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AGE-STRUCTURED MODELING FOR THE DIRECTLY TRANSMITTED INFECTIONS – I: CHARACTERIZING THE BASIC REPRODUCTION NUMBER*

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One of the main features of directly transmitted infections is the strong dependency of the risk of infection with age. We propose and analyze a simple mathematical model where the force of infection (per-capita incidence rate) is age-depending. The existence and stability of the non-trivial equilibrium point are determined based on the basic reproduction number. For this reason we deal with a characterization of the basic reproduction number by applying the spectral radius theory.

1. Introduction

Directly transmitted childhood infections, like rubella and measles, have been used as good examples for the application of mathematical models to the study and comprehension of the epidemiology of these diseases. The models are formulated basically by taking into account the force of infection depending on the contact rate, which is related to the pattern of contacts among susceptible and infectious individuals. Therefore, the assumptions on the contact rate lead to quite different approaches when one deals with the models[2].

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A first assumption, and also the simplest, is to consider a constant contact rate among individuals over all ages and time. Consequently the force of infection becomes constant. The resulting mathematical model is described by a time-depending system of differential equations without age structure. This assumption can generate non-realistic outputs when modeling childhood diseases with a strong age depending pattern. A second and better assumption is, therefore, to take into account the age dependency in the pattern of contacts. A mathematical model with this assumption yields a time- and age-depending system of differential equations (see Dietz[5] who was the first author to apply this formalism to epidemiology with constant contact rate), resulting in the well established concept of the age depending force of infection[8].

When dealing with a constant contact rate modeling, there are classical results related to the basic reproduction number and the lower value (threshold) for the vaccination rate above which the disease can be considered eradicated[1]. With respect to age-structured modeling, results related to the basic reproduction number R_0 and the threshold vaccination rate are more complex. For instance, Greenhalgh[7] and Inaba[10] showed the existence and uniqueness of the non-trivial solution for the Hammerstein equation similar to that presented in Yang[18]. They showed that the bifurcation from the trivial to non-trivial solution of the Hammerstein integral equation occurs when the spectral radius assumes unity value. Furthermore, they related this spectral radius with the basic reproduction number, and stated that whenever $R_0 < 1$ the disease fades out in the community, and when $R_0 > 1$, the disease can be settle at an endemic level. Following the same arguments, they showed the procedure to calculate the threshold vaccination rate.

Two attempts of representing the age-structured contact rate can be found in the literature: a matrix with constant elements and a constant value for different age classes. Anderson and May[1] developed the concept they called *Who-Acquires-Infection-From-Whom* matrix (WAIFW). Briefly, this is a matrix where the elements of rows and columns are the contact rates, constant values, over the discrete age classes of susceptible and infectious individuals. Schenzle[15] developed an age-structured contact pattern where constant values on several age intervals are assigned and, then, structured the dynamics in a coupled differential equations to estimate the contact rate from notified data. Although both methods represent good approaches to modeling the dynamics of direct transmitted diseases, they are applicable to the description of different kinds of data collection:

the WAIFW method is appropriate to analyze seroprevalence data, while Schenzle's method is better applied to incidence records.

The purpose of this paper is to develop a model with age-structured contact rate. However, as pointed by Tudor[17], data on contact rates do not exist, although most parameters related to the disease transmission can be estimated directly. This fundamentally theoretical paper is divided as follows. In section 2 the general model is presented and analyzed. In section 3 we present a characterization of the basic reproduction number. Discussion and conclusion are presented in section 4.

2. The model

Farrington[6] obtained an age depending force of infection from cumulative distribution function of age at infection. Here, the age depending force of infection is obtained from a compartmental model taking into account an age-structured contact rate. Let a closed community be subdivided into four groups $X(a,t)$, $H(a,t)$, $Y(a,t)$ and $Z(a,t)$ which are, respectively, susceptible, exposed, infectious and immune individuals, distributed according to age a at time t. According to the natural history of infection, susceptible individuals are infected at a rate $\lambda(a,t)$, known as force of infection (per-capita incidence rate), and transferred to exposed class. The age-specific force of infection at time t is defined by

$$\lambda(a,t) = \int_0^L \beta(a,a')\, Y(a',t)\, da', \tag{1}$$

where $\beta(a,a')$ is the age-structured contact rate, that is, the contact among susceptible individuals of age a with infectious individuals of age a', and L is the maximum age attainable by human population. The exposed individuals are moved to the infectious class at a constant rate σ, and enters to the immune class at a rate γ. All individuals are under a constant mortality rate μ. We remark that the additional mortality due to the disease, the loss of immunity and the protective action of maternal antibodies in newborns are not considered in the model.

