ldots, m_j)\);

- Admissible \(U(k)\) solutions of the Hitchin’s equations over the dual torus \(\hat{T}\), such that the Higgs field has at most simple poles at \([\xi_1, \ldots, \xi_j]\); moreover, its residue at \(\xi_j\) is semi-simple and has rank \(\leq m_j\).

8. The Instanton Spectral Data

Our goal now is to construct an algebraic curve \(S \hookrightarrow \hat{T} \times \mathbb{C}\) associated to a doubly-periodic instanton \(A\); let \(E\) be the associated instanton bundle.

Let \(D^*_{A_{\xi}(w)}\) denote the restriction of the coupled Dirac operator \(D_{A_{\xi}}\) to the torus \(T_w\). We define:

\[
S = \{ (\xi, w) \in \hat{T} \times \mathbb{C} \mid \ker\{D^*_{A_{\xi}(w)}\} \neq 0 \}.
\]

Since \(D_{A_{\xi}(w)} = \overline{\partial}_{A_{\xi}}|_{T_w} - \overline{\partial}_{A_{\xi}}|_{T_w}\), it is easy to see that:

\[
\ker\{D^*_{A_{\xi}(w)}\} = H^1(T_w, E(\xi)|_{T_w}) = H^1(T_w, \mathcal{E}(\xi)|_{T_w}).
\]

Note also that \(S\) can be compactified to a curve \(\overline{S} \hookrightarrow \hat{T} \times \mathbb{P}^1\) by adding the two points \((\pm \xi_0, \infty)\) corresponding to the asymptotic states.

Assuming that the instanton bundle \(E\) is fibrewise semistable\(^2\), we conclude that \(\overline{S}\) is a branched double cover of \(\mathbb{P}^1\); the branch points correspond to those \(w \in \mathbb{C}\) such that \(E|_{T_w}\) is an extension of the trivial line bundle by itself.

**Lemma 10.** If \(E\) is fibrewise semistable, the natural projection map \(\pi_1 : \overline{S} \to \hat{T}\) is a branched \(k\)-fold covering map. Furthermore, the projection \(\pi_2 : \overline{S} \to \mathbb{P}^1\) is a branched double covering map with \(4k\) branch points, counted with multiplicity.

It follows that all spectral curves belong to the linear system \([k \cdot [\hat{T}] + 2 \cdot [\mathbb{P}^1]] \subset \hat{T} \times \mathbb{P}^1\). Moreover, if \(E\) is regular, then \(\overline{S}\) is a smooth curve of genus \(g(S) = 2k - 1\), by the Riemann–Hurwitz formula.

**Proof.** The proof of the first statement is a simple application of the Riemann–Roch theorem for the family of Dolbeault operators \(\overline{\partial}_{\xi}\) on \(E(\xi)|_{T_w}\), parametrised by \(\mathbb{P}^1\).

For generic \(w \in \mathbb{P}^1\), \(\dim\ker\overline{\partial}_{\xi} = 0\); this dimension jumps precisely when \(E(\xi)|_{T_w}\) is an extension of the trivial line bundle by itself. Thus the cardinality of \(\pi_1^{-1}(\xi)\) coincides with the number of jumping points (counted with multiplicity).

\(^2\) In general, \(E\) is only generically fibrewise semistable, so that \(\overline{S}\) may contain whole fibres.
From index theory, we know that the number of jumping points is precisely the first Chern class of the index bundle; therefore:

\[
\#(\text{jumping points}) = c_1(\text{Ind}[\mathcal{I}_w]) = \int_{\mathbb{P}^1} c_1(\mathcal{E}(\xi)) \cdot td(K_T)/[T] \\
= -\int_{T \times \mathbb{P}^1} c_2(\mathcal{E}(\xi)) = -k.
\]  

(45)

