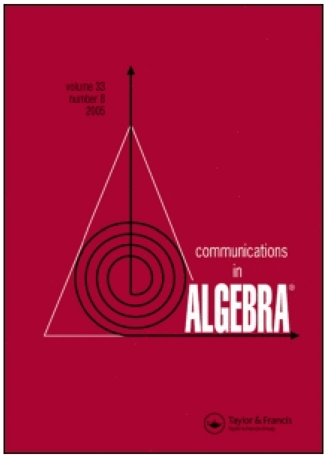

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Marcos Jardim [a]; Rosa M. Miró-Roig [b]
[a] IMECC-UNICAMP, Department of Mathematics, Campinas-SP, Brazil
[b] Faculty of Mathematics, Department of Algebra and Geometry, Barcelona, Spain

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ON THE SEMISTABILITY OF INSTANTON SHEAVES OVER CERTAIN PROJECTIVE VARIETIES

Marcos Jardim[1] **and Rosa M. Miró-Roig**[2]
[1]*IMECC–UNICAMP, Department of Mathematics, Campinas-SP, Brazil*
[2]*Faculty of Mathematics, Department of Algebra and Geometry, Barcelona, Spain*

We show that instanton bundles of rank $r \leq 2n-1$, defined as the cohomology of certain linear monads, on an n-dimensional projective variety with cyclic Picard group are semistable in the sense of Mumford–Takemoto. Furthermore, we show that rank $r \leq n$ linear bundles with nonzero first Chern class over such varieties are stable. We also show that these bounds are sharp.

Key Words: Monads; Semistable sheaves.

2000 Mathematics Subject Classification: 14J60; 14F05.

1. INTRODUCTION

Let X be a nonsingular projective variety of dimension n over an algebraically closed field $\mathbb{F}$ of characteristic zero, and let $\mathscr{L}$ denote a very ample invertible sheaf on X; let $\mathscr{L}^{-1}$ denote its inverse.

Given (finite-dimensional) $\mathbb{F}$-vector spaces V, W, and U, a *linear monad* on X is a complex of sheaves

$$M_\bullet : 0 \to V \otimes \mathscr{L}^{-1} \xrightarrow{\alpha} W \otimes \mathscr{O}_X \xrightarrow{\beta} U \otimes \mathscr{L} \to 0 \tag{1}$$

which is exact on the first and last terms, i.e., $\alpha \in \operatorname{Hom}(V, W) \otimes H^0(\mathscr{L})$ is injective while $\beta \in \operatorname{Hom}(W, U) \otimes H^0(\mathscr{L})$ is surjective. The coherent sheaf $E = \ker \beta / \operatorname{Im} \alpha$ is called the *cohomology of the monad* $M_\bullet$; it is locally-free if and only if $\alpha(x) \in \operatorname{Hom}(V, W)$ is injective for every $x \in X$.

A torsion-free sheaf E on X is said to be a *linear sheaf* on X if it can be represented as the cohomology of a linear monad and it is said to be an *instanton sheaf* on X if in addition it has $c_1(E) = 0$.

Linear monads and instanton sheaves have been extensively studied for the case $X = \mathbb{P}^n$ during the past 30 years. Linear bundles on $\mathbb{P}^2$ and $\mathbb{P}^3$ have been studied since the late 1970s (cf. Barth and Hulek, 1978; Okonek et al., 1980),

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Address correspondence to Rosa M. Miró-Roig, Departamenta de Algebra i Geometria, Universitat de Barcelona, Barcelona 08007, Spain; Fax: +34-4021601; E-mail: miro@ub.edu

