

Research Article

Contagious Criminal Career Models Showing Backward Bifurcations: Implications for Crime Control Policies

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Received 7 March 2018; Revised 30 May 2018; Accepted 3 July 2018; Published 1 August 2018

Academic Editor: Zhidong Teng

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We provide a theoretical framework to study how criminal behaviors can be treated as an infectious phenomenon. There are two infectious diseases like models that mimic the role of convicted criminals in contaminating individuals not yet engaged in the criminal career. Equilibrium analyses of each model are studied in detail. The models proposed in this work include the social, economic, personal, and pressure from peers aspects that can, theoretically, determine the probability with which a susceptible individual with criminal propensity engages in a criminal career. These crime-inducing parameters are treated mathematically and their inclusion in the model aims to help policy-makers design crime control strategies. We propose, to the best of our knowledge by the first time in quantitative criminology, the existence of thresholds for the stability of crime-endemic equilibrium which are the equivalent to the “basic reproduction number” widely used in the mathematical epidemiology literature. Both models presented the phenomena of backward bifurcation and breaking-point when the contact rates are chosen as bifurcation parameters. The finding of backward bifurcation in both models implies that there is an endemic equilibrium of criminality even when the threshold parameter for contagion is below unit, which, in turn, implies that control strategies are more difficult to achieve considerable impact on crime control.

1. Introduction

There is now substantial support in the specialized literature on economics, sociology, criminology, and social psychology to the attempts to explain how and why an individual's propensity to engage in criminal behavior is influenced by his/her social context [1]. One interesting metaphor is the one that states that criminal behavior is contagious or that individuals can be susceptible to what economists call endogenous effects [2]. According to this effect, the social milieu in which individuals live may change the individual's propensity to engage in that same criminal behavior as their peers [3, 4]. In addition, the individual's criminal behavior can be affected by other attributes of his/her neighbors, like in the “role model” theory [5] or the peer pressure to maintain local order [6]. Moreover, the institutional or other characteristics of neighborhoods, including crime

prevalence, may induce criminal behavior in susceptible individuals [7].

However, in spite of the large theoretical literature on whether or not crime is contagious, the empirical support for this hypothesis is still limited. Some authors (see [8, 9]) reported an excess in the variation in crime rates across areas which cannot be explained only by the variation in standard sociodemographic determinants of criminal behavior. This suggests that social interactions are more important for less serious than more serious crimes. In a famous study, Crane (1991) (see [10]) showed that in the presence of endogenous residential sorting such reports may be biased by the causal effects of environmental and some individual or family characteristics on the selection of the neighborhood. As pointed out by Kling and Ludwig (see [1]), however, even in the absence of the biased selection problem, it would be very difficult to determine which of the theoretical perspectives

above are responsible for any observed neighborhood effects on criminal behavior.

The influence of others' behavior on criminal offences of susceptible individuals is called "behavioral contagion" [11, 12]. Behavioral contagion has been defined as the spread of any attitude or behavior from one individual or group of individuals to another individual or group of individuals throughout a social network of varying structures [11].

Social contagion arises among people interacting in social structures of diverse nature [13–15]. In such interactions information, behavioral innovation, belief, or meme is transmitted in a similar way to infectious diseases spread in groups of susceptible individuals [16]. Contagion occurs when susceptible individuals interact with contagious people in such a way that this interaction results in a new case [17]. In social networks, contact is defined by the communication and/or imitation of influential processes that make transmission potentially effective (see [13] pp. 1288–1289).

The social contagion theory of violence describes the spread of criminal behavior as similar to the spread of infectious diseases [18]. In such a context, the contagious nature of criminal behavior can be understood, described, and analyzed with the tools developed for studying infectious diseases spreading, in particular the use of mathematical models [19]. However, in contrast with directly transmitted diseases, the infectivity of violence does not require direct contact between susceptible and infected individuals.

In this paper, we provide a different approach to the contagious effect of criminal behavior in susceptible individuals. Rather than considering direct contagious effect by the social environment, we consider a contagious effect by criminals already convicted and who control criminal activities from inside prison, a common effect of some countries like Brazil.

The advent of organized crime in Brazilian prisons, especially in the state of São Paulo, and its role in the contagion of criminality to susceptible individuals outside the prisons constitute the object of this article. The gang leaders the Capital's First Command (PCC, Primeiro Comando da Capital) (see [20]) unleashed a series of attacks in May 2006, resulting in deaths, brought cities to a halt, and cornered authorities in charge preventing them from applying law and order and these are the starting as well as reference points taken. In addition, the gang leaders were sustained by an organization maintained by a hierarchical structure of disciplined and obedient employees capable of executing orders without questioning them. Operating from a base of support networks disseminated in distinct mobile points throughout the state, they revealed that they had an able and agile communication system among leaders, followers, and those who took orders, through protected channels barely permeated by external interference by means of cell phones, telephone exchanges, and carrier pigeons [21].

Criminal contagion from inside the prison system of Brazil to outside susceptible individuals with criminal propensity will be considered as an infectious event in a dynamical system context. The models proposed in this work include the social, economic, personal, and pressure from peers aspects that can, theoretically, determine the probability with which a susceptible individual with criminal propensity

engage in a criminal career. These crime-inducing parameters are treated mathematically and their inclusion in the model aims to help policy-makers design crime control strategies.

2. The Models

The models presented here are based on the criminal activity within a population in an effort to understand how the criminal careers change and evolve over time. We assume a population-based approach, similar to those models of the spreading of infections that confer temporary/permanent immunity. By introducing the key epidemiological concept of a threshold from mass action law [22, 23], we illustrate the fundamental relationship between incarceration and recidivism within a population and use it to show how the criminal activity could be controlled to reduce the likelihood of an individual to engage in a criminal career. We compute and analyze this threshold considering the spread of the crime and the dynamics of incarceration and recidivism. Keeping in mind both the threshold and the perspective of criminal dynamics, it is possible to evaluate what programs of rehabilitation or prevention contribute to the reduction of the recidivism and the number of contacts among individuals susceptible to crime, offenders, and ex-offenders.

In epidemiology theory, the core groups are conceptualized as being the individuals in a population who will infect more than one person over the duration of infection. Core groups are recognized as playing a central role in sustaining the infection in a population and interventions targeting these groups are central to an effective prevention response. Building on this, we extend the concept of core groups to the inmate population.

Especially in inmate population, most criminality is found among core groups and the criminality will only become more generalized if the contact spreads throughout other networks. Hence, it is very likely that incarcerated individuals will contact those susceptible individuals who have never been incarcerated but have an intrinsic criminal propensity. The basic problem is to find out when this contact occurs, who regulates contact, what types of contact are feasible and desirable, and what are the effects of contact (or lack thereof) on susceptible individuals.

In this way, the total population size, denoted by $N(t)$, is then characterized by three classes: susceptible (S), incarcerated (C), and desisting (D) individuals. We specify first-time ($i = 1$) and multiple-time ($i = 2, 3, \dots, n$) incarceration by adding subscripts to model variables and parameters.

2.1. The Partially Contagious Criminality Model (PCCM) Formulation. Let the susceptible in core group population be divided into two categories: S_0 , those individuals who have a criminal propensity but have never been incarcerated, not criminally active but susceptible to crime, and S_i , those individuals susceptible to criminal activities who were once incarcerated (S_1) and those susceptible who were multiple-time incarcerated (S_i , $i = 2, 3, \dots, n$) and became criminally active again. Similarly, let the incarcerated population be divided into two categories: C_1 , those who are first-time

incarcerated, and C_i ($i = 2, 3, \dots, n$), those who were multiple-time incarcerated at a given time. Finally, D_0 represents those individuals who have a criminal propensity but desisted from criminal life either by their own or as a result of early interventions when discharged out of prison (desisting offenders/criminal desisters), and D_i ($i = 1, 2, 3, \dots, n$) represents those individuals who were either first-time or multiple-time incarcerated but desisted from criminal behavior (desisting offenders/criminal desisters) due to variety of reasons.

It is assumed that individuals susceptible to the crime S_0 move to either incarcerated (C_1) or desisting (D_0) class. The rate of initial participation in crime (δ_0) at which individuals move from state S_0 (not criminally active) to C_1 (criminally active and incarcerated) is proportional to intrinsic criminal propensity of individuals. It is also assumed that special intervention programmes (γ_1) may change their basic propensity traits and affect the decisions to engage in crime [24]. As a consequence, we should expect that certain interventions discourage participation in crime, resulting in desistance from criminal activity and reintegration back into society. Primary prevention is an attempt to reduce the risk of behaviors that potentially lead to incarceration. This point is one of the great interests here.

The average length of the primary incarceration term is given by $1/\tau_1$; that is, the rate at which inmates move from state C_1 to S_1 (formerly incarcerated but not criminally active, i.e., ex-offenders) is τ_1 . In the same way, the average length of the multiple incarceration term is given by $1/\tau_2$, with $1/\tau_2 > 1/\tau_1$.

It is worth mentioning that the representation of the cycles of criminal dynamics, with a focus on modeling the recidivism process (criminal careers), could be extended including S_i , C_i , and D_i ($i = 2, 3, \dots, n$). However, we do not intend to develop complex models that account for these cycles and we will explore two cycles only ($i = 1, 2$).

In addition, it is assumed that the recidivism prevention takes place during incarceration and after release back into society. Its purpose is to reduce the risk of an individual reoffending and eventually returning to the prison system. However, some individuals may resume criminal activity very soon after being released from prisons depending on having contact with those individuals still incarcerated and criminally active. It should be mentioned that Walsh and Graig (see [25]) indicated that recidivism is also called falling back into a previous criminal behavior.

We also define β_i ($i=1,2$) as the rate of imprisonment, which captures the return to criminal activity of the individuals released from prison, such that β_1 and β_2 describe the rates with which individuals engage into criminal activity, depending on having had any contagious contact with those incarcerated criminals. They are analogous to the effective contact rate in infectious diseases models.