This shows that \( \mathcal{S} \) is a \( k \)-fold covering of \( \hat{T} \). Since the branch points of the projection \( \pi_2 \) are exactly the pre-images of the elements of order two in \( \hat{T} \), there are \( 4k \) branch points, counted with multiplicity. \( \square \)

**Line bundle with connection.** Let \( \pi_1 : \hat{T} \times \mathbb{P}^1 \to \hat{T} \) and \( \pi_2 : \hat{T} \times \mathbb{P}^1 \to \mathbb{P}^1 \) be the natural projection maps; we will also use \( \pi_1 \) and \( \pi_2 \) to denote the projections \( \mathcal{S} \to \hat{T} \) and \( \mathcal{S} \to \mathbb{P}^1 \).

To each \( s \in \mathcal{S} \), we attach the vector space:

\[
\mathcal{L}_s = \ker \left\{ D_{A_{\pi_1(s)(\pi_2(s))}} \right\} = H^1(T_{\pi_2(s)}, \mathcal{E}(\pi_1(s)|_{T_{\pi_2(s)}}).
\]  

(46)

If \( \mathcal{E} \) is generically fibrewise semistable, then \( \mathcal{L} \) is only a coherent sheaf on the (possibly singular, non-reduced) spectral curve. However, when the instanton bundle is regular \( \mathcal{L} \) becomes a line bundle over the smooth spectral curve.

So now let us assume that \( A \) is a regular doubly-periodic instanton, and consider the bundle \( \pi_1^* \mathcal{H} \to S \). There is a bundle map \( T : \pi_1^* \mathcal{H} \to \mathcal{L} \), which is given by the following composition on each fibre:

\[
L_s^1(\Omega^{0,1} E(\pi_1(s))) \xrightarrow{\rho} \ker \left\{ D_{A_{\pi_1(s)}} \right\} \xrightarrow{\mathcal{L}} \ker \left\{ D_{A_{\pi_1(s)(\pi_2(s))}} \right\}.
\]  

(47)

where \( \rho \) denotes the restriction map. Let \( \iota_{\mathcal{L} \to \mathcal{H}} \) denote the inclusion \( \mathcal{L} \hookrightarrow \pi_1^* \mathcal{H} \), which makes sense in terms of distributions. A connection \( \Gamma \) on the line bundle \( \mathcal{L} \to S \) is defined by:

\[
\nabla_{\Gamma} = T \circ \pi_1^* d \circ \iota_{\mathcal{L} \to \mathcal{H}}.
\]  

(48)

9. Hitchin’s Spectral Data

We now look at the other side of the correspondence in Theorem 1 and review Hitchin’s construction of spectral curves associated to Higgs bundles [12].

Recall that \( V \to \hat{T} \setminus \{ \pm \xi_0 \} \) is a rank \( k \) vector bundle, and \( \Phi \) is an endomorphism valued \((1, 0)\)-form with simple poles at \( \pm \xi_0 \). So, for any fixed \( \xi \in \hat{T} \setminus \{ \pm \xi_0 \} \), \( \Phi(\xi) \) is a \( k \times k \) matrix and one can compute its \( k \) eigenvalues. As we vary \( \xi \), we get a \( k \)-fold covering, possibly branched, of \( \hat{T} \setminus \{ \pm \xi_0 \} \) inside \( \hat{T} \times \mathbb{C} \). This curve of eigenvalues is what we want to define as our Higgs spectral curve; more precisely:

\[
C = \left\{ (\xi, w) \in \hat{T} \times \mathbb{C} \mid \det(\Phi(\xi) - w \cdot 1_k) = 0 \right\}.
\]

(49)

In other words, \( C \) is the set of points \( (\xi, w) \in \hat{T} \times \mathbb{C} \) such that \( w \) is an eigenvalue of the endomorphism \( \Phi(\xi) : V_\xi \to V_\xi \).