Based on the above considerations, the dynamics of directly transmitted infectious diseases model considering age-structured contact rate is described by a system of partial differential equations[16],

$$\begin{cases} \frac{\partial}{\partial t}X(a,t)+\frac{\partial}{\partial a}X(a,t)=-\left[\lambda(a,t)+\nu(a)+\mu\right]X(a,t)\\ \frac{\partial}{\partial t}H(a,t)+\frac{\partial}{\partial a}H(a,t)=\lambda(a,t)X(a,t)-(\mu+\sigma)H(a,t)\\ \frac{\partial}{\partial t}Y(a,t)+\frac{\partial}{\partial a}Y(a,t)=\sigma H(a,t)-(\mu+\gamma)Y(a,t)\\ \frac{\partial}{\partial t}Z(a,t)+\frac{\partial}{\partial a}Z(a,t)=\nu(a)X(a,t)+\gamma Y(a,t)-\mu Z(a,t), \end{cases} \tag{2}$$

where $\nu(a)$ is the age depending vaccination rate. Note that the last equation is decoupled from the system, hence, hereafter, we will omit the equation for the immune individuals, $Z(a,t)$. Defining the total population as $N(a,t)=X(a,t)+H(a,t)+Y(a,t)+Z(a,t)$, we obtain the equation for the age-distribution of the population irrespective of the disease as

$$\frac{\partial}{\partial a}N(a,t)+\frac{\partial}{\partial t}N(a,t)=-\mu N(a,t).$$

Using this equation, we can obtain the decoupled variable as $Z(a,t)=N(a,t)-X(a,t)-H(a,t)-Y(a,t)$.

The boundary conditions of (2) are $X(0,t)=X_b$, which is the newborn rate, and $H(0,t)=Y(0,t)=0$, which come out from the assumptions of the model. Let us assume that a vaccination strategy is introduced at $t=0$ in a non-vaccinated population ($\nu(a)=0$) in which the disease encounters in the steady state. Then the initial conditions of (2) are the solutions of

$$\begin{cases} \frac{d}{da}X_0(a)=-\left[\lambda_0(a)+\mu\right]X_0(a)\\ \frac{d}{da}H_0(a)=\lambda_0(a)X_0(a)-(\sigma+\mu)H_0(a)\\ \frac{d}{da}Y_0(a)=\sigma H_0(a)-(\gamma+\mu)Y_0(a), \end{cases} \tag{3}$$

where $\lambda_0(a)=\int_0^L \beta(a,a')Y_0(a')da'$ is equation (1) in the steady state. From the boundary conditions, the initial conditions are $X_0(0)=X_b$ and $H_0(0)=Y_0(0)=0$.

3. A characterization of $\mathbf{R_0}$ in the steady state

Let us characterize the reproduction number R_ν as a spectral radius of a integral operator. The basic reproduction number R_0 is defined as the average number of secondary infections produced by one susceptible individual in a completely homogeneous and susceptible population in the absence of any kind of constraint ($\nu=0$). Hence, R_0 describes the epidemiological situation in a non-vaccinated population. Greenhalgh[7] considered an age-structured contact rate $\beta(a,a')$ being a separable function

$$\beta(a,a')=\sum_{i=1}^{n}p_i(a)\,q_i(a'),$$

and applied results from the spectral radius of linear operators on Banach spaces in a finite dimension. In our case, we consider $\beta(a,a') \in C[0,L]$ and use results from bifurcation points and positive operators in cones[3] and fixed points[11]. In a companion paper[4], as Inaba[10], we discuss about the stability of the trivial solution and the uniqueness of the non-trivial solution, and provide the estimations for upper and lower bounds of R_0 for special contact rates.

Let us introduce some definitions.

We consider a Banach space X with a solid cone K and an operator $T : X \to X$. A **cone** is a proper convex closed subset (K is proper if $K \cap -K = \emptyset$) such that for all $k > 0$ we have $kK \subset K$. A cone is **solid** if it has $int\,(K) \neq \emptyset$ (particularly, if a cone is solid it is **reproducing**, that is, $X = K - K$). K establishes on X a partial ordering relation, that is, if $x, y \in X$, we say that $x \leq y$ if $y - x \in K$ and $x < y$ if $y - x \in K$ and $x \neq y$. Particularly, we say that $0 \leq x$ if $x \in K$ and $0 < x$ if $x \in K$ and $x \neq 0$. A cone K is called **normal** if exists a $\delta > 0$ such that $\|x_1 + x_2\| \geq \delta$ for $x_1, x_2 \in K$ and $\|x_1\| = \|x_2\| = 1$. For example, the cone of non-negative continuous real functions in a closed interval with the sup-norm is normal. An operator $T : X \longrightarrow X$ is **positive** if $T(K) \subset K$, and **strongly positive** if for $0 \neq x \in K$, then $T\,(x) \in int\,(K)$ (see Deimling[3]).