Finally, μ is the natural mortality rate and a_i ($i=1,2$) is the incarceration-related additional deaths rate (inmate mortality rate can be caused by illness, such as AIDS-related, suicide, accidental self-injury, execution, or any other unspecified cause). Moreover, since the model monitors human populations, all parameters are assumed as nonnegative. We also assumed homogeneous population without any differences in

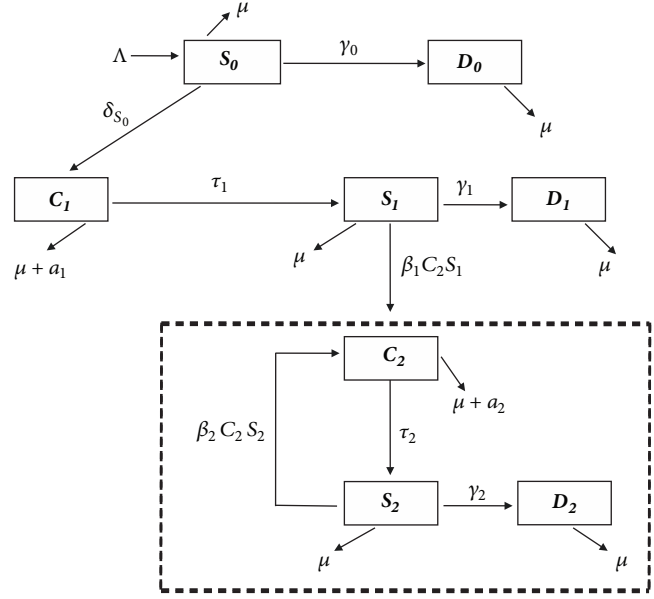


FIGURE 1: The flow diagram for the low-high criminality model (1).

age or in crime type occurrences. In addition, we used the number of incarcerations as a proxy for reoffending.

The flow diagram of the PCCM for two stages is depicted in Figure 1. The variables and parameters are described in Tables 1 and 2, respectively.

Combining the above derivations and assumptions, it follows that the model for transmission dynamics of criminality for two stages is given by the following nonlinear system of differential equations:

$$\begin{aligned}
 \frac{dS_0}{dt} &= \Lambda - (\delta_0 + \gamma_0 + \mu) S_0 \\
 \frac{dD_0}{dt} &= \gamma_0 S_0 - \mu D_0 \\
 \frac{dC_1}{dt} &= \delta_0 S_0 - (\tau_1 + \mu + a_1) C_1 \\
 \frac{dS_1}{dt} &= \tau_1 C_1 - (\gamma_1 + \mu) S_1 - \beta_1 C_2 S_1 \\
 \frac{dD_1}{dt} &= \gamma_1 S_1 - \mu D_1 \\
 \frac{dC_2}{dt} &= (\beta_1 S_1 + \beta_2 S_2) C_2 - (\tau_2 + \mu + a_2) C_2 \\
 \frac{dS_2}{dt} &= \tau_2 C_2 - \beta_2 C_2 S_2 - (\gamma_2 + \mu) S_2 \\
 \frac{dD_2}{dt} &= \gamma_2 S_2 - \mu D_2
 \end{aligned} \tag{1}$$

with generic initial conditions $S_0(0) \geq 0, D_0(0) \geq 0, C_1(0) \geq 0, S_1(0) \geq 0, D_1(0) \geq 0, C_2(0) \geq 0, S_2(0) \geq 0$, and $D_2(0) \geq 0$.

TABLE 1: Models variables and their biological meaning.

Variables	Description
S_0	Individuals not criminally active but susceptible to crime (core group)
C_1	First-time incarcerated individuals
S_1	First-time ex-offenders individuals who are again susceptible to crime
D_i	Individuals who desist from criminal behavior ($i = 0,1,2$) (desisting offenders/criminal desisters)
C_2	Individuals who were two or multiple times incarcerated (recidivists, reoffenders)
S_2	Second-time ex-offenders individuals and susceptible to crime (at least two incarcerations)

TABLE 2: Models parameters and their biological meaning.

Parameters	Description
Λ	Rate of recruitment of individuals into the core group
δ_0	Basic flow to the criminality ($time^{-1}$) (criminal propensity)
β_1	Rate of cooptation of first-time ex-offenders (contact rate between C_2 and S_1)
β_2	Rate of cooptation of second-time ex-offenders (contact rate between C_2 and S_2)
γ_0	Rate of early desistance from crime
γ_1	Rate of desistance from crime when in the first cycle
γ_2	Rate of desistance from crime when in at least the second cycle
μ	Natural inmate mortality rate
a_1	Additional inmate mortality rate when in C_1
a_2	Additional inmate mortality rate when in C_2
τ_1	Release rate from incarceration of first-time offenders
τ_2	Release rate from incarceration of at least second-time offenders

By summing up the above equations, the total population size $N(t)$ is variable with

$$\frac{dN}{dt} = \Lambda - \mu N - a_1 C_1 - a_2 C_2 \tag{2}$$

Thus, in the absence of additional inmate mortalities, that is, $a_1 = a_2 = 0$, the population size evolves as an immigration model with natural mortality, that is, according to $dN/dt = \Lambda - \mu N$. This equation has a single equilibrium $N = N(0) = \Lambda/\mu$ for any initial value of $N(0)$. Thus, in the long run, the population size settles to this constant value. It follows from (2) that $\lim_{t \rightarrow \infty} N(t) \leq \Lambda/\mu = N(0)$.

The differential equation for N implies that solutions of (1), starting in the positive orthant \mathbb{R}_+^8 , either approach, enter, or remain in the subset \mathbb{R}_+^8 defined by

$$\Omega = \left\{ (S_0, D_0, C_1, S_1, D_1, C_2, S_2, D_2) \in \mathbb{R}_+^8 : S_0 + D_0 + C_1 + S_1 + D_1 + C_2 + S_2 + D_2 \leq N(0) \right\}. \tag{3}$$

Thus it suffices to consider solutions in the region Ω . Solutions of the initial value problem starting in Ω and defined by (1) exist and are unique on a maximal interval [26]. Since solutions remain bounded in the positively invariant region Ω , the initial value problem is then both mathematically and epidemiologically well posed [27]. Hence, it is sufficient to consider the dynamics of the flow generated by model (1) in Ω .

2.2. *Analysis of the PCCM Model.* In this section, system (1) is qualitatively analyzed to investigate the existence of its equilibria [28] and the control strategies of its dynamical behavior.

From system (1), with the right-hand side equal to zero, it can be seen from the first five equations that the coordinates of the equilibrium point are given, respectively, by

$$\begin{aligned} S_0 &= \frac{\Lambda}{(\delta_0 + \gamma_0 + \mu)}, \\ D_0 &= \frac{\gamma_0}{\mu} S_0, \\ C_1 &= \frac{\delta_0}{(\tau_1 + \mu + a_1)} S_0, \\ S_1 &= \frac{\tau_1 \delta_0}{(\tau_1 + \mu + a_1)(\mu + \gamma_1 + \beta_1 C_2)} S_0, \\ D_1 &= \frac{\gamma_1}{\mu} S_1. \end{aligned} \tag{4}$$

Moreover, from seventh and eighth equations of system (1), we obtain

$$\begin{aligned} S_2 &= \frac{\tau_2 C_2}{\mu + \gamma_2 + \beta_2 C_2}, \\ D_2 &= \frac{\gamma_2}{\mu} S_2. \end{aligned} \tag{5}$$

From the sixth equation of system (1), one gets

- (i) $C_2 = 0$,
- (ii) $C_2 \neq 0$, which implies $\beta_1 S_1 + \beta_2 S_2 - (\tau_2 + \mu + a_2) = 0$.

If $C_2 = 0$, model (1) has a low-criminality equilibrium $P_l = (S_0, D_0, C_1, S_1, D_1, 0, 0, 0)$ which indicates the existence of offenders who are incarcerated only once in life (C_1), given by

$$\begin{aligned} S_0 &= \frac{\Lambda}{(\delta_0 + \gamma_0 + \mu)}, \\ D_0 &= \frac{\gamma_0}{\mu} S_0, \\ C_1 &= \frac{\delta_0}{(\tau_1 + \mu + a_1)} S_0, \\ S_1 &= \frac{\tau_1 \delta_0}{(\tau_1 + \mu + a_1)(\mu + \gamma_1)} S_0, \\ D_1 &= \frac{\gamma_1}{\mu} S_1. \end{aligned} \tag{6}$$

To determine the stability of this equilibrium, the Jacobian of system (1) is computed and evaluated at P_l . Hence, the low-criminality equilibrium $P_l = (S_0, D_0, C_1, S_1, D_1, 0, 0, 0)$ is locally asymptotically stable if $R_1^* < 1$, where R_1^* , defined as Criminality Reproduction Number (CRN), is given by

$$R_1^* = \frac{\beta_1}{\beta_1^*} \tag{7}$$

with

$$\beta_1^* = \frac{(\tau_1 + \mu + a_1)(\mu + \gamma_1)(\tau_2 + \mu + a_2)}{\tau_1 \delta_0 S_0}. \tag{8}$$

Note that, in analogy to the spread of infectious diseases models [29], the CRN, R_1^* , represents the ‘‘average expected number of new offenders originated by a single persisting offender C_2^* , whilst in a criminal career.’’ In other words, one person C_2^* who was incarcerated two times gets into contact with S_1 susceptible individuals, just released from first imprisonment, successfully and induces R_1^* persons to commit crime. In other words, R_1^* is the average number of individuals who commit crime influenced by one inmate C_2^* .