Since we are assuming that \( \Phi \) has simple poles at \( \pm \xi_0 \), the curve \( C \hookrightarrow \hat{T} \times \mathbb{C} \) can be compactified to a curve \( \overline{C} \hookrightarrow \hat{T} \times \mathbb{P}^1 \) by adding the points \( (\pm \xi_0, \infty) \).

The following proposition is a familiar fact from the theory of Higgs bundles.
Theorem 5. If \( \xi_0 \neq -\xi_0 \), the spectral curve associated to a generic Higgs bundle \((V, B, \Phi)\) is smooth. Otherwise, if \( \xi_0 = -\xi_0 \), then all spectral curves have a double-point at \((\pm \xi_0, \infty)\), but are generically smooth elsewhere.

Defining the spectral bundle. As before, we will denote the projections \( \overline{C} \to \hat{T} \) and \( C \to \mathbb{P}^1 \) by \( \pi_1 \) and \( \pi_2 \). We define a coherent sheaf \( \mathcal{N} \) on \( \overline{C} \) with stalks given by:

\[
\mathcal{N}_\xi = \text{coker} \{ \Phi[\pi_1(\xi)] - \pi_2(\xi) \cdot \text{Id}_k \}.
\]

i.e. the dual of the \( \pi_2(\xi) \)-eigenspace of \( \Phi[\pi_1(\xi)] \). Generically, one expects the eigenvalues to be distinct, so that \( \mathcal{N} \) becomes a line bundle over the smooth curve \( \overline{C} \).

Assuming that Higgs bundle \((V, B, \Phi)\) is generic, we define a connection \( \Lambda \) on the line bundle \( \mathcal{N} \to C \). First note that \( \mathcal{N} \) is naturally a subbundle of \( \pi_1^* V \); let \( i_{\mathcal{N} \to V} \) be the inclusion and \( E : \pi_1^* V \to \mathcal{N} \) the fibrewise projection. We define:

\[
\nabla_\Lambda = E \circ \pi_1^* \nabla_B \circ i_{\mathcal{N} \to V}.
\]

10. Matching the Spectral Data

We are finally in a position to state and prove the second main result of this paper:

Theorem 5. If \((V, B, \Phi)\) is the Nahm transform of a regular instanton \((E, \Lambda)\), then the instanton spectral data \((\mathcal{E}, \mathcal{L}, \Gamma)\) is equivalent to the Higgs spectral data \((\mathcal{C}, \mathcal{N}, \Lambda)\), in the sense that the curves \( \mathcal{S} \) and \( \mathcal{C} \) coincide pointwise and there is a natural isomorphism \( \mathcal{L} \to \mathcal{N} \) preserving the connections.

Proof. Clearly, both spectral curves already have the points \((\pm \xi_0, \infty)\) in common. So let \( \xi \neq \pm \xi_0 \) and suppose that \( \alpha \) is an eigenvector of \( \Phi[\xi] \) with eigenvalue \( \epsilon < \infty \). In particular, the point \((\xi, \epsilon) \in \hat{T} \times \mathbb{C}\) belongs to the Higgs spectral curve \( \mathcal{C} \). By definition, we have:

\[
\Phi[\xi](\alpha) = \epsilon \cdot \alpha \quad \Rightarrow \quad \alpha \otimes (s_0 - \epsilon \cdot s_\infty) = 0.
\]

Clearly, \( s_\epsilon = s_0 - \epsilon \cdot s_\infty \) is a holomorphic section in \( H^0(\mathbb{P}^1, \mathcal{O}(1)) \) vanishing at \( \epsilon \in \mathbb{P}^1 \setminus \{\infty\} \). Therefore it induces the following exact sequence:

\[
0 \to \mathcal{E}(\xi) \otimes \mathcal{E}(\xi) \to \mathcal{E}(\xi)|_{\mathcal{T}_\epsilon} \to 0
\]

which in turn induces the cohomology sequence:

\[
0 \to H^0(T_{\mathcal{C}}, \mathcal{E}(\xi)|_{\mathcal{T}_\epsilon}) \to H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) \to H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) \to 0.
\]