An operator $T : X \longrightarrow X$ is (strongly) **Fréchet differentiable at the point** $u_0 \in X$ **in the directions of the cone** K if there exist a linear operator $T'\,(u_0) : X \to X$ and an operator $\omega\,(u_0, \cdot) : K \longrightarrow X$ so that

$$T\,(u_0 + h) = T\,(u_0) + T'\,(u_0)\,h \mp \omega\,(u_0, h)\,, \forall h \in K,$$

where $\lim\limits_{\|h\| \to 0} \frac{\|\omega(u_0,h)\|}{\|h\|} = 0$. $T'\,(u_0)$ is called (strong) **Fréchet derivative with respect to the cone** K **at the point** u. A function $y : t \in \mathbf{R} \to y\,(t) \in X$ is called **differentiable at infinity** if the ratio $\left(\frac{1}{t}\right) y\,(t)$ converges to some element $y'\,(\infty) \in X$ as $t \to \infty$, and it is usual to speaking about **strong** or **weak differentiability at infinity** depending on whether $\left(\frac{1}{t}\right) y\,(t)$ converges strongly or weakly to $y'\,(\infty)$. The operator T is called (strongly) **differentiable at infinity in the directions of the cone** K if for all directions $h \in K$, $h \neq 0$, we have the derivative $y'\,(\infty)$ of $T\,(th)$ is representable in the form $y'\,(\infty) = T'\,(\infty)\,h$, where $T'\,(\infty)$ is some continuous linear operator, which is the derivative at infinity with respect to the cone K. The operator $T'\,(\infty)$ is called the **strong asymptotic derivative with respect to the cone** K, and the operator T is called

strongly asymptotically linear with respect to the cone K, if

$$\lim_{R\to\infty} \sup_{\|x\|\geq R, x\in K} \frac{\|Tx - T'(\infty)\, x\|}{\|x\|} = 0$$

(see Krasnosel'skii[11]).

If X is not a complex Banach space, we can consider its complexification $X_{\mathbf{C}}$, the complex Banach space of all pairs (x, y) with $x, y \in X$, where

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$(\lambda_1 + i\lambda_2)(x, y) = (\lambda_1 x - \lambda_2 y, \lambda_1 y + \lambda_2 x),$$

with norm given by $\|(x, y)\| = \sup\limits_{\theta\in[0,2\pi]} \|x\cos\theta + y\sin\theta\|$. In this case, X is isometrically isomorphic to the real subspace $\widehat{X} = \{(x, 0); x \in X\}$ of $X_{\mathbf{C}}$. If $T : X \to X$ is a linear operator, its complexification $T_{\mathbf{C}} : X_{\mathbf{C}} \to X_{\mathbf{C}}$ is defined by

$$T_{\mathbf{C}}(x, y) = (Tx, Ty),$$

and $\|T_{\mathbf{C}}\| = \|T\|$ (see Deimling[3]).

Let $T : X \longrightarrow X$ be a linear operator on a complex normed space X, and λ is a complex number, $\lambda \in \mathbf{C}$. Then we have associate operators

$$T_\lambda = T - \lambda I$$

and

$$\Re(\lambda) = T_\lambda^{-1}$$

when the inverse operator exists. The application $\Re$ which for $\lambda \in \mathbf{C}$ associates $\Re(\lambda)$, when it is possible, is called the **resolvent operator of** T. $\lambda \in \mathbf{C}$ is called a **regular value of** T if there exists $\Re(\lambda)$, $\Re(\lambda)$ is bounded linear operator and $\overline{Dom(\Re(\lambda))} = X$. The set of all regular values of T is called resolvent set of T, or simply, the **resolvent of** T, which is denoted by $\rho(T)$. $\sigma(T) = \mathbf{C} - \rho(T)$ is called the **spectrum of** T. The set of all $\lambda \in \mathbf{C}$ so that $R(\lambda)$ does not exist is called the **point spectrum of** T, $\sigma_p(T)$, and their elements are called **eigenvalues**, so $\lambda \in \sigma_p(T)$ if and only if there exists $x \in X$, $x \neq 0$, so that $Tx = \lambda x$, and x is called an **eigenvector of T associates to eigenvalue** λ, or simply, an eigenvector of T. Following the definition of spectrum of T , $\lambda \in \sigma(T)$ if one of the above conditions is not true, that is, whether T_λ^{-1} does not exist, or if there exists but it is not bounded, or $\overline{Dom\,(R\,(\lambda))} \neq X$.

Let us consider that X is a complex Banach space and T is bounded. Then if $\Re\,(\lambda)$ exists, is defined on the whole space X, and is bounded. The classical results show that $\rho\,(T)$ is open and the natural domain of the analyticity of $\Re$ (a **domain** in the complex plane $\mathbf{C}$ is an open subset such that every pair of points can be joined by a broken line consisting of finitely many straight line segments such that all points of they belong to it) and $\sigma\,(T)$ is non-avoid closed bounded set. Since T is a bounded linear operator and $\sigma\,(T)$ is bounded, we have the definition of the **spectral radius** $r\,(T)$,

$$r\,(T) = \sup_{\lambda \in \sigma(T)} |\lambda|\,,$$

and it is known that

$$r\,(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}\,,$$

which is called Gelfand's formula (see Kreyszig[13]).