motivated by the work of Atiyah, Drinfeld, Hitchin, and Manin who, using the Penrose–Ward transform, showed that rank 2 instanton bundles on $\mathbb{P}^3$ satisfying a reality condition correspond to self-dual Yang–Mills $Sp(1)$-connections on the four-dimensional sphere S^4 (Atiyah et al., 1978). Later, Salamon generalized this to a correspondence between instanton bundles on $\mathbb{P}^{2n+1}$ and quaternionic instanton connections on $\mathbb{HP}^n$ (Mamone Capria and Salamon, 1988). Such generalization motivated Okonek and Spindler (1986) to introduce the notion of mathematical instanton bundles on $\mathbb{P}^{2n+1}$, which are examples of rank $2n$ linear bundles on $\mathbb{P}^{2n+1}$. The existence of instanton bundles on $\mathbb{P}^{2n+1}$ was given by Okonek and Spindler (1986) answering a question posed by Salamon in Mamone Capria and Salamon (1988). Recently, more general instanton sheaves on $\mathbb{P}^n$ were considered in Jardim (2006). The existence of the moduli space $MI_{\mathbb{P}^{2n+1}}(k)$ of mathematical instanton bundles on $\mathbb{P}^{2n+1}$ with charge k was also established by Okonek and Spindler (1986, Theorem 2.6). Determining the irreducibility and smoothness of $MI_{\mathbb{P}^{2n+1}}(k)$ is a long standing question far from being solved, see Coandă et al. (2003) and Jardim (2006) for a survey on the topic.

Spindler and Trautmann (1990) asked whether every locally-free rank $2n$ instanton bundle on $\mathbb{P}^{2n+1}$ is stable in the sense of Mumford–Takemoto. This is known to be true for $n = 1, 2$ and other special cases (cf. Ancona and Ottaviani, 1994), but it is still an open question in general. Indeed, this question also makes sense for instanton sheaves on other projective varieties. In a recent work, Costa and Miró-Roig (2007) have initiated the study of linear monads and locally-free instanton sheaves over smooth quadric hypersurfaces Q_n within $\mathbb{P}^{n+1}$ ($n \geq 3$) (Costa and Miró-Roig, 2007). They have asked whether every such locally free sheaf of rank $n-1$ is stable (in the sense of Mumford–Takemoto) (Costa and Miró-Roig, 2007, Question 5.1). Other authors have also shown interest in the study of monads over more general varieties; for instance, Buchdahl has studied monads over arbitrary blow-ups of $\mathbb{P}^2$ (Buchdahl, 2004), and the authors of Jardim and Martins (2007) considered linear monads over arithmetically Cohen–Macaulay varieties.

The main goal of this article is to address the question of stability of linear bundles in a more general context, showing that locally-free instanton sheaves of rank $r \leq 2n-1$ on an n-dimensional smooth projective variety with cyclic Picard group are semistable, while locally-free linear sheaves of rank $r \leq n$ and $c_1 \neq 0$ on such varieties are stable. Furthermore, we also show that the bounds on the rank are sharp by providing examples of rank $2n$ instanton sheaves and rank $n+1$ linear sheaves on $\mathbb{P}^n$ which are not semistable. Our technique amounts to a generalization of the one used in Ancona and Ottaviani (1994).

We conclude the article by studying the semistability of special sheaves on Q_n, as introduced by Costa and Miró-Roig. Theorem 12 provides a partial answer to Question 5.2 in Costa and Miró-Roig (2007), showing that every rank $r \leq 2n-1$ locally-free special sheaf E on Q_n with $c_1 = 0$ is semistable, while every rank $r \leq n$ locally-free special sheaf on Q_n with $c_1 \neq 0$ is stable.

2. SEMISTABILITY OF INSTANTON SHEAVES

Fixed an ample invertible sheaf $\mathscr{L}$ with $c_1(\mathscr{L}) = \ell$ on a projective variety X of dimension n, recall that the slope $\mu(E)$ with respect to $\mathscr{L}$ of a torsion-free sheaf E

on X is defined as follows:

$$\mu(E) := \frac{c_1(E)\ell^{n-1}}{\mathrm{rk}(E)}.$$

We say that E is *semistable* with respect to $\mathscr{L}$ if for every coherent sheaf $0 \neq F \hookrightarrow E$ we have $\mu(F) \leq \mu(E)$. Furthermore, if for every coherent sheaf $0 \neq F \hookrightarrow E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ we have $\mu(F) < \mu(E)$, then E is said to be stable. A sheaf E is said to be *properly semistable* if it is semistable but not stable. It is also important to recall that E is (semi)stable if and only if E^* is (semi)stable if and only if $E \otimes \mathscr{L}^{\otimes k}$ is (semi)stable.