Now \( H^0(T_{\mathcal{C}}, \mathcal{E}(\xi)|_{\mathcal{T}_\epsilon}) \) is nonempty (since it contains \( \alpha \)), thus \( H^1(T_{\mathcal{C}}, \mathcal{E}(\xi)|_{\mathcal{T}_\epsilon}) = H^1(T_{\mathcal{C}}, \mathcal{E}(\xi)|_{\mathcal{T}_\epsilon}) \) is also nonempty since \( h^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) = h^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) = k \). Therefore, \( \text{ker}(D^*_L(e)) \) is also nonempty since it can be identified with \( H^1(T_{\mathcal{C}}, \mathcal{E}(\xi)|_{\mathcal{T}_\epsilon}) \) (see (44)). Hence \((\xi, \epsilon) \in \hat{T} \times \mathbb{C}\) also belongs to the instanton spectral curve \( \mathcal{S} \). The same argument provides the converse statement. Thus the curves \( \mathcal{S} \) and \( \mathcal{C} \) must coincide pointwise.
It also follows from the cohomology sequence (52) that the dual of the \( \epsilon \)-eigenspace of \( \Phi[\xi] \) is exactly \( H^1(T, \mathcal{E}(\xi)) = H^1(T, \mathcal{E}(\xi)) \). In other words, there are canonical identifications between the fibres \( N(\xi, \epsilon) \) and \( L(\xi, \epsilon) \); therefore, the line bundles are isomorphic.

Finally, let us check that the connection \( \nabla_A \) and \( \nabla_T \) also coincide. Noting that the projection \( E : \pi_1^*V \to \mathcal{N} = \mathcal{L} \) is just the restriction map \( r : \ker \{ D_{A_{\pi_1(\xi)}}^* \mathcal{N} \to \mathcal{L} \} \) on each \( s \in S = C \), it is easy to see that \( T = E \circ \pi_1^*P \). Therefore, we have:

\[
\nabla_T = T \circ \pi_1^*d \circ i_{\mathcal{N}^*} \mathcal{H} = E \circ (\pi_1^*P \circ \pi_1^*d \circ i_{\mathcal{N}^*} \mathcal{V}) \circ i_{\mathcal{N}^*} \mathcal{V} = E \circ \pi_1^*\nabla_B \circ i_{\mathcal{N}^*} \mathcal{V} = \nabla_A .
\]

Remark 2. More generally, the above argument shows that the pairs \((S, \mathcal{L})\) and \((C, \mathcal{N})\) also coincide (i.e. the curves \( S \) and \( C \) coincide pointwise, and \( \mathcal{L} \) and \( \mathcal{N} \) are isomorphic as coherent sheaves) when \( \mathcal{E} \) is fibrewise semistable.

Remark 3. Cherkis and Kapustin used a similar argument to establish the analogous result for periodic monopoles [5]. More precisely, they considered monopoles on \( S^1 \times \mathbb{R}^2 \), so that the Nahm transformed object is a Higgs pair on \( S^1 \times \mathbb{R}^2 \). Each of these objects can be associated to a spectral pair consisting of an algebraic curve on \( \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\}) \) plus a line bundle over it. If the Higgs pair is the Nahm transform of a periodic monopole, Cherkis and Kapustin have shown that both spectral data coincide.

11. Conclusion

11.1. An analytical remark. The attentive reader might have noticed that the assumptions on doubly-periodic instantons used on this paper (namely extensibility) do coincide with the conclusions of the first paper of the series. However, it is important to point out at this stage the small gap remaining between the conclusions of the present paper and the assumptions in [14].