An operator $T : X \to X$ is a **compact operator** if bounded sets are mapped in relatively compact sets. If T is a compact linear operator their properties closely resemble those of operators on finite dimensional spaces. For example, if T is a compact linear operator its set of the eigenvalues is countable (perhaps finite or even empty) and $\lambda = 0$ is the only possible point of accumulation of this set, the dimension of any eigenspace of T is finite and every spectral value $\lambda \neq 0$ is an eigenvalue. Furthermore, if $\lambda \neq 0$ is an eigenvalue there exists a natural number $r = r\,(\lambda)$ such that

$$X = N\,(T_\lambda^r) \oplus T_\lambda^r\,(X)\,,$$

where

$$N\left(T_\lambda^0\right) \subset N\,(T_\lambda) \subset N\left(T_\lambda^2\right) \subset \cdots \subset N\,(T_\lambda^r) = N\left(T_\lambda^{r+1}\right) = \cdots\,,$$

and

$$T_\lambda^0\,(X) \supset T_\lambda\,(X) \supset T_\lambda^2\,(X) \supset \cdots \supset T_\lambda^r\,(X) = T_\lambda^{r+1}\,(X) = \cdots\,.$$

The $\dim\,(N\,(T_\lambda^r))$ is the **algebraic multiplicity of** λ and $\dim\,(N\,(T_\lambda))$ is the **geometric multiplicity of** λ (in the case that $X = \mathbf{R}^n$, $\dim\,(N\,(T_\lambda^r))$ and $\dim\,(N\,(T_\lambda))$ are the multiplicities of λ as a zero of the characteristic polynomial and minimal polynomial of T). Particularly, the order of eingevalue $\lambda \neq 0$ as pole of resolvent operator $\Re(\cdot)$ is its algebraic multiplicity[3].

Let us consider the Banach space $C[0,L]$ of all continuous real functions defined on $[0,L]$, the normal solid cone $C[0,L]^{+} = \{f \in X; f(s) \geq 0, s \in [0,L]\}$, and the usual norm, that is, $\|f\| = \sup\{|f(s)|; s \in [0,L]\}$.

The steady state solutions of (2), letting zero the derivatives with respect to time, are

$$\begin{cases} X_\infty(a) = X_b e^{-[\mu a + \Lambda(a) + N(a)]} \\ H_\infty(a) = X_b e^{-(\mu+\sigma)a} \int\limits_0^a e^{\sigma\zeta - N(\zeta)} \lambda_\infty(\zeta) e^{-\Lambda(\zeta)} d\zeta \\ Y_\infty(a) \ = X_b e^{-(\mu+\gamma)a} \int\limits_0^a \sigma e^{(\gamma-\sigma)s} ds \int\limits_0^s e^{\sigma\zeta - N(\zeta)} \lambda_\infty(\zeta) e^{-\Lambda(\zeta)} d\zeta, \end{cases}$$

where $\Lambda(\zeta) = \int_0^\zeta \lambda_\infty(s) ds$ and $N(\zeta) = \int_0^\zeta \nu(s) ds$. Substituting the resulting $Y_\infty(a)$ into the equation (1) at equilibrium, after some calculations we obtain

$$\lambda_\infty(a) = \int\limits_0^L B(a,\zeta) \times M(\zeta, \lambda_\infty(\zeta), \nu(\zeta)) \times \lambda_\infty(\zeta) d\zeta, \tag{4}$$

where the function $M(\zeta, \lambda(\zeta), \nu(\zeta))$ is

$$M(\zeta, \lambda(\zeta), \nu(\zeta)) = e^{-\int_0^\zeta \lambda(s)ds} \times e^{-\int_0^\zeta \nu(s)ds}$$

and the kernel $B(a,\zeta)$ is

$$B(a,\zeta) = \sigma X_b e^{-N(\zeta)} \int\limits_\zeta^L e^{-\sigma(s-\zeta)} e^{\gamma s} \left[\int\limits_s^L \beta(a,a') e^{-(\mu+\gamma)a'} da' \right] ds. \tag{5}$$

Equation (4) is a Hammerstein equation[9]. Notice that the force of infection corresponding to the initial conditions, solutions of (3), is

$$\lambda_0(a) = \int\limits_0^L B'(a,\zeta) M(\zeta, \lambda_0(\zeta), 0) \lambda_0(\zeta) d\zeta,$$

from which we characterize the basic reproduction number R_0.

Let us assume that:

(a) $\beta(a,a')$ is continuous and $\beta(a,a') > 0$ for every $a, a' \in [0,L]$, except for $a = a' = 0$, where $\beta(a,a') = 0$.

(b) $\nu(a)$ is continuous or piecewise continuous with only finitely many discontinues and is bounded.

Let us consider the operator T on $C\left[0,L\right]$ defined by

$$Tu\left(a\right)=\int\limits_{0}^{L}B\left(a,\zeta\right)M\left(\zeta,u\left(\zeta\right),\nu\left(\zeta\right)\right)u\left(\zeta\right)d\zeta, \tag{6}$$

where $B\left(a,\zeta\right)$ and $M\left(\zeta,u,\nu\right)$ are real functions satisfying the conditions:

(c) $B\left(a,\zeta\right)$ is defined on $\left[0,L\right]\times\left[0,L\right]$, which is positive and continuous in a and ζ.