For $C_2 \neq 0$, that is, for $\beta_1 S_1 + \beta_2 S_2 - (\tau_2 + \mu + a_2) = 0$, replacing both expressions for S_1 given by (4) and S_2 given by (5), an expression for $C_2 = C_2^* > 0$ is obtained as

$$b_2 (C_2^*)^2 + b_1 C_2^* + b_0 = 0, \tag{9}$$

where

$$\begin{aligned} b_2 &= \beta_1 \beta_2 (\mu + a_2), \\ b_1 &= (\mu + \gamma_1)(\tau_2 + \mu + a_2) \\ &\cdot \beta_2 \left\{ \left[\frac{\beta_1 (\mu + \gamma_2)}{\beta_2 (\mu + \gamma_1)} + \frac{(\mu + a_2)}{(\tau_2 + \mu + a_2)} \right] - R_1^* \right\}, \\ b_0 &= (\mu + \gamma_1)(\mu + \gamma_2)(\tau_2 + \mu + a_2)(1 - R_1^*), \end{aligned} \tag{10}$$

Let us now determine the conditions under which the quadratic equation (9) has positive real roots; that is, we search for the existence of multiple equilibria of system (1). However, the expression for the discriminant of the quadratic equation (9) is very complex, so we will analyze the signs of its coefficients to ensure the existence of real solutions. Thus, the conditions under which this equation has either one or two positive real roots can be determined, and these results translated into nontrivial equilibrium of system (1) which is biologically feasible (i.e., positive).

Hence, substituting the positive real solutions of the quadratic equation (9) (i.e., positive values of C_2^*) into the expressions in (5), model (1) has a high-criminality equilibrium, $P_h = (S_0, D_0, C_1, S_1^*, D_1^*, C_2^*, S_2^*, D_2^*)$, where there is coexistence of both offenders C_1 and C_2^* , given by

$$\begin{aligned} S_1^* &= \frac{\tau_1 \delta_0}{(\tau_1 + \mu + a_1) + (\mu + \gamma_1) + \beta_1 C_2^*} S_0 \\ D_1^* &= \frac{\gamma_1}{\mu} S_1^* \\ S_2^* &= \frac{\tau_2 C_2^*}{(\mu + \gamma_2) + \beta_2 C_2^*} \\ D_2^* &= \frac{\gamma_2}{\mu} S_2^*, \end{aligned} \tag{11}$$

with S_0, D_0 , and C_1 given by (6). Thus, the following result is then established.

Theorem 1. *Model (1) has*

- (i) *a unique positive equilibrium P_h if $b_0 < 0 \iff R_1^* > 1$;*
- (ii) *a unique positive equilibrium P_h if $b_0 = 0$ and $b_1 < 0$;*
- (iii) *two positive equilibria, P_h , if $b_0 > 0$ and $b_1 < 0$ and $b_1^2 - 4b_2 b_0 > 0$;*
- (iv) *no positive equilibrium, otherwise.*

Since all model parameters are assumed as nonnegative, it follows from (10) that the coefficient b_2 is always positive, $b_0 < 0$ for $R_1^* > 1$ and $b_0 > 0$ for $R_1^* < 1$. Thus, it is clear from Theorem 1 that model (1) has a unique positive equilibrium, P_h , when $b_0 < 0$, that is, when $R_1^* > 1$ (case (i)).

Now, for $b_0 > 0$ and $R_1^* < 1$, the quadratic equation (9) has two positive solutions if $b_1 < 0$ and $b_1^2 - 4b_2 b_0 > 0$ (case (iii)). Hence, assuming that (9) has two positive real solutions, let C_2^- and C_2^+ be the smaller and higher value of C_2^* , respectively. Translating it into equilibrium of system (1), the question is to address what means the positive high-criminality equilibria P_h for $R_1^* < 1$. It is important to note that, in this case, system (1) can have two equilibria, which are biologically feasible, even though $R_1^* < 1$. This idea is explored more deeply below.

It is instructive at this point to explore some qualitative features for $R_1^* = 1$ and $R_1^* > 1$. Firstly, for $R_1^* = 1$, it follows that $b_0 = 0$; the quadratic equation (9) has either a unique positive root (if $b_1 < 0$) or no positive root (if $b_1 > 0$). In other words, for $R_1^* = 1$ and $b_1 < 0$, model (1) has a unique positive high-criminality equilibrium given by P_h (case (ii)).

Moreover, note that when $\beta_2 = 0$, we have $b_2 = 0$ and $b_1 = (\mu + \gamma_2)(\tau_2 + \mu + a_2)\beta_1$. Thus, if $R_1^* > 1$, then

$$C_2^*(\beta_2 = 0) = \frac{(\mu + \gamma_1)}{\beta_1}(R_1^* - 1), \quad (12)$$

and model (1) has a unique positive high-criminality equilibrium, P_h , for $\beta_2 = 0$.

For $\beta_2 \rightarrow \infty$, we have $b_1 = (\mu + \gamma_1)(\tau_2 + \mu + a_2)\beta_2[(\mu + a_2)/(\tau_2 + \mu + a_2) - R_1^*]$ and $b_2 = \beta_1\beta_2(\mu + a_2)$ with $b_0 \ll |b_1|$ and $b_0 \ll b_2$ such that

$$C_2^*(\beta_2 \rightarrow \infty) = \frac{(\mu + \gamma_1)(\tau_2 + \mu + a_2)}{\beta_1(\mu + a_2)} \left[R_1^* - \frac{(\mu + a_2)}{(\tau_2 + \mu + a_2)} \right], \quad (13)$$

and $C_2^* > 0$ if only if $R_1^* > (\mu + a_2)/(\tau_2 + \mu + a_2)$. Thus, model (1) has a unique high-criminality equilibrium P_h if $\beta_2 \rightarrow \infty$.

Unfortunately, this high-criminality equilibrium cannot be studied from its closed form, so we carried out its local stability using numerical methods. The results are provided next.

Finally, if (i), (ii), and (iii) do not occur, then there are no endemic equilibria for system (1).

In what follows, model (1) admits two realistic scenarios: the best-case scenario, where offenders were incarcerated once and it is still possible to fight crime and recidivism is given by the low-criminality equilibrium $P_l = (S_0, D_0, C_1, S_1, D_1, 0, 0, 0)$, and the worst-case scenario, where the offenders were incarcerated at least once (recidivism), which can potentially lead to increased criminal activities, given by the high-criminality equilibrium $P_h = (S_0, D_0, C_1, S_1^*, D_1^*, C_2^*, S_2^*, D_2^*)$.

Having found the scenarios in which there exist the equilibria for system (1), it is instructive to analyze whether or not these equilibria are stable under any of these scenarios. Moreover, together with the CRN, R_1^* (see (7)), and the parameter β_2 , we see that each scenario can be used as a check for the existence and the stability of the equilibria. Another crucial question is which of the two incidence rates β_1 and β_2 is more likely to affect the criminal dynamics in general and criminal contacts in particular. These points are of great interest here and we then explore how our model behaves under control strategies when $R_1^* < 1$. We fix β_2 and explore the system behavior by varying β_1 . We also explore what happens when we fix β_1 while β_2 is varied. These control measures are designed to fight recidivism. In both cases, the recidivism will depend upon the initial sizes of the subpopulation C_2 and indicate the possibility of backward bifurcation.

Firstly, it is worth remembering that the quantity β_1 measures the average number of new contacts generated by a typical incarcerated individual C_2 with those susceptible individuals S_1 who are ex-offenders and can become susceptible to the crime again. β_2 , in turn, measures the average number of new contacts generated by C_2 with those individuals S_2 who were reoffenders (recidivist behavior) and are susceptible to the crime once again.

In this way, there are two groups of offenders: those who are under much lower risk, most of whom will go to prison once and not come back (ex-offenders), and those who repeatedly do crimes and come back multiple times to the prison (reoffenders). The strong implication of the findings is that individuals who are incarcerated are extremely likely to reoffend once they are free.

Note that when system (1) has a small influx of reoffenders (C_2), it does not generate high criminality rates and, for $R_1^* < 1$, it is still possible to minimize the spreading of crime (best-case scenario). We will see that, in this case, the corresponding low-criminality equilibrium P_l could be locally asymptotically stable. On the other hand, the criminality will persist and increase if $R_1^* > 1$ (worst-case scenario). In this case, the corresponding high-criminality equilibrium P_h could be locally asymptotically stable. This phenomenon, where the possibility of fighting the spread of the crime is lost and the criminality is potentially active, that is, where P_l loses its stability and a unique P_h appears as R_1^* increases through one, is known as forward bifurcation in epidemiology.

For models that exhibit this type of bifurcation, the requirement $R_1^* < 1$ is necessary and sufficient for the high-criminality elimination. In contrast, other models undergo another type of bifurcation, known as "backward bifurcation" in epidemiology, where two equilibria P_h^+ and P_h^- coexist with the low-criminality equilibrium, P_l ; that is, there are three steady states when R_1^* is immediately less than one. Thus, the requirement $R_1^* < 1$ is necessary but is not sufficient to fight the spread of crime.

2.2.1. Analytic Strategy: β_2 Fixed with β_1 Increasing. The physical implication of backward bifurcation is that the C_2 population can engage in a high criminality level even when R_1^* crosses unity downwards. In other words, in the presence of recidivism, decreasing R_1^* below one is not a sufficient condition to make the criminality level decrease. This has very important consequence for crime control, as will be discussed later in this paper.

To check the possibility of backward bifurcation in model (1), it is necessary to know other subthreshold, and we will refer to this limit point, as expressed on the R_1^* scale, as R_1^{thr} . Hence, for $R_1^* < R_1^{thr}$, system (1) will present only the locally asymptotically stable low-criminality equilibrium P_l . For $R_1^{thr} < R_1^* < 1$, system (1) will present the locally asymptotically stable low-criminality equilibrium P_l plus two positive high-criminality equilibria, P_h^+ and P_h^- , which will correspond to the solutions of (9): C_2^+ , the higher solution, which corresponds to the stable equilibrium, and C_2^- , the smaller solution, which corresponds to the unstable equilibrium.

Although the critical value of the bifurcation could not be found analytically due to the high dimension of system (1), this task can be performed numerically. In this way, our simulations show that there exists a critical value $R_1^{thr} < R_1^* = 1$, where model (1) undergoes backward bifurcation.

A schematic diagram of the backward and the forward bifurcations for system (1) is given in Figure 2, where β_1 is chosen as a bifurcation parameter; that is, we fixed β_2 , whereas β_1 increases.