In the case at hand, note that if E is the cohomology of a linear monad as in (1), then

$$\mathrm{rk}(E) = w - v - u \quad \text{and} \quad c_1(E) = (v - u) \cdot \ell,$$

where $w = \dim W$, $v = \dim V$, $u = \dim U$, and $\ell = c_1(\mathscr{L})$.

A smooth projective variety X is said to be *cyclic* if $\mathrm{Pic}(X) = \mathbb{Z}$. Examples of cyclic varieties are projective spaces, Grassmannians, and complete intersection subvarieties of dimension $n \geq 3$ within P^N, $N \geq 4$. In this case, we can assume without loss of generality that $\mathscr{L} \cong \mathscr{O}_X(l)$ for some $l \geq 1$ and any instanton sheaf E can be represented as the cohomology of a monad of the type

$$0 \to \mathscr{O}_X(-l)^{\oplus c} \xrightarrow{\alpha} \mathscr{O}_X^{\oplus r+2c} \xrightarrow{\beta} \mathscr{O}_X(l)^{\oplus c} \to 0, \tag{2}$$

where r is the rank and c, called the *charge* of E, is its second Chern class.

Remark 1. For $X = \mathbb{P}^n$, instanton sheaves exist for $r \geq n-1$ and all c (Jardim, 2006). For X being a smooth quadric hypersurface of dimension $n \geq 3$, instanton sheaves exist for $r \geq n-1$ and all c (Costa and Miró-Roig, 2007). It would be very interesting to obtain existence results for a wider class of varieties.

Proposition 2. *Every rank 2 torsion-free instanton sheaf on a cyclic variety is semistable.*

Proof. First, take E to be the cohomology of the linear monad (2), let $K = \ker \beta$; it is a locally-free sheaf of rank $r + c$ fitting into the sequences

$$0 \to K(k) \to \mathscr{O}_X(k)^{\oplus r+2c} \xrightarrow{\beta} \mathscr{O}_X(k+l)^{\oplus c} \to 0 \quad \text{and} \tag{3}$$

$$0 \to \mathscr{O}_X(k-l)^{\oplus c} \xrightarrow{\alpha} K(k) \to E(k) \to 0. \tag{4}$$

It follows easily from the Kodaira vanishing theorem and the associated long exact sequences in cohomology that $H^0(E(k)) = H^0(E^*(k)) = 0$ for all $k \leq -1$.

Now let us consider a rank 2 reflexive sheaf F on X with $c_1(F) = 0$ and $H^0(F(-1)) = 0$; we argue that F is semistable. Indeed, if F is not semistable, then any destabilizing sheaf $L \hookrightarrow F$ with torsion-free quotient F/L must be reflexive (see Okonek et al., 1980, p. 158). But every rank 1 reflexive sheaf is locally-free, thus $L = \mathscr{O}_X(d)$ with $d = c_1(L) > 0$ since $\mathrm{Pic}(X) = \mathbb{Z}$. It follows that $H^0(F(-d)) \neq 0$, hence $H^0(F(-1)) \neq 0$ as well.

Now if E is a rank 2 torsion-free sheaf with $c_1(E) = 0$ and $H^0(E^*(-1)) = 0$, then $F = E^*$ is a rank 2 reflexive sheaf with $c_1 F = 0$ and $H^0(F(-1)) = 0$. But we've seen that such F is semistable, hence E is also semistable. Since instanton sheaves do satisfy $H^0(E(-1)) = H^0(E^*(-1)) = 0$, the desired result follows. □

For instanton sheaves of higher rank, we have our first main result.