(d) $M\left(\zeta,u,\nu\right)$ is defined on $\left[0,L\right]\times\left[0,\infty\right)\times\left[0,\infty\right)$, which is positive, continuous in ζ for each u and v, strictly monotone decreasing for u for each ζ and ν, and there exists $k_1\geq 0$ such that

$$\left|M\left(\zeta,u_1\left(\zeta\right),\nu\left(\zeta\right)\right)-M\left(\zeta,u_2\left(\zeta\right),\nu\left(\zeta\right)\right)\right|\leq k_1\left\|u_1-u_2\right\|+R\left(u_1,u_2\right),$$

with $\lim\limits_{\left\|u_1-u_2\right\|\to 0}R\left(u_1,u_2\right)=0$.

(e) there exists a real number $m>0$ such that $\left|M\left(\zeta,u,\nu\right)\right|\leq m$ for every ζ, u and ν.

Notice that $\left|M\left(\zeta,\lambda\left(\zeta\right),\nu\left(\zeta\right)\right)\right|\leq 1$ for all $\zeta\in\left[0,L\right]$, and

$$\left|M\left(\zeta,\lambda_1\left(\zeta\right),\nu\left(\zeta\right)\right)-M\left(\zeta,\lambda_2\left(\zeta\right),\nu\left(\zeta\right)\right)\right|\leq\left|1-e^{-\int_0^L\left(\lambda_1(s)-\lambda_2(s)\right)ds}\right|\to 0,$$

when $\left\|\lambda_1-\lambda_2\right\|\to 0$.

Definition 3.1. An operator A is **completely continuous** if it is a compact continuous operator.

The following three theorems are used to proof lemmas below.

Theorem 3.1. *(Krasnosel'skii[12]) Let us consider the Banach spaces E_1 and E_2, the operator $\overline{f}:E_1\to E_2$ which is continuous and bounded and also $\overline{B}:E_2\to E_1$ which is completely continuous linear operator. Then the operator $A=\overline{B}\ \overline{f}:E_1\to E_1$ is completely continuous.*

Theorem 3.2. *(**Ascoli's Theorem,** Kreyszig[13]) A bounded equicontinuous sequence $\left(x_n\right)_n$ in $C\left[0,L\right]$ has a subsequence which converges in the norm on $C\left[0,L\right]$ (a sequence $\left(y_n\right)_n$ in $C\left[0,L\right]$ is said to be **equicontinuous** if for every $\varepsilon>0$ there is a $\delta>0$, depending only on ε, such that for all y_n and all $a,a'\in\left[0,L\right]$ satisfying $\left|a-a'\right|<\delta$ we have $\left|y_n\left(a\right)-y_n\left(a'\right)\right|<\varepsilon$).*

Theorem 3.3. *(**Compactness criterion,** Kreyszig[13]) Let $S:Y\to Z$ be a linear operator where Y and Z are normed spaces. Then S is compact operator if and only if it maps every bounded sequence $\left(y_n\right)_n$ in Y onto a sequence in Z which has a convergent subsequence.*

Lemma 3.1. *T is completely continuous positive operator.*

Proof: If $u \in C[0,L]^+$ then $Tu \in C[0,L]^+$. Let be $a_1, a_2 \in [0,L]$ then

$|Tu(a_1) - Tu(a_2)| \leq_0^L |M(\zeta, u(\zeta), \nu(\zeta))| |u(\zeta)| |B(a_1,\zeta) - B(a_2,\zeta)| d\zeta \leq$
$m \|u\|_0^L |B(a_1,\zeta) - B(a_2,\zeta)| d\zeta.$

Note that B is continuous on compact set $[0,L] \times [0,L]$, hence, given $\varepsilon > 0$ there is $\delta > 0$ such that if $\|(a_1,\zeta) - (a_1,\zeta)\| = \sqrt{(a_1 - a_2)^2} \leq \delta$, then $|B(a_1,\zeta) - B(a_2,\zeta)| \leq \frac{\varepsilon}{m\|u\|L}$. Therefore, if $|a_1 - a_2| \leq \delta$ then $|Tu(a_1) - Tu(a_2)| \leq \varepsilon$.
We show that T is continuous. Let $u, u_0 \in C[0,L]$ and $a \in [0,L]$, then

$$|Tu(a) - Tu_0(a)| \leq$$
$$\int_0^L |M(\zeta, u(\zeta), \nu(\zeta)) u(\zeta) - M(\zeta, u_0(\zeta), \nu(\zeta)) u_0(\zeta)| |B(a,\zeta)| d\zeta.$$