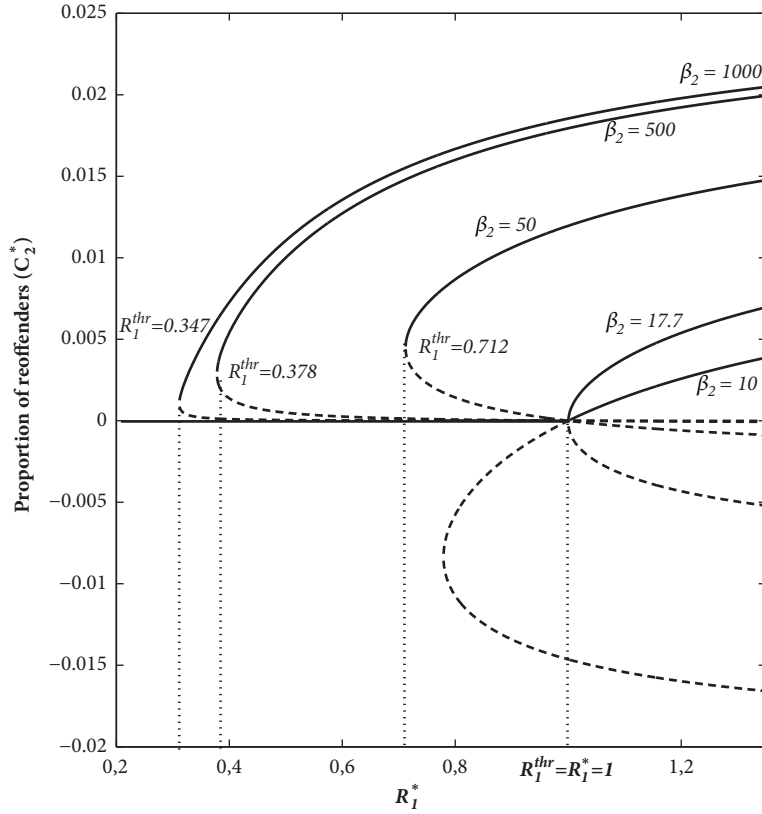


FIGURE 2: Backward and forward bifurcations diagram for proportion of reoffenders (C_2^*) for model (1), where β_1 is chosen as a bifurcation parameter (β_2 fixed and β_1 increasing). The higher solution corresponds to the stable equilibrium (solid curve); the smaller solution corresponds to the unstable equilibrium (dashed curve). Backward bifurcation for (a) $\beta_2 = 1000$, $\beta_1 = 5.371$, and $R_1^{thr} = 0.347$; (b) $\beta_2 = 500$, $\beta_1 = 5.845$, and $R_1^{thr} = 0.378$; and (c) $\beta_2 = 50$, $\beta_1 = 11$, and $R_1^{thr} = 0.712$. Forward bifurcation for (a) $\beta_2 = 17.7$, $\beta_1 = 15.45$, and $R_1^{thr} = R_1^* = 1$ and (b) $\beta_2 = 10.0$, $\beta_1 = 15.45$, and $R_1^{thr} < R_1^* = 1$. Parameters' values used are as given in Table 3.

Figure 2 shows the profile of the proportion of both reoffenders C_2^+ (solid curve) and C_2^- (dashed curve) as a function of R_1^* with decreasing values of $\beta_2 = 1000; 500; 50; 17.7; 10$ ($year^{-1}$) as R_1^* increases (i.e., as β_1 increases). The solid curve stands for the stable high-criminality equilibrium, C_2^+ , and the dashed curve stands for the unstable high-criminality equilibrium, C_2^- . For $\beta_2 = 1000$ and $\beta_1 = 5.371$, one has $R_1^{thr} = 0.347$. Thus, for $0.347 < R_1^* < 1$, model (1) has two positive high-criminality equilibria, while for $R_1^* > 1$, model (1) has one positive high-criminality equilibrium. Similarly, for $\beta_2 = 500$ and $\beta_1 = 5.845$, one gets $R_1^{thr} = 0.378$. Thus, for $0.378 < R_1^* < 1$, model (1) has two positive high-criminality equilibria, while for $R_1^* > 1$, model (1) has one positive high-criminality equilibrium. For $\beta_2 = 50$ and $\beta_1 = 11$, one gets $R_1^{thr} = 0.712$. Thus, for $0.712 < R_1^* < 1$, model (1) has two positive high-criminality equilibria, while for $R_1^* > 1$, model (1) has one positive high-criminality equilibrium. Finally, for $\beta_2 = 17.7$ and $\beta_1 = 15.45$ and for $\beta_2 = 10.0$ and $\beta_1 = 15.45$, we have $R_1^{thr} \leq R_1^* = 1$, and the model has one positive high-criminality equilibrium for $R_1^* > 1$ and no positive equilibrium for $R_1^* < 1$. Consequently, model (1) exhibits the forward bifurcation at $R_1^* = 1$. As it should be expected, it

TABLE 3: Baseline values for model (1).

Variable	Description
Λ	0.015 ($years^{-1}$)
δ_0	0.02 ($years^{-1}$)
β_1	variable ($years^{-1}$)
β_2	variable ($years^{-1}$)
γ_0	0.2 ($years^{-1}$)
γ_1	0.1 ($years^{-1}$)
γ_2	0.08 ($years^{-1}$)
μ_0	0.015 ($years^{-1}$)
a_1	0.03 ($years^{-1}$)
a_2	0.025 ($years^{-1}$)
τ_1	0.2 ($years^{-1}$)
τ_2	0.1 ($years^{-1}$)

can be seen in Figure 2 that R_1^{thr} increases with increasing β_1 and a greater reduction in recidivism prevalence is recorded for decreasing values of β_2 .

As stated earlier, the physical significance of the phenomenon of backward bifurcation is that the classical requirement of $R_1^* < 1$ is no longer sufficient for avoiding the

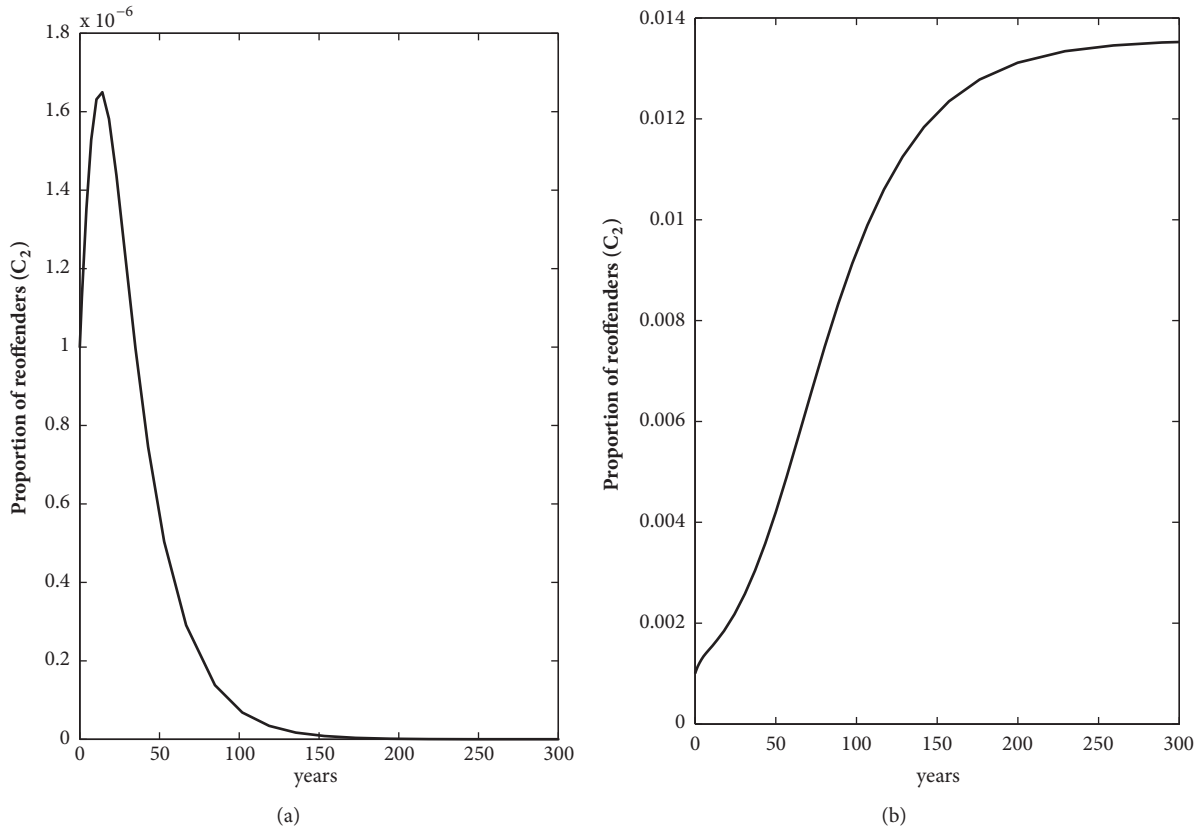


FIGURE 3: For $\beta_1 = 11, \beta_2 = 50$, and $R_1^{thr} < R_1^* < 1$. The other parameters' values are given in Table 3. Profile of the population of reoffenders (C_2). (a) If the recidivism is low (or $C_2(0)$ small), then the equilibrium point P_l is locally asymptotically stable; (b) if the recidivism is higher (or $C_2(0)$ large), then the equilibrium point P_h is locally asymptotically stable.

recidivism prevalence, as it is for the forward bifurcation. In such a scenario, the recidivism would depend on the initial sizes of the subpopulation C_2 of the model. That is, the presence of backward bifurcation in model (1) suggests that the possibility of avoiding the recidivism event when $R_1^* < 1$ could be dependent on the initial sizes of the subpopulation C_2 . This scenario is illustrated numerically in Figure 3. Thus, if the recidivism is low, then $C_2(0)$ is small, such that the low-criminality equilibrium P_l is stable (Figure 3(a)). In contrast, if the recidivism is higher, then $C_2(0)$ is large, so the high-criminality equilibrium P_h is stable (Figure 3(b)). Therefore, the stability of these equilibrium points depends on the initial condition of system (1). This clearly indicates the coexistence of two locally asymptotically stable equilibria when $R_1^* < 1$, confirming that model (1) undergoes the phenomenon of backward bifurcation with one stable high-criminality equilibrium P_h^+ (higher, solid curve in Figure 2), one unstable high-criminality equilibrium P_h^- (lowest dashed curve in Figure 2), and one low-criminality equilibrium P_l .