Theorem 3. *Let E be a rank r instanton sheaf on a cyclic variety X of dimension n. If E is locally-free and $r \leq 2n - 1$, then E is semistable.*

The proof of Theorem 3 is based on a very useful criterion to decide whether a locally-free sheaf on cyclic variety is (semi)stable. First, recall that for any rank r locally-free sheaf E on a cyclic variety X, there is a uniquely determined integer k_E such that $-r + 1 \leq c_1(E(k_E)) \leq 0$. We set $E_{\mathrm{norm}} := E(k_E)$ and we call E *normalized* if $E = E_{\mathrm{norm}}$. We then have the following criterion (Hoppe, 1984, Lemma 2.6).

Proposition 4. *Let E be a rank r locally-free sheaf on a cyclic variety X. If $H^0((\wedge^q E)_{\mathrm{norm}}) = 0$ for $1 \leq q \leq r - 1$, then E is stable. If $H^0((\wedge^q E)_{\mathrm{norm}}(-1)) = 0$ for $1 \leq q \leq r - 1$, then E is semistable.*

Remark 5. Note that if E is a locally-free linear sheaf on X represented as the cohomology of the linear monad

$$M_\bullet : 0 \to (\mathscr{L}^{-1})^{\oplus a} \xrightarrow{\alpha} \mathscr{O}_X^{\oplus b} \xrightarrow{\beta} \mathscr{L}^{\oplus c} \to 0,$$

its dual E^* is also a linear sheaf, being represented as the cohomology of the dual monad

$$M_\bullet^* : 0 \to (\mathscr{L}^{-1})^{\oplus c} \xrightarrow{\alpha} \mathscr{O}_X^{\oplus b} \xrightarrow{\beta} \mathscr{L}^{\oplus a} \to 0.$$

In particular, if E is a locally-free instanton sheaf on X, then its dual E^* is also an instanton. In general, however, there are non-locally-free instanton sheaves whose duals are not instantons; the simplest example of this situation is a non-locally-free nullcorrelation bundle on $\mathbb{P}^3$.

Proof of Theorem 3. We argue that every instanton sheaf on an n-dimensional cyclic variety X satisfying the conditions of the theorem fulfills Hoppe's criterion. Indeed, let E be a rank r locally-free instanton sheaf on X. Assume that E can be represented as the cohomology of the linear monad as in (2). Considering the long exact sequence of symmetric powers associated to the sheaf sequence (3), we have

$$0 \to \wedge^q K(k) \to \wedge^q \big(\mathscr{O}_X^{\oplus r+2c}\big)(k) \to \cdots$$
$$\cdots \to \mathscr{O}_X^{\oplus r+2c}(k) \otimes S^{q-1}\big(\mathscr{O}_X(l)^{\oplus c}\big) \to S^q(\mathscr{O}_X(l)^{\oplus c}) \to 0.$$

Cutting into short exact sequences, passing to cohomology, and using the fact that $H^p(\mathcal{O}_X(k)) = 0$ for $p \leq n-1$ and $k \leq -1$ (Kodaira vanishing theorem), we conclude that

$$H^p(\wedge^q K(k)) = 0 \quad \text{for } 1 \leq q \leq r + c = \text{rk}(K), \qquad p \leq n-1 \quad \text{and} \quad k \leq -pl - 1.$$

Now consider the long exact sequence of exterior powers associated to the sheaf sequence (4) with $k = -1$:

$$\begin{aligned} 0 \to \mathcal{O}_X(-ql-1)^{\oplus\binom{c+q-1}{q}} \to K((-q+1)l-1)^{\oplus\binom{c+q-2}{q-1}} \to \cdots \\ \cdots \to \wedge^{q-1}K(-1-l)^{\oplus c} \to \wedge^q K(-1) \to \wedge^q E(-1) \to 0. \end{aligned} \tag{5}$$

Cutting into short exact sequences and passing to cohomology, we obtain that

$$H^0(\wedge^p E(-1)) = 0 \qquad \text{for } 1 \leq p \leq n-1. \tag{6}$$

If $\text{rk}(E) \leq n$, this proves that E is semistable by Proposition 4. If $\text{rk}(E) = n+1$, we have, since $c_1(E) = 0$ and E is locally-free,

$$H^0(\wedge^n E(-1)) \simeq H^0(E^*(-1)) = 0, \tag{7}$$

thus E is also semistable.