As B is continuous over a compact set, so there is $m_1 > 0$ such that $|B(a,\zeta)| \leq m_1$ for all $(a,\zeta) \in [0,L] \times [0,L]$. Furthermore,

$$\begin{aligned}
&|M(\zeta, u(\zeta), \nu(\zeta)) u(\zeta) - M(\zeta, u_0(\zeta), \nu(\zeta)) u_0(\zeta)| \leq |M(\zeta, u(\zeta), \nu(\zeta)) u(\zeta) - \\
&M(\zeta, u(\zeta), \nu(\zeta)) u_0(\zeta)| + |M(\zeta, u(\zeta), \nu(\zeta)) u_0(\zeta) - \\
&M(\zeta, u_0(\zeta), \nu(\zeta)) u_0(\zeta)| \leq |M(\zeta, u(\zeta), \nu(\zeta))| |u(\zeta) - u_0(\zeta)| + \\
&|M(\zeta, u(\zeta), \nu(\zeta)) - M(\zeta, u_0(\zeta), \nu(\zeta))| |u_0(\zeta)| \leq \\
&|M(\zeta, u(\zeta), \nu(\zeta))| \|u - u_0\| + [k_1 \|u - u_0\| \|u_0\| + R(u(\zeta), u_0(\zeta))] \|u_0\| \leq \\
&m \|u - u_0\| + k_1 \|u_0\| \|u - u_0\| + \|u_0\| R(u(\zeta), u_0(\zeta)),
\end{aligned}$$

from which

$$\begin{aligned}
&|Tu(a) - Tu_0(a)| \leq \\
&\int_0^L m_1 [m \|u - u_0\| + k_1 \|u_0\| \|u - u_0\| + \|u_0\| R(u(\zeta), u_0(\zeta))] d\zeta \leq \\
&m_1 (m + k_1 \|u_0\|) \|u - u_0\| L + m_1 \|u_0\| \int_0^L R(u(\zeta), u_0(\zeta)) d\zeta.
\end{aligned}$$

Since $\lim\limits_{\|\lambda_1 - \lambda_2\| \to 0} R(\lambda_1, \lambda_2) = 0$, we have that $\|Tu - Tu_0\| \to 0$ when $\|u - u_0\| \to 0$.
Now we show that T is compact. To prove this we will use the Theorem 3.1. Let us consider the operators

$$\begin{cases} \overline{B} : C[0,L] \to C[0,L] \\ \overline{B}u(a) = \int_0^L B(a,\zeta) u(\zeta) d\zeta \end{cases}$$

and

$$\begin{cases} \overline{f} : C\left[0, L\right] \to C\left[0, L\right] \\ \overline{f}u\left(\zeta\right) = M\left(\zeta, u\left(\zeta\right), \nu\left(\zeta\right)\right) u\left(\zeta\right). \end{cases}$$

It is enough to verify that $\overline{B}$ is completely continuous and $\overline{f}$ is continuous and bounded.
We show that $\overline{B}$ is completely continuous, that is, $\overline{B}$ is compact and continuous. Let be $u \in C\left[0, L\right]$ and $a, a_0 \in \left[0, L\right]$, then

$$\left|\overline{B}u\left(a\right) - \overline{B}u\left(a_0\right)\right| = \left|\int_0^L B\left(a, \zeta\right) u\left(\zeta\right) d\zeta - \int_0^L B\left(a_0, \zeta\right) u\left(\zeta\right) d\zeta\right| \leq$$
$$\left\|u\right\| \int_0^L \left|B\left(a, \zeta\right) - B\left(a_0, \zeta\right)\right| d\zeta.$$

As B is continuous on compact, given $\varepsilon > 0$ there is $\delta > 0$ such that $\left|a - a_0\right| \leq \delta$, then $\left|B\left(a, \zeta\right) - B\left(a_0, \zeta\right)\right| \leq \frac{\varepsilon}{\|u\| L}$. Thus $\overline{B}u \in C\left[0, L\right]$.
First we show that $\overline{B}$ is continuous. Let be $u, u_0 \in C\left[0, L\right]$ and $a \in \left[0, L\right]$, then

$$\begin{aligned} &\left|\overline{B}u\left(a\right) - \overline{B}u_0\left(a\right)\right| = \\ &\left|\int_0^L B\left(a, \zeta\right) u\left(\zeta\right) d\zeta - \int_0^L B\left(a, \zeta\right) u_0\left(\zeta\right) d\zeta\right| \leq m_1 L \left\|u - u_0\right\|. \end{aligned}$$

So $\left\|\overline{B}u - \overline{B}u_0\right\| \to 0$ when $\left\|u - u_0\right\| \to 0$.
Second, $\overline{B}$ is compact. Let $\left(u_n\right)_n$ be a bounded sequence in $C\left[0, L\right]$, that is, there is $m_2 \in \mathbf{R}$ such that $\left\|u_n\right\| = \sup\limits_{a \in [0, L]} \left|u_n\left(a\right)\right| \leq m_2$ for every n. Hence, given $a_1, a_2 \in \left[0, L\right]$, we have

$$\left|\overline{B}u_n\left(a_1\right) - \overline{B}u_n\left(a_2\right)\right| = \left|\int_0^L B\left(a_1, \zeta\right) u_n\left(\zeta\right) d\zeta - \int_0^L B\left(a_2, \zeta\right) u_n\left(\zeta\right) d\zeta\right| \leq$$
$$m_2 \int_0^L \left|B\left(a_1, \zeta\right) - B\left(a_2, \zeta\right)\right| d\zeta.$$