2.2.2. Analytic Strategy: β_1 Fixed with β_2 Increasing. Alternatively, from now on we explore the implications of the parameter β_2 on the criminal dynamics. It is worth mentioning that, from expression (10), it is easy to verify that $b_1 < 0$ if and only if

$$\beta_2 > \frac{\beta_1 (\mu + \gamma_2)}{(\mu + \gamma_1) [R_1^* - (\mu + a_2) / (\tau_2 + \mu + a_2)]} > 0, \quad (14)$$

and thus, in such case, system (1) has two positive high-criminality equilibria (case (iii), see Theorem 1.) for $b_0 > 0, b_1^2 - 4b_2b_0 > 0$, and $R_1^* > (\mu + a_2) / (\tau_2 + \mu + a_2)$.

A schematic diagram of this bifurcation phenomenon for system (1) is given in Figure 4, where β_2 is chosen as a bifurcation parameter. As mentioned previously, this bifurcation phenomenon of model (1) is only illustrated numerically.

Figure 4 shows the profile of the proportion of both reoffenders C_2^+ (solid curve) and C_2^- (dashed curve) as a function of β_2 with decreasing values of $\beta_1 = 30; 15.449; 11; 7.5; 5.845 (year^{-1})$ as β_2 increases. The solid curve stands for the stable equilibrium and the dashed curve stands for the unstable equilibrium. Note that, for $\beta_1 = 30$ and $R_1^* > 1, \beta_2 = 0$ such that system (1) has a unique positive high-criminality equilibrium (see case (i) of Theorem 1 and (12)).

It should be also noticed in Figure 4 that R_1^* increases with decreasing β_2 , such that greater reduction in recidivism prevalence is recorded for decreasing values of β_1 . One aspect to differentiate both parameters β_1 and β_2 is the way that they account for the transition to criminal activity.

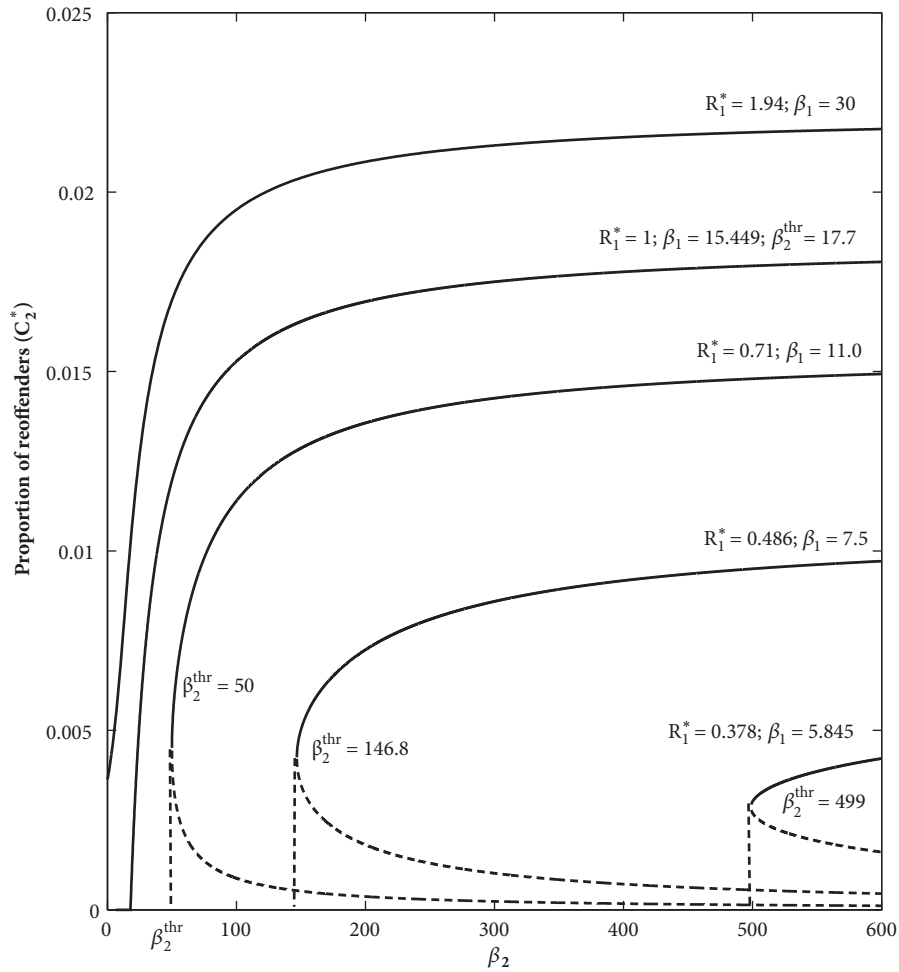


FIGURE 4: Breaking point and forward bifurcation diagrams for proportion of reoffenders (C_2^*) for model (1), where β_2 is chosen as a bifurcation parameter (β_1 fixed with β_2 increasing). The higher solution corresponds to the stable equilibrium (solid curve); the smaller solution corresponds to the unstable equilibrium (dashed curve). The phenomenon of the breaking point for (a) $\beta_1 = 11$, $R_1^* = 0.71$, and $\beta_2^{thr} = 50$; (b) $\beta_1 = 7.5$, $R_1^* = 0.486$, and $\beta_2^{thr} = 146.8$; and (c) $\beta_1 = 5.845$, $R_1^* = 0.378$, and $\beta_2^{thr} = 499$; forward bifurcation for $\beta_1 = 15.449$, $R_1^* = 1$, and $\beta_2^{thr} = 17.7$. For $\beta_1 = 30$ and $R_1^* = 1.94$, $\beta_2 = 0$ such that system (1) has a unique positive high-criminality equilibrium P_h . Parameters' values used are as given in Table 3.

In the low-criminality scenario, the only way someone becomes criminally active is on his/her own, without interacting with another person who is already criminally active. This restriction is necessary in order to illustrate the dynamics of a system which would lead to low-criminality equilibrium. If new criminal could only emerge on his/her own, then reducing the number of individuals who are not criminally active (core group) through either intervention or spontaneous desistance could lead to the lowest criminal scenario. For this, we calculated the threshold between the low-criminality equilibrium and the high-criminality equilibrium. Therefore, for $R_1^* < R_1^{thr}$, system (1) presented the locally asymptotically stable low-criminality equilibrium P_l . Although not a realistic possibility, understanding the intrinsic criminal propensity that leads to such kind of deviant behavior is a necessary step for understanding the high-criminality prevalence in the model, which will be presented in the next section.

In addition, in the high-criminality scenario, the way someone can become criminally active is to interact with another person who is already criminally active. Thus, if new reoffenders could only emerge by interaction between reoffenders (C_2^*) and those second-time offenders susceptible to the crime (S_2^*) at a rate β_2 , then reducing the number of contacts between C_2^* and S_1 , that is, reducing β_1 , could lead to the low-criminality scenario if the proportion of reoffenders C_2^* is small (see Figure 3). Hence, to reach the low-criminality equilibrium, a community needs to reduce β_1 . However, if β_1 is large, then the proportion of reoffenders C_2^* , as well as S_2^* , increases, which leads to higher crime prevalence, such that the high-criminality scenario emerges.

Figure 5(a) shows the long-term behavior of $\ln \beta_2^{thr}$ plotted versus β_1 . The threshold β_2^{thr} is expressed as a minimum value where system (1) has two positive equilibrium points. Thus, for each fixed β_1 , there is a corresponding unique

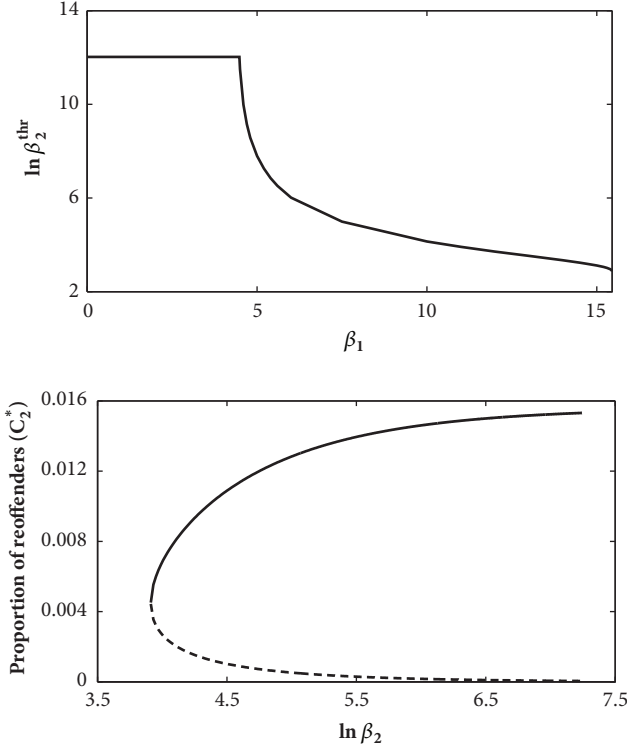


FIGURE 5: (a) The long-term behavior of $\ln \beta_2^{thr}$ plotted versus β_1 . (b) For $\beta_1 = 11$, $R_1^* = 0.71$, and $\ln \beta_2 = 3.91$, system (1) has two positive equilibrium points. The solid curve stands for the stable equilibrium and the dashed curve stands for the unstable equilibrium. The other parameters' values are given in Table 3.

threshold β_2^{thr} . For one such example (see Figure 5(b)), $\beta_1 = 11$, $R_1^* = 0.71$, and $\beta_2^{thr} = 50$ (or $\ln 50 = 3.91$) are the minimum values where system (1) has two positive value. For $\beta_2 > \beta_2^{thr}$, we have $C_2^+ > 0$ and $C_2^- > 0$. In this case, system (1) has a stable equilibrium point, C_2^+ (the solid curve in Figure 5(b)), and an unstable equilibrium point, C_2^- (the dashed curve in Figure 5(b)). On the other hand, for $\beta_2 < \beta_2^{thr}$, we have $C_2^+ = C_2^- = 0$, which means that there are no positive equilibrium points for system (1). Moreover, for $\beta_1^{thr} = 15.4492917$, we have $R_1^* = 1$ and $\beta_2^{thr} = 17.867$ (or $\ln 17.867 = 2.883$) such that, for $\beta_2 < \beta_2^{thr}$, one has $C_2^+ = 0$ and $C_2^- < 0$. Otherwise, for $\beta_2 > \beta_2^{thr}$, one has $C_2^+ > 0$ and $C_2^- = 0$ (see Figure 6).