Assume $\text{rk}(E) > n+1$. The dual E^* is also a locally-free instanton sheaf on X, so

$$H^0(\wedge^q(E^*)(-1)) = 0 \qquad \text{for } 1 \leq q \leq n-1. \tag{8}$$

But $\wedge^p(E) \simeq \wedge^{r-p}(E^*)$, since $\det(E) = \mathcal{O}_X$; it follows that:

$$\begin{aligned} H^0(\wedge^p E(-1)) = H^0(\wedge^{r-p}(E^*)(-1)) = 0 \\ \text{for } 1 \leq r-p \leq n-1 \Longrightarrow r-n+1 \leq p \leq r-1. \end{aligned} \tag{9}$$

Together, (8) and (9) imply that if E is a rank $r \leq 2n-1$ locally-free instanton sheaf, then

$$H^0(\wedge^p E(-1)) = 0 \qquad \text{for } 1 \leq p \leq 2n-2,$$

hence E is semistable by Proposition 4. □

On the other hand, we have the following proposition.

Proposition 6. *Let $H = h^0(\mathcal{O}_X(l))$. For $r > (H-2)c$, there are no stable rank r instanton sheaves of charge c on X.*

In particular, for $X = \mathbb{P}^n$ and $l = 1$, it follows that every locally-free instanton sheaf on $\mathbb{P}^n$ of charge 1 and rank r with $n \leq r \leq 2n-1$ must be properly semistable; for $X = Q_n$ and $l = 1$, every locally-free instanton sheaf on Q_n of charge 1 and rank r with $n+1 \leq r \leq 2n-1$ must be properly semistable.

Proof. Note that if E is a stable torsion-free sheaf over a cyclic variety with $c_1(E) = 0$, then $H^0(E) = 0$. Indeed, if $H^0(E) \neq 0$, then there is a map $\mathcal{O}_X \to E$, which contradicts stability.

It follows from the sequences (3) and (4) for $k = 0$ that

$$H^0(E) \simeq H^0(K) \simeq \ker\{H^0\beta : H^0(\mathcal{O}_X^{\oplus r+2c}) \to H^0(\mathcal{O}_X(l)^{\oplus c})\}.$$

If $r > (H-2)c$, then the map $H^0\beta$ cannot be injective, $H^0(E) \neq 0$, and E cannot be stable. □

Now dropping the $c_1(E) = 0$ condition, we obtain the following theorem.

Theorem 7. *Let E be a rank $r \leq n$ linear locally-free sheaf on a cyclic variety X of dimension n. If $c_1(E) \neq 0$, then E is stable.*

Proof. Since E is a linear sheaf, it is represented as the cohomology of a linear monad

$$0 \to \mathcal{O}_X(-l)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \xrightarrow{\beta} \mathcal{O}_X(l)^{\oplus c} \to 0,$$

so that $c_1(E) = (a-c)l$. Assuming $a - c > 0$, we have $\mu(\wedge^q E) = ql(a-c)/(b-a-c) > 0$, hence $(\wedge^q E)_{\text{norm}} = (\wedge^q E)(t)$ for some $t \leq -1$. Arguing as in the proof of Theorem 3, we get

$$H^0((\wedge^q E)(-1)) = 0 \qquad \text{for all } q \leq n-1. \tag{10}$$

Therefore, if E is a rank $r \leq n$ locally-free sheaf represented as the cohomology of a linear monad and $c_1(E) > 0$, then

$$H^0((\wedge^p E)_{\text{norm}}) = 0 \qquad \text{for } 1 \leq p \leq r-1.$$

Hence E is stable by Proposition 4.