More precisely, we assumed in [14] that the harmonic metric associated with the Higgs pair \((B, \Phi)\) on the bundle \( V \to \hat{T} \setminus \{\pm \xi_0\} \) is non-degenerate along the kernel of the residues of \( \Phi \), and \( h \sim O(r^{1/2}) \) along the image of the residues of \( \Phi \), for some \( 0 \leq \alpha < 1/2 \), in a holomorphic trivialisation of \( V \) over a sufficiently small neighbourhood around \( \pm \xi_0 \).

The gap is closed in [2], where it is shown that the Nahm transformed Higgs pairs here constructed do satisfy the above condition.

The analytical features of extensible doubly-periodic instantons are further studied by Olivier Biquard and the author in [2]. In particular, we show that if \( |F_A| \sim O(r^{-2}) \) then \( A \) is extensible, and the asymptotic behaviour is completely determined. Moreover, we give a deformation theory description of the moduli space of rank two doubly-periodic instanton connections as a hyperkähler manifold of complex dimension \( 4k - 2 \). It is also shown that the Nahm transform is a hyperkähler isometry between the moduli space of doubly-periodic instantons and the moduli space of singular Higgs pairs.

11.2. Relation with Fourier–Mukai transform. The instanton spectral pair \((\mathcal{S}, \mathcal{L})\) could also be constructed via Fourier–Mukai transform in the following way.
Let $F$ be a sheaf on $T \times \mathbb{P}^1$ and consider the diagram:

\[
\begin{array}{ccc}
T \times \hat{T} \times \mathbb{P}^1 & \xrightarrow{\pi} & T \times \mathbb{P}^1 \\
\downarrow \downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow \\
\hat{T} \times \mathbb{P}^1 & \xrightarrow{\hat{\pi}} & T \times \mathbb{P}^1
\end{array}
\]

The Fourier–Mukai transform of $F$ is given by

$$\Psi(F) = R\hat{\pi}_* (\pi^* F \otimes \mathcal{P}),$$

where $\mathcal{P}$ denotes the pullback of the Poincaré bundle from $T \times \hat{T}$ to $T \times \hat{T} \times \mathbb{P}^1$. If $F$ is torsion-free and generically fibrewise semistable, then $\Psi(F)$ is a torsion sheaf on $\hat{T} \times \mathbb{P}^1$.

It is simple to show that if $F$ is locally-free and generically fibrewise semistable (as the instanton bundles considered in this paper are), then $\Psi(F)$ is supported exactly over the spectral curve $\mathcal{S}$ and the restriction of $\Psi(F)$ to its support coincides with $\mathcal{L}$ [18]. Furthermore, $V = \pi_{\mathcal{L}}(R^1\hat{\pi}_* (\pi^* F \otimes \mathcal{P}))$. A more careful study of doubly-periodic instantons from the point of view of its Fourier–Mukai transform is done in [17].

Therefore, the holomorphic version of the Nahm transform can be seen as a Fourier–Mukai transform composed with Hitchin’s correspondence. However, the Nahm transform (and the spectral construction of Sect. 8) also contains some differential-geometric information (i.e. the instanton $A$, the transformed connection $B$, and the spectral connection $\tilde{A}$) in addition to the holomorphic information encoded into the Fourier–Mukai transform.

Of course, such differential-geometric information is usually encoded into the holomorphic data in the form of a stability condition. Such a condition is well-known for Higgs bundles [11]. For doubly-periodic instantons, the appropriate concept of stability for the corresponding instanton bundles is established in [2]. It is less clear, though, what is the stability condition to be imposed on the spectral pairs ($\mathcal{S}$, $\mathcal{L}$); such a question is addressed in [18] in a more general context.

Acknowledgement. This work is part of my Ph.D. project [13], which was funded by CNPq, Brazil. I am grateful to my supervisors, Simon Donaldson and Nigel Hitchin, for their constant support and guidance. I also thank Brian Steer and Olivier Biquard for valuable suggestions in the later stages of this project.

References


Communicated by R. Dijkgraaf