Being B continuous on compact, given $\varepsilon > 0$ there is $\delta > 0$ such that if $\left|a_1 - a_2\right| \leq \delta$, then $\left|B\left(a_1, \zeta\right) - B\left(a_2, \zeta\right)\right| \leq \frac{\varepsilon}{m_2 L}$. So $\left(\overline{B}\left(u_n\right)\right)_n$ is equicontinuous. That $\left(\overline{B}\left(u_n\right)\right)_n$ is bounded sequence is checked straightforwardly. As $\left(\overline{B}\left(u_n\right)\right)_n$ is equicontinuous and bounded sequence on $C\left[0, L\right]$, it has a convergent subsequence (see Theorem 3.2). Since $\overline{B}$ maps a bounded sequence into a sequence which has a convergent subsequence it is a compact operator (see Theorem 3.3).

In relation to $\overline{f}$, we show that $\overline{f}u \in C[0,L]$ if $u \in C[0,L]$. Let be $\zeta, \zeta_0 \in [0,L]$, then

$$\left|\overline{f}u(\zeta) - \overline{f}u(\zeta_0)\right| = \left|e^{-\int_0^\zeta u(s)ds} e^{-\int_0^\zeta \nu(s)ds} u(\zeta) - e^{-\int_0^{\zeta_0} u(s)ds} e^{-\int_0^{\zeta_0} \nu(s)ds} u(\zeta_0)\right|.$$

Let us suppose that $\zeta < \zeta_0$, then, when $\zeta \to \zeta_0$, we have

$$\begin{aligned}&\left|\overline{f}u(\zeta) - \overline{f}u(\zeta_0)\right| = \\ &e^{-\int_0^\zeta u(s)ds} e^{-\int_0^\zeta \nu(s)ds} \left|u(\zeta) - e^{-\int_\zeta^{\zeta_0} u(s)ds} e^{-\int_\zeta^{\zeta_0} \nu(s)ds} u(\zeta_0)\right| \to 0.\end{aligned}$$

Now, we will see that $\overline{f}$ is continuous. Let be $u, u_0 \in C[0,L]$ and $\zeta \in [0,L]$, in a way that

$$\begin{aligned}&\left|\overline{f}u(\zeta) - \overline{f}u_0(\zeta)\right| = \left|M(\zeta, u(\zeta), \nu(\zeta))\, u(\zeta) - M(\zeta, u_0(\zeta), \nu(\zeta))\, u_0(\zeta)\right| \leq \\ &\{\left|M(\zeta, u(\zeta), \nu(\zeta))\, u(\zeta) - M(\zeta, u(\zeta), \nu(\zeta))\, u_0(\zeta)\right| + \\ &\left|M(\zeta, u(\zeta), \nu(\zeta))\, u_0(\zeta) - M(\zeta, u_0(\zeta), \nu(\zeta))\, u_0(\zeta)\right|\} \leq \\ &\left|M(\zeta, u(\zeta), \nu(\zeta))\right| \|u - u_0\| + \left|M(\zeta, u(\zeta), \nu(\zeta)) - M(\zeta, u_0(\zeta), \nu(\zeta))\right| \|u_0\| \leq \\ &m\|u - u_0\| + \|u_0\| \left[k_1 \|u - u_0\| + R(u(\zeta), u_0(\zeta))\right].\end{aligned}$$

Being $\lim_{\|\lambda_1 - \lambda_2\| \to 0} R(\lambda_1, \lambda_2) = 0$, we have $\left\|\overline{f}u - \overline{f}u_0\right\| \to 0$ when $\|u - u_0\| \to 0$. That $\overline{f}$ is bounded is checked straightforwardly, i.e.,

$$\begin{aligned}&\left\|\overline{f}u\right\| = \sup_{\zeta\in[0,L]} \left|\overline{f}u(\zeta)\right| = \sup_{\zeta\in[0,L]} \left|M(\zeta, u(\zeta), \nu(\zeta))\, u(\zeta)\right| = \\ &\sup_{\zeta\in[0,L]} \left|M(\zeta, u(\zeta), \nu(\zeta))\right| \left|u(\zeta)\right| \leq m\|u\|.\end{aligned}$$

Lemma 3.2. *T is Fréchet differentiable at the point $0 \in C[0,L]$ in the directions of the cone $C[0,L]^+$ and*

$$T'(0)\, h(a) = \int_0^L B(a,\zeta)\, M(\zeta, 0, \nu(\zeta))\, h(\zeta)\, d\zeta. \tag{7}$$

Furthermore $T'(0)$ is strongly positive completely continuous operator.