Now if E is a locally-free linear sheaf with $c_1(E) < 0$, then E^* is a locally-free linear sheaf with $c_1(E^*) > 0$. By the argument above, E^* is stable; hence E is stable whenever $c_1(E) \neq 0$, as desired. □

We will end this section with two examples illustrating that the upper bounds in rank given by Theorems 3 and 7 are sharp. To establish them, we first need to provide the following useful cohomological characterization of linear sheaves on projective spaces (Jardim, 2006, Proposition 2 and Theorem 3).

Proposition 8. *Let F be a torsion-free sheaf on $\mathbb{P}^n$. F is the cohomology of a linear monad*

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus b} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c} \to 0$$

if and only if the following cohomological conditions hold:

(i) *For* $n \geq 2$, $H^0(F(-1)) = 0$ *and* $H^n(F(-n)) = 0$;
(ii) *For* $n \geq 3$, $H^1(F(k)) = 0$ *for* $k \leq -2$ *and* $H^{n-1}(F(k)) = 0$ *for* $k \geq -n+1$;
(iii) *For* $n \geq 4$, $H^p(F(k)) = 0$ *for* $2 \leq p \leq n-2$ *and all* k.

We are finally ready to construct rank $2n$ locally-free instanton sheaves on $\mathbb{P}^n$ which are not semistable; in other words, the bound $r \leq 2n-1$ in the second part of Theorem 3 is sharp.

Example 9. Let $X = \mathbb{P}^n$, $n \geq 4$. By Fløystad's (2000) theorem, there is a linear monad

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 2} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1) \to 0 \tag{11}$$

whose cohomology F is a locally-free sheaf of rank n on $\mathbb{P}^n$ and $c_1(F) = 1$.

Dualizing, we get a linear monad

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\beta^*} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+3} \xrightarrow{\alpha^*} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2} \to 0$$

whose cohomology is F^*, hence it is a locally-free linear sheaf of rank n on $\mathbb{P}^n$ and $c_1(F^*) = -1$.

Take an extension E of F^* by F

$$0 \to F \to E \to F^* \to 0.$$

Using the cohomological criterion given in Proposition 8, it is easy to see that the extension of linear sheaves is also a linear sheaf. Moreover, $c_1(E) = 0$, i.e., E is a rank $2n$ locally-free instanton sheaf of charge 3 which is not semistable.

Such extensions are classified by $\mathrm{Ext}^1(F^*, F) = H^1(F \otimes F)$. We claim that there are nontrivial extensions of F^* by F. Indeed, we consider the exact sequences

$$0 \to K = \ker(\beta) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1) \to 0, \tag{12}$$

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 2} \to K \to F \to 0 \tag{13}$$

associated to the linear monad (11). We apply the exact covariant functor $\cdot \otimes F$ to the exact sequences (12) and (13) and we obtain the exact sequences

$$0 \to K \otimes F \to F^{\oplus n+3} \to F(1) \to 0,$$

$$0 \to F(-1)^{\oplus 2} \to K \otimes F \to F \otimes F \to 0.$$

Passing to cohomology, we obtain $H^i(K \otimes F) = H^i(F \otimes F) = 0$ for all $i \geq 3$. Hence, $\chi(F \otimes F) = h^0((F \otimes F)) - h^1((F \otimes F)) + h^2((F \otimes F))$. On the other hand,

$$\begin{aligned}\chi(F \otimes F) &= \chi(K \otimes F) - 2\chi(F(-1)) \\ &= (n+3)\chi(F) - \chi(F(1)) - 2\chi(F(-1)) = 8 - \frac{n^2}{2} - \frac{n}{2} < 0, \qquad \text{if} \quad n \geq 4.\end{aligned}$$

Thus if $n \geq 4$, we must have $h^1((F \otimes F)) > 0$, hence there are nontrivial extensions of F^* by F.