2.3. The Full Contagion Criminal Model (FCCM) Formulation.

In this section, we relax the assumption made in the PCCM model (1) that the susceptible individuals (S_0) can enter to crime only on their own, so we make $\delta_0 = 0$. Hence, in the modified model, the flow of the susceptible individuals S_0 into the criminal activity depends only on the contact with those individuals who are incarcerated, C_1 . We also define β_i ($i = 0,1,2$) as the rate of imprisonment; it captures the return to criminal activity of those released from prison, such that β_0 , β_1 , and β_2 are the flows into criminal activity which depend on having had contact with those first-time and second-time

incarcerated individuals (C_1 and C_2), respectively. Following the idea of the previous model (1), here β_i ($i = 0,1,2$) is analogous to the effective contact rate in infectious diseases model. This derivation adopts a standard incidence formulation, where the contact rate is assumed to be constant, unlike the case of the mass action formulation, where the contact rate depends on the size of the total population (see, e.g., [27, 30], for detailed derivation of these incidences' functions).

The flow diagram of the FCCM is depicted in Figure 7. The variables and parameters' values are given in Table 3, except for β_0 that is the contact rate between C_1 and S_0 (cooptation rate).

The model is represented by the following nonlinear system of differential equations:

$$\begin{aligned}
 \frac{dS_0}{dt} &= \Lambda - \beta_0 C_1 S_0 - (\gamma_0 + \mu) S_0 \\
 \frac{dD_0}{dt} &= \gamma_0 S_0 - \mu D_0 \\
 \frac{dC_1}{dt} &= \beta_0 C_1 S_0 - (\tau_1 + \mu + a_1) C_1 \\
 \frac{dS_1}{dt} &= \tau_1 C_1 - (\gamma_1 + \mu) S_1 - \beta_1 C_2 S_1 \\
 \frac{dD_1}{dt} &= \gamma_1 S_1 - \mu D_1 \\
 \frac{dC_2}{dt} &= (\beta_1 S_1 + \beta_2 S_2) C_2 - (\tau_2 + \mu + a_2) C_2 \\
 \frac{dS_2}{dt} &= \tau_2 C_2 - \beta_2 C_2 S_2 - (\gamma_2 + \mu) S_2 \\
 \frac{dD_2}{dt} &= \gamma_2 S_2 - \mu D_2
 \end{aligned} \tag{15}$$

2.4. The Existence and Local Stability of Equilibria

2.4.1. Crime-Free Equilibrium. In the absence of crime, that is, for $C_1 = C_2 = 0$, model (15) has a crime-free equilibrium $P_0 = (S_0^0, D_0^0, 0, 0, 0, 0, 0, 0)$ which is obtained by setting the right-hand sides of system (15) to zero, where

$$\begin{aligned}
 S_0^0 &= \frac{\Lambda}{(\mu + \gamma_0)}, \\
 D_0^0 &= \frac{\gamma_0}{\mu} S_0^0.
 \end{aligned} \tag{16}$$

To analyze the local stability of this equilibrium, the Jacobian of system (15) is computed and evaluated at P_0 , which is locally asymptotically stable if the real parts of the eigenvalues of the Jacobian matrix are all negative.

Thus, the local stability of the crime-free equilibrium P_0 is governed by the Jacobian matrix

$$M_{P_0} = \begin{bmatrix} A_{P_0} & 0 \\ 0 & B_{P_0} \end{bmatrix}, \tag{17}$$

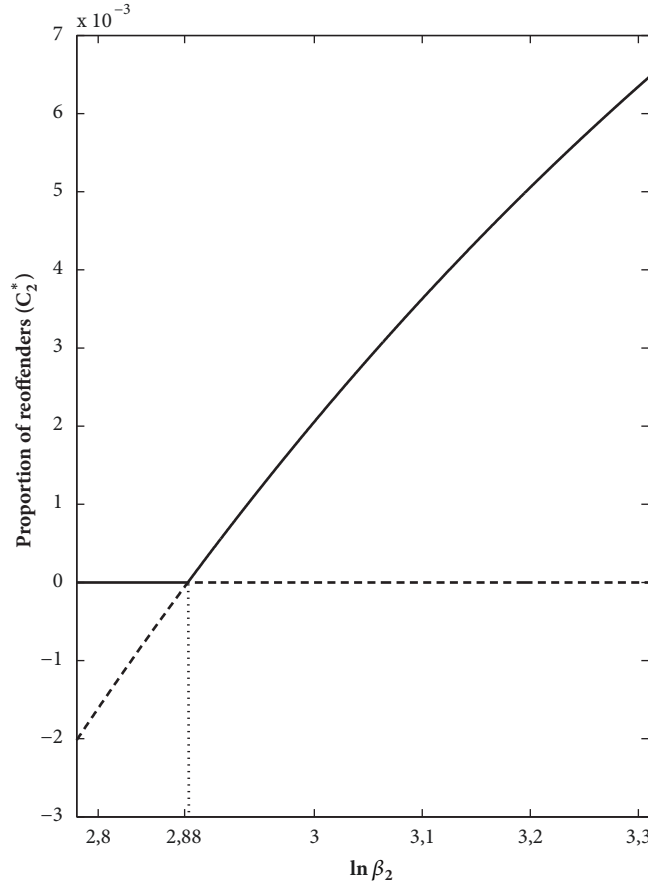


FIGURE 6: Long-term physical outcome C_2^* plotted versus $\ln \beta_2$. For $\beta_1^{thr} = 15.449$, $R_1^* = 1$, and $\beta_2 > \beta_2^{thr}$, system (1) has one positive equilibrium point. The solid curve stands for the stable equilibrium and the dashed curve stands for the unstable equilibrium. The other parameters' values are given in Table 3.

where

$$A_{P_0} = \begin{bmatrix} -(\mu + \gamma_0) & -\beta_0 S_0^0 \\ 0 & \beta_0 S_0^0 - (\tau_1 + \mu + a_1) \end{bmatrix}, \quad (18)$$

$$B_{P_0} = \begin{bmatrix} -(\mu + \gamma_1) & 0 & 0 \\ 0 & -(\tau_2 + \mu + a_2) & 0 \\ 0 & \tau_2 & -(\mu + \gamma_2) \end{bmatrix}. \quad (19)$$

It is easy to verify that the three eigenvalues of matrix (19) are always negative. In the same way, it is also straightforward to verify that one of the eigenvalues of matrix (18) is always negative, while the other is negative whenever

$$R_0 = \frac{\beta_0}{(\tau_1 + \mu + a_1)} S_0^0 < 1. \quad (20)$$

Hence, all the eigenvalues of matrix (17) are negative or have negative real parts if and only if $R_0 < 1$. In summary, the crime-free equilibrium P_0 of system (15) is locally asymptotically stable if the Basic Criminality Reproduction Number $R_0 < 1$. Therefore, we have established the following result.

Lemma 2. *The crime-free equilibrium P_0 of system (15) is locally asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$.*

2.4.2. Low-Criminality Equilibrium. In the absence of the reoffenders, that is, for $C_2 = 0$, model (15) has a low-criminality equilibrium $P_1^* = (S_0, D_0, C_1, S_1, D_1, 0, 0, 0)$, where

$$\begin{aligned} S_0 &= \frac{(\tau_1 + \mu + a_1)}{\beta_0} \\ D_0 &= \frac{\gamma_0}{\mu} S_0 \\ C_1 &= \frac{\Lambda \beta_0 - (\mu + \gamma_0)(\tau_1 + \mu + a_1)}{(\tau_1 + \mu + a_1) \beta_0} \\ &= \frac{(\mu + \gamma_0)}{\beta_0} (R_0 - 1) \\ S_1 &= \frac{\tau_1 C_1}{(\mu + \gamma_1)} \\ D_1 &= \frac{\gamma_1}{\mu} S_1. \end{aligned} \quad (21)$$

Note that P_1^* exists if $C_1 > 0$, that is, if $R_0 > 1$. Now consider the resulting model (15). The local stability

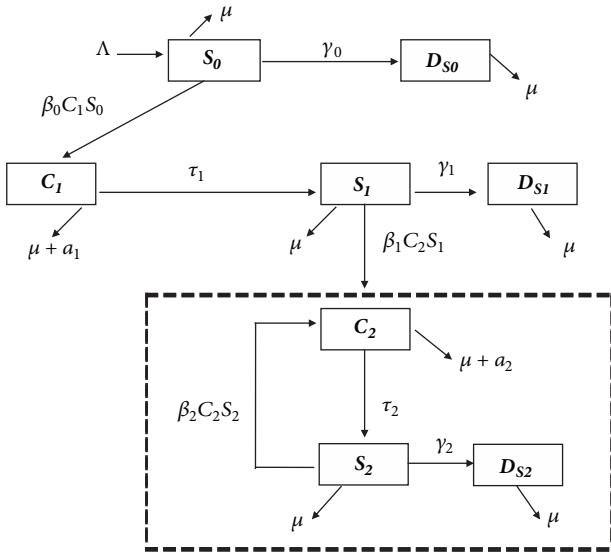


FIGURE 7: The flow diagram for the free-low-high criminality model (15).

of the low-criminality equilibrium P_l , which is examined by linearizing system (15) around P_l , is governed by the Jacobian matrix

$$M_{P_l} = \begin{bmatrix} M_{11} & -\beta_0 S_0 & 0 & 0 & 0 \\ \beta_0 C_1 & M_{22} & 0 & 0 & 0 \\ 0 & \tau_1 & M_{33} & -\beta_1 S_1 & 0 \\ 0 & 0 & 0 & M_{44} & 0 \\ 0 & 0 & 0 & \tau_2 & M_{55} \end{bmatrix}, \quad (22)$$

with $M_{11} = -\beta_0 C_1 - (\mu + \gamma_0)$; $M_{22} = \beta_0 S_0 - (\tau_1 + \mu + a_1)$; $M_{33} = -(\mu + \gamma_1)$; $M_{44} = \beta_1 S_1 - (\tau_2 + \mu + a_2)$; and $M_{55} = -(\mu + \gamma_2)$.