For $X = \mathbb{P}^n$, $2 \leq n \leq 3$, arguing as above, we can construct a rank $2n$ locally-free instanton which is not semistable as a nontrivial extension E of F^* by F, where F is a linear sheaf represented as the cohomology of the linear monad

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 4} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+7} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 3} \to 0.$$

To conclude this section, we show that the upper bound in the rank given in Theorem 7 is also sharp.

Example 10. Now let $X = \mathbb{P}^n$, $n \geq 2$. By Fløystad's (2000) theorem, there is a linear monad

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 4} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+9} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 5} \to 0, \tag{14}$$

whose cohomology G is a locally-free sheaf of rank n on $\mathbb{P}^n$ and $c_1(G) = -1$.

Now G^* is the cohomology of the dual monad

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 5} \xrightarrow{\beta^*} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+9} \xrightarrow{\alpha^*} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 4} \to 0.$$

It follows that

$$H^1(G^*) = H^1(\ker \alpha^*) = \operatorname{coker}\{H^0\alpha^* : H^0(\mathcal{O}_{\mathbb{P}^n}^{\oplus n+9}) \to H^0(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 4})\}.$$

Since $n \geq 2$ forces $4n + 4 > n + 9$, the generic map α will have $\operatorname{coker}(H^0\alpha^*) \neq 0$. In other words, there exists a rank n locally-free linear sheaf G on $\mathbb{P}^n$ with $c_1(G) = -1$ and $H^1(G^*) \neq 0$.

Take an extension E of such a linear sheaf G by $\mathcal{O}_{\mathbb{P}^n}$:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to E \to G \to 0. \tag{15}$$

Using the cohomological criterion given in Proposition 8, it is easy to see that E is a rank $n+1$ locally-free linear sheaf with $c_1(E) = c_1(G) = -1$. It is not stable, since $H^0(E) \neq 0$.

Note also that there are nontrivial extensions of G by $\mathcal{O}_{\mathbb{P}^n}$ since $H^1(G^*) \neq 0$. Furthermore, the dual E^* is an example of a rank $n+1$ locally-free linear sheaf with $c_1(E) > 0$ which is not stable.

We do not know how to establish the semistability of non-locally-free instanton sheaves of rank higher than 3. However, for each $n \geq 2$, it is easy to show, using the same technique as in the examples above, that there are unstable torsion-free instanton sheaves of rank $n+1$ in $\mathbb{P}^n$, see Jardim (2006, Example 3). The natural, sharp conjecture would be that every torsion-free instanton sheaf of rank $r \leq n$ on a cyclic variety X of dimension n is semistable; this statement is true for $n = 2$, see Proposition 2 above.

For reflexive linear sheaves, one can construct rank $n+2$ reflexive instanton sheaves which are not semistable; in this case, one can expect that every reflexive instanton sheaf of rank $r \leq n+1$ on a cyclic variety X of dimension n is semistable; this statement is also true for $n=2$, since every reflexive sheaf on surface is locally-free.

Even more generally, a natural conjecture would be that a rank r instanton sheaf which is a kth locally syzygy sheaf will be semistable provided $r \leq k+n-1$ and that this bound is sharp.

3. SPECIAL SHEAVES ON SMOOTH QUADRIC HYPERSURFACES

Now we restrict ourselves to the set-up in Costa and Miró-Roig (2007), and assume that Q_n is a smooth quadric hypersurface within $\mathbb{P}^{n+1}$, $n \geq 3$; such varieties are cyclic.

Recall that a *special sheaf* E on Q_n is defined in Costa and Miró-Roig (2007, Definition 3.4) as either the cohomology of a linear monad

$$0 \to \mathcal{O}_{Q_n}(-1)^{\oplus a} \to \mathcal{O}_{Q_n}^{\oplus b} \to \mathcal{O}_{Q_n}(1)^{\oplus c} \to 0, \tag{M1}$$

or the cohomology of a monad of the type

$$0 \to \Sigma(-1)^{\oplus a} \to \mathcal{O}_{Q_n}^{\oplus b} \to \mathcal{O}_{Q_n}(1)^{\oplus c} \to 0, \qquad \text{if } n \text{ is odd}, \tag{M2.1}$$

$$0 \to \Sigma_1(-1)^{\oplus a_1} \oplus \Sigma_2(-1)^{\oplus a_2} \to \mathcal{O}_{Q_n}^{\oplus b} \to \mathcal{O}_{Q_n}(1)^{\oplus c} \to 0, \qquad \text{if } n \text{ is even}, \tag{M2.2}$$

where Σ is the spinor bundle for n odd, and Σ_1, Σ_2 are the spinor bundles for n even. Clearly, instanton sheaves on Q_n are special sheaves of the first kind with zero degree.

Proposition 11. *Every rank 2 torsion-free special sheaf E on Q_n with $c_1(E)=0$ is semistable.*

Proof. Since every torsion-free special sheaf E on Q_n satisfies $H^0(E(k)) = H^0(E^*(k)) = 0$, simply use the argument in the proof of Proposition 2. □

Finally, for higher rank locally-free special sheaves on Q_n, we have the following theorem.

Theorem 12. *Let E be a rank r locally-free special sheaf on Q_n.*

a) *If $r \leq 2n-1$ and $c_1(E)=0$, then E is semistable.*
b) *If $r \leq n$ and $c_1(E) \neq 0$, then E is stable.*

Proof. For locally-free special sheaves which are represented as cohomologies of the monad (M1), the statement follows from Theorems 3 and 7 and for locally-free special sheaves which are represented as cohomologies of the monad (M2.1) and (M2.2) an analogous argument works. □

Note that using the Fløystad type existence theorem for linear sheaves on Q_n established in Costa and Miró-Roig (2007, Proposition 4.7), and the cohomological characterization of linear bundles on quadric hypersurfaces proved in Jardim and Martins (2007, Theorem 4.4), one can produce examples of rank $2n$ locally-free instanton sheaves on Q_n as well as rank $n+1$ locally-free linear sheaves on Q_n which are not semistable, following the ideas in Examples 9 and 10, showing that the bounds in Theorem 12 are sharp for locally-free special sheaves which are represented as cohomologies of the monad (M1). However, we do not know whether the bounds in the rank are sharp for locally-free sheaves on Q_n which are the cohomology of monads of type (M2.1) and (M2.2). For instance, is there an unstable rank $2n$ locally-free sheaf on Q_n which can be represented as the cohomology of a nonlinear special monad?

4. CONCLUSION

In this article we have studied the semistability of torsion-free sheaves on nonsingular projective varieties with cyclic Picard group that arise as cohomologies of a particular type of monad. Many interesting questions regarding linear monads and instanton sheaves remain unanswered.

First of all, one would like to have a generalizations of Fløystad's (resp. Costa and Miró-Roig's) existence result (Fløystad, 2000 resp. Costa and Miró-Roig, 2007) and of Proposition 8, establishing the existence of instanton sheaves over varieties other than $\mathbb{P}^n$ (resp. Q_n) and their intrinsic cohomological characterization.

The semistability of instanton sheaves of low rank indicate the existence of a well-behaved moduli space of instanton sheaves on cyclic varieties. One approach to study the moduli space of instanton sheaves would be the construction of the moduli space of linear monads, using methods from geometric invariant theory. This task is probably deeply linked with the theory of representation of quivers, since a monad can be regarded as the representation of a quiver, the one whose underlying graph is the Dynkyn diagram for A_3, into the category of sheaves, see King (1994).

This also brings up the question of a reasonable stability condition for monads, meaning compatible with geometric invariant theory, and how does it compare with the slope stability of its cohomology sheaf. Notice that a monad can also be regarded as an element in the derived category $D^b(X)$ of bounded complexes of coherent sheaves on X; the concept of stability on triangulated categories has been recently introduced by Bridgeland (2007), but it is still unclear what it has to do with moduli spaces. We hope that the study of the moduli space of instanton sheaves will shed some light on this topic.

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