The eigenvalues of matrix (22) are $\lambda_1 = M_{55}$, $\lambda_2 = M_{44}$, $\lambda_3 = M_{33}$, and the roots of

$$\phi^2 + z_1 \phi + z_0 = 0, \quad (23)$$

with $z_1 = \Lambda \beta_0 / (\tau_1 + \mu + a_1) > 0$ and $z_0 = \Lambda \beta_0 - (\mu + \gamma_0)(\tau_1 + \mu + a_1) > 0 (\iff R_0 > 1)$.

The eigenvalues λ_1 and λ_3 are real negative from definition of the parameters μ , γ_1 , and γ_2 . The eigenvalue λ_2 is negative if

$$\beta_1 < \frac{(\tau_2 + \mu + a_2)}{S_1} = \frac{\beta_0 (\tau_2 + \mu + a_2) (\mu + \gamma_1)}{\tau_1 (\mu + \gamma_0) (R_0 - 1)} \quad (24)$$

or, equivalently, if

$$R_1 = \frac{\beta_1 \tau_1 (\mu + \gamma_0) (R_0 - 1)}{(\tau_2 + \mu + a_2) (\mu + \gamma_1) \beta_0} < 1 \quad (25)$$

Finally, the eigenvalues λ_4 and λ_5 are real negative if $z_0 > 0$ and $z_1 > 0$ by applying the Routh-Hurwitz criteria [31, 32] on polynomial (23). In this sense, $z_0 > 0$ and $z_1 > 0$ if and only if $R_0 > 1$.

Thus, we have established the following result.

Lemma 3. *The low-criminality equilibrium P_l^* of system (15) exists and it is locally asymptotically stable if $R_0 > 1$ and $R_1 < 1$. Otherwise, P_l^* is unstable.*

It should be mentioned that the consequence of the above result is that when P_l^* becomes unstable, two scenarios emerge: one for the case where the criminality is eliminated from population and P_0 is stable and the other where new criminals could emerge and there is high-criminality prevalence in the population such that model (15) has a high-criminality equilibrium with coexistence of both offenders: $C_1 \neq 0$ and $C_2 \neq 0$. In this way, it is instructive to determine the possible interventions on β_0 and β_1 in order to reduce R_0 and R_1 below one, that is, to guarantee the conditions under which the criminality is eliminated, or at least having its incidence reduced.

2.4.3. High-Criminality Equilibrium. In what follows, if $R_1 > 1$, such that the low-criminality equilibrium P_l^* is unstable, system (15) has the positive high-criminality equilibrium, $P_h^* = (S_0, D_0, C_1, S_1^*, D_1^*, C_2^*, S_2^*, D_2^*)$, where

$$\begin{aligned} S_1^* &= \frac{\tau_1 C_1}{\mu + \gamma_1 + \beta_1 C_2^*}, \\ S_2^* &= \frac{\tau_2 C_2^*}{\mu + \gamma_2 + \beta_2 C_2^*}, \\ D_1^* &= \frac{\gamma_1 S_1^*}{\mu}, \\ D_2^* &= \frac{\gamma_2 S_2^*}{\mu}. \end{aligned} \quad (26)$$

Remembering from (21) that $C_1 > 0 \iff R_0 > 1$, so P_h^* exists if and only if $R_0 > 1$. Now replacing the expressions for S_1^* and S_2^* in the sixth equation of system (15), the positive high-criminality equilibrium, which cannot be expressed cleanly in closed form, can then be obtained by solving for C_2^* the following expression:

$$\eta_2 (C_2^*)^2 + \eta_1 C_2^* + \eta_0 = 0, \quad (27)$$

where

$$\begin{aligned} \eta_2 &= \beta_1 \beta_2 (\mu + a_2), \\ \eta_1 &= (\mu + \gamma_1) (\tau_2 + \mu + a_2) \\ &\cdot \beta_2 \left\{ \left[\frac{\beta_1 (\mu + \gamma_2)}{\beta_2 (\mu + \gamma_1)} + \frac{(\mu + a_2)}{(\tau_2 + \mu + a_2)} \right] - R_1 \right\} \\ \eta_0 &= (\mu + \gamma_1) (\mu + \gamma_2) (\mu + \gamma_2 + a_2) (1 - R_1). \end{aligned} \quad (28)$$

Since all the model's parameters are nonnegative, it follows from (28) that the coefficient η_2 is always positive and $\eta_0 < 0$ for $R_1 > 1$. Thus, it is clear that model (15) has a unique positive equilibrium P_h^* when $R_1 > 1$ and $R_0 > 1$. For $\eta_0 = 0$ and $\eta_1 < 0$, model (15) also has a unique positive equilibrium

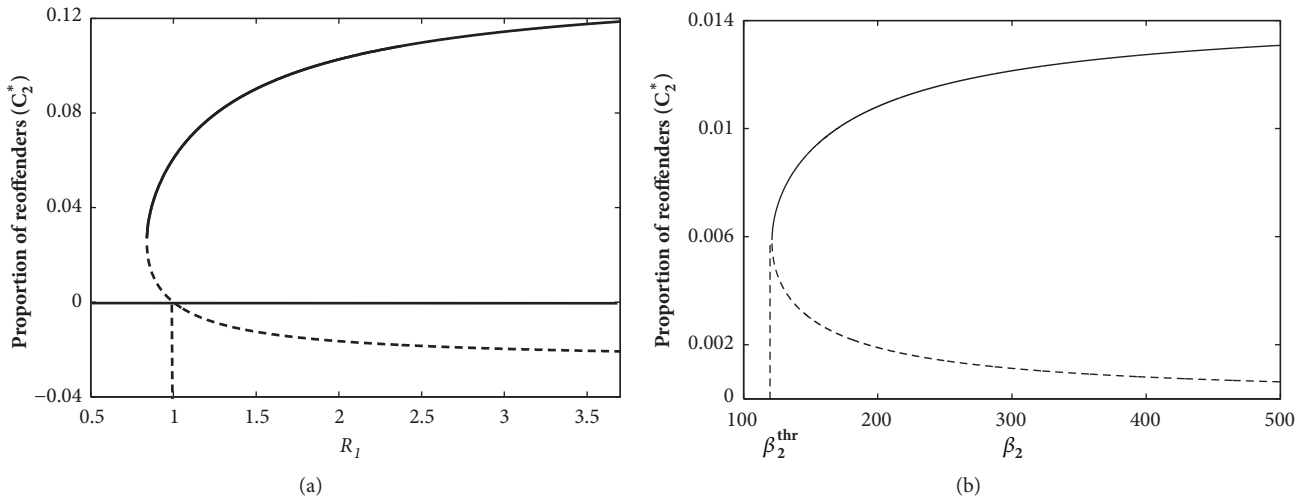


FIGURE 8: The phenomena of backward bifurcation and breaking point, when β_1 and β_2 are chosen as a bifurcation parameter. (a) Backward bifurcation for $\beta_2 = 5$ with β_0 and β_1 increasing. (b) Breaking point for $\beta_0 = \beta_1 = 5$ with β_2 increasing, where $\beta_2^{thr} = 121.2$ and $R_0 = 1.139$ and $R_1 = 0.464$. The higher solution corresponds to the stable equilibrium (solid curve); the smaller solution corresponds to the unstable equilibrium (dashed curve). Parameters' values used are as given in Table 3.

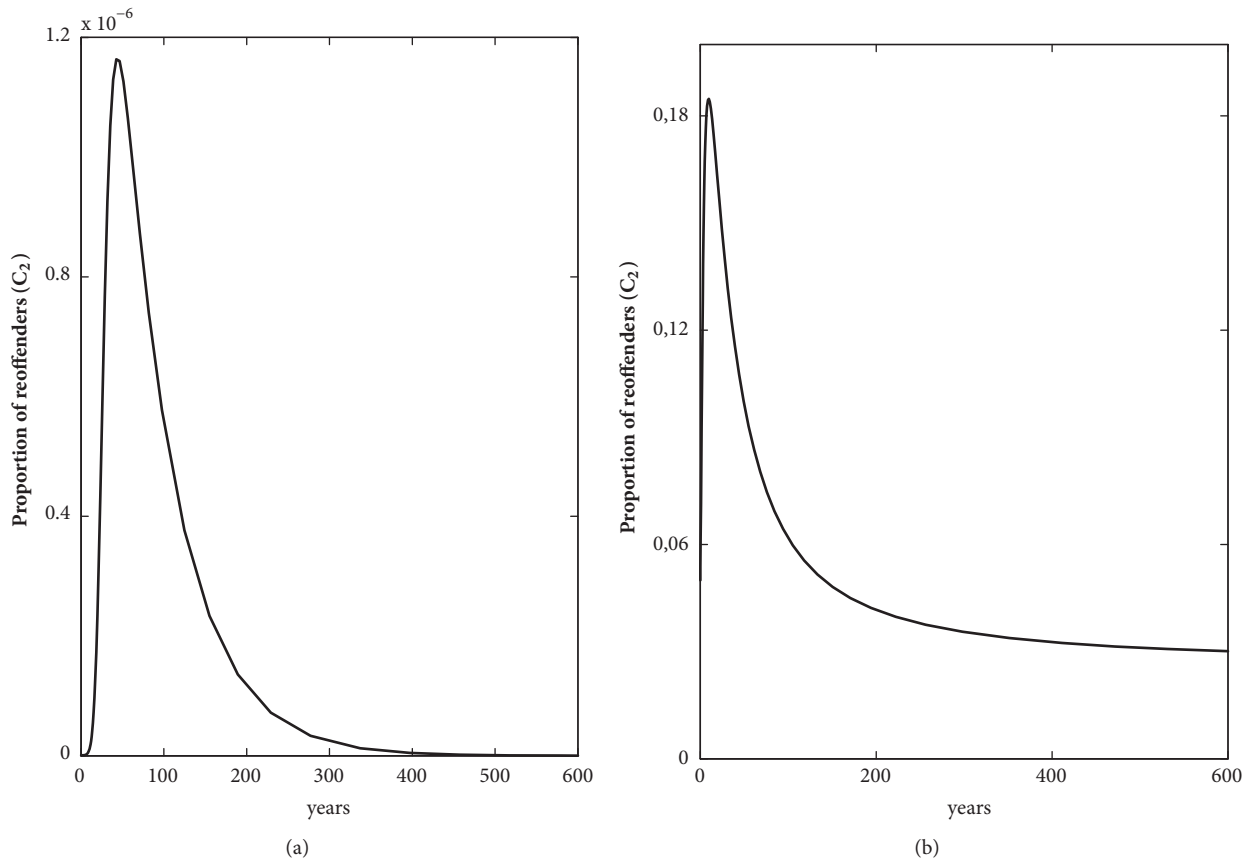


FIGURE 9: For $\beta_1 = 2.0$, $\beta_2 = 4.4$ ($\beta_2 > \beta_2^{thr}$), $R_0 = 2.39$, and $R_1 = 0.89$. The other parameters' values are given in Table 3. Profile of population of reoffenders (C_2). (a) If the recidivism is low (or $C_2(0)$ small), then the equilibrium point P_l^* is locally asymptotically stable. (b) If the recidivism is higher (or $C_2(0)$ large), then the equilibrium point P_h^* is locally asymptotically stable.

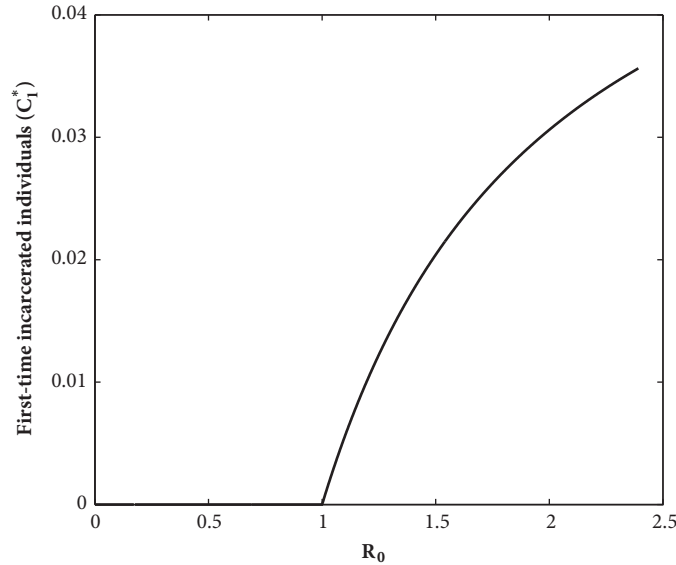


FIGURE 10: The phenomenon of forward bifurcation when β_0 is chosen as a bifurcation parameter. The crime-free equilibrium P_0 of system (15) is locally asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$. Parameters' values used are as given in Table 3.

P_h^+ . Moreover, whenever $R_0 > 1$, if $R_1 = 1$, then $\eta_0 = 0$ and $\eta_1 < 0$ if only if

$$\beta_2 > \frac{(\tau_2 + \mu + a_2)(\mu + \gamma_2)\beta_1}{\tau_2(\mu + \gamma_1)}, \quad (29)$$

such that model (15) also has a unique positive equilibrium P_h^* . Finally, for $\eta_0 > 0$, $\eta_1 < 0$, and $\eta_1^2 - 4\eta_2\eta_0 > 0$, that is, for $R_0 > 1$ and $R_1 < 1$, model (15) has two positive equilibria P_h^+ and P_h^- . Note that these equilibrium points cannot be studied in its closed form, so we carried out its local stability using numerical methods.

Thus, the following result is then established.

Theorem 4. *Model (15) has*

- (i) *a unique positive equilibrium P_h^+ if $\eta_0 < 0 \iff R_1 > 1$;*
- (ii) *a unique positive equilibrium P_h^+ if $\eta_0 = 0$ and $\eta_1 < 0$;*
- (iii) *two positive equilibria, P_h^+ and P_h^- , if $\eta_0 > 0$ and $\eta_1 < 0$ and $\eta_1^2 - 4\eta_2\eta_0 > 0$;*
- (iv) *no positive equilibrium, otherwise.*

As explored in model (1), model (15) also exhibits the phenomenon of backward bifurcation (Figure 8(a)) and breaking point (Figure 8(b)) when β_1 and β_2 are chosen as bifurcation parameters.

Figure 8(a) shows the backward bifurcation for $\beta_2 = 5$ and $\beta_0 = 8.4$ ($R_0 = 2.39$) with β_1 increasing. Figure 8(b) shows the phenomenon of breaking point for $\beta_1 = \beta_0 = 5$, with β_2 increasing, where $\beta_2^* = 121.2$ and $R_0 = 1.139$ and $R_1 = 0.464$. As it should be expected, Figures 8(a) and 8(b) present the same results shown in Figures 2 and 4. In summary, if $R_0 > 1$ and $R_1 > 1$, model (15) has the locally asymptotically stable high-criminality equilibrium P_h^+ .

Finally, if $R_0 > 1$ and $R_1 < 1$ and $\beta_2 > \beta_2^{thr}$, then there is coexistence of two locally asymptotically stable equilibria, P_l^* and P_h^+ , such that the stability of these equilibrium points depends on the initial condition of system (15) (see Figure 9).

In contrast to model (1), when β_0 is chosen as a bifurcation parameter, model (15) has the crime-free equilibrium P_0 given by (16), which indicates the possibility of the forward bifurcation (Figure 10). If $R_0 < 1$, then the crime-free equilibrium P_0 of system (15) is locally asymptotically stable; if $R_0 > 1$, P_0 becomes unstable and P_l^* is locally asymptotically stable if $R_1 < 1$. Thus, the substitution of the assumption that the susceptible individuals (S_0) can enter to crime only on their own (i.e., δ_0) by the standard incidence (i.e., $\beta_1 C_1 S_0$) in model (1) includes the forward bifurcation phenomenon to model (15).

3. Results

To illustrate the theoretical results contained in this paper, models (1) and (15) are simulated using baseline parameters values/ranges summarized in Table 3 (unless otherwise stated). The parameters are chosen for simulations purposes only, so we could illustrate our qualitative results. Moreover, it is worth mentioning that if the assumption made in the PCCM model (1), that is, susceptible individuals can enter to crime only on their own (i.e., $\delta_0 \neq 0$), and the assumptions made in FCCM model (15), that is, susceptible individuals get into criminal behavior depending only on the contact with those individuals who are incarcerated, C_1 (i.e., $\beta_0 C_1 S_0$), were considered in a single model, this new model would be more realistic. However, the analytical results and the phenomena of backward bifurcation and breaking point of this new model would be similar to model (1), except for the fact that R_0 would no longer exist. For this reason, we did not study this more realistic mixed model.

4. Discussion

In this paper, we attempt to present theoretical models of criminal careers using the dynamical system approach traditionally used in the study of infectious diseases spreading in a homogeneous population. The models consider crime dynamics as contagious phenomena in which a susceptible cohort of individuals with criminal propensity is “infected” by criminal individuals who have been convicted for their crimes and are arrested in a prison. The contagion occurs by several ways, in particular through the sending of messages by mobile phones (a very widespread habit in Brazilian prisons) and through “carrier pigeons” represented by relatives and lawyers.

We propose, to the best of our knowledge by the first time in quantitative criminology, the existence of thresholds for the stability of crime-endemic equilibrium which are the equivalent to the “basic reproduction number” widely used in the mathematical epidemiology literature [22], as shown in Lemmas 2 and 3. Both model (1) and model (15), however, exhibit the phenomena of backward bifurcation and breaking point when the contact rates β_1 and β_2 are chosen as bifurcation parameters. Since data strongly suggests that standard incidence formulation is more suited for modeling human diseases [22, 23, 27] and we adopted this formulation in the current study, the above results show that the phenomena of the backward bifurcation and the breaking point could be important properties of the criminality model.

As in other criminal career models, our models seek to provide a theoretical framework to analyse the longitudinal behavior of individuals who commit criminal offenses [33]. In addition, we centered our analysis, although in an implicit way, in the parameter considered to be the most important for the analysis of criminal career, namely, the rate at which offenders commit crime, denoted in the specialized literature by the Greek letter λ , coincidentally, the same symbol used to denote the force-of-infection in epidemiology of transmissible diseases.

Other dynamical system models proposed the analysis of criminal dynamics including differential equations, like the works by Farrington synthesized in Refs. [16, 34]. However, these models are either linear (the former) or related to other kinds of infectious contagion (the latter).

Finally, it is noteworthy that the models proposed in the paper are intended only to provide a theoretical framework upon which other works can provide empirical support for the assumptions and values for the parameters determinant of the dynamical behavior of the systems here studied. The present work, therefore, is intended to provide the first step in the study of criminal careers as determined by contagious events like the ones related to the phenomenon of incarcerated criminals influencing the behavior of susceptible juveniles outside prisons with a criminal propensity. The finding of backward bifurcation in both models, however, implies that there is an endemic equilibrium of criminality even when the threshold parameter for contagion is below unit which, in turn, implies that control strategies are more difficult to achieve any considerable impact on crime control in situations similar to the ones here analyzed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work has been funded by LIM01-HCFMUSP, FAPESP, CNPq and by the Office of Naval Research, USA, Grant Number 2059558. An earlier version of this work has been presented as at EMap Seminars according to the following link: <https://emap.fgv.br/seminarios/contagious-criminal-career-models-showing-backward-bifurcations-implications-crime>.

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