Proof: Let be $u, h \in C[0, L]$. Then, from equation (6), we have

$$T(u+h)(a) - Tu(a) =$$
$$\int_0^L B(a,\zeta)\, M(\zeta,(u+h)(\zeta),\nu(\zeta))\,(u+h)(\zeta)\,d\zeta -$$
$$\int_0^L B(a,\zeta)\, M(\zeta,u(\zeta),\nu(\zeta))\,u(\zeta)\,d\zeta =$$
$$\int_0^L B(a,\zeta)\left[M(\zeta,(u+h)(\zeta),\nu(\zeta)) - M(\zeta,u(\zeta),\nu(\zeta))\right]u(\zeta)\,d\zeta +$$
$$\int_0^L B(a,\zeta)\, M(\zeta,(u+h)(\zeta),\nu(\zeta))\,h(\zeta)\,d\zeta.$$

So we have at $u \equiv 0$,

$$T(h)(a) - T(0)(a) = \int_0^L B(a,\zeta)\, M(\zeta,h(\zeta),\nu(\zeta))\,h(\zeta)\,d\zeta =$$
$$\int_0^L B(a,\zeta)\left[M(\zeta,h(\zeta),\nu(\zeta)) - M(\zeta,0,\nu(\zeta))\right]h(\zeta)\,d +$$
$$\int_0^L B(a,\zeta)\, M(\zeta,0,\nu(\zeta))\,h(\zeta)\,d\zeta.$$

Defining $\omega(a,h)$ by equation

$$\omega(a,h) = \int_0^L B(a,\zeta)\left[M(\zeta,h(\zeta),\nu(\zeta)) - M(\zeta,0,\nu(\zeta))\right]h(\zeta)\,d\zeta,$$

we observe that

$$|\omega(a,h)| = \left|\int_0^L B(a,\zeta)\left[M(\zeta,h(\zeta),\nu(\zeta)) - M(\zeta,0,\nu(\zeta))\right]h(\zeta)\,d\zeta\right| \le$$
$$\int_0^L |B(a,\zeta)|\left[k_1|h(\zeta)| + R(h(\zeta),0)\right]|h(\zeta)|\,d\zeta,$$

and $\lim\limits_{\|h\|\to 0} R(h(\zeta),0) = 0$, then

$$\lim_{\|h\|\to 0} \frac{\|\omega(a,h)\|}{\|h\|} = 0.$$

Hence, we have (7), the definition of Fréchet derivative,

$$T'(0)\,h(a) = \int_0^L B(a,\zeta)\, M(\zeta,0,\nu(\zeta))\,h(\zeta)\,d\zeta.$$

Now we show that $T'(0)$ is strongly positive. Let us consider $0 \neq h \in C[0,L]^{+}$, that is, there exists $\zeta^{*} \in [0,L]$ such that $h(\zeta^{*}) \neq 0$. If $T'(0)\,h(a^{*}) = 0$ for some $a^{*} \in [0,L]$, then

$$\int_{0}^{L} B(a^{*},\zeta)\, M(\zeta,0,\nu(\zeta))\, h(\zeta)\, d\zeta = 0.$$

Since $B(a,\zeta)\, M(\zeta,0,\nu(\zeta))\, h(\zeta)$ is positive continuous function in ζ for each a, we have

$$B(a^{*},\zeta)\, M(\zeta,0,\nu(\zeta))\, h(\zeta) = 0$$

for all ζ, particularly for ζ^{*}, we have

$$B(a^{*},\zeta^{*})\, M(\zeta^{*},0,\nu(\zeta^{*}))\, h(\zeta^{*}) = 0.$$

Therefore, we have

$$B(a^{*},\zeta^{*}) = 0,$$

which implies that $\beta(a^{*},a') = 0$ for all $a' \in [\zeta^{*},L]$ and this is not possible (see condition (a) on $\beta(a,a')$).
Since $T'(0)$ is a linear operator, to verify that it is completely continuous, it is sufficient to proceed like the case of operator T in Lemma 3.1.

To demonstrate that $R_{\nu} = r(T'(0))$, we use three theorems stated below. In their enunciates, X, Y, K and T will be general spaces and operator, respectively. Notice that R_0 is calculated by (7) letting $\nu = 0$.

Theorem 3.4. *(Krasnosel'skii[11]) Let the positive operator T ($T0 = 0$) have a strong Fréchet derivative $T'(0)$ with respect to a cone and a strong asymptotic derivative $T'(\infty)$ with respect to a cone. Let the spectrum of the operator $T'(\infty)$ lie in the circle $|\mu| \leq \rho < 1$. Let the operator $T'(0)$ have in K an eigenvector h_0; then*

$$T'(0)\, h_0 = \mu_0 h_{0,}$$

where $\mu_0 > 1$, and $T'(0)$ does not have in K eigenvectors to which an eigenvalue equals to 1. Then if T is completely continuous, the operator T has one non-zero fixed point in the cone.

Theorem 3.5. *(Deimling[3]) Let be X a Banach space, $K \subset X$ a solid cone, that is, $int(K) \neq \emptyset$, and $T : X \to X$ strongly positive compact linear operator. Then:*
(i) $r(T) > 0$, $r(T)$ is a simple eigenvalue with eigenvector $v \in int(K)$ and

there is not eigenvalue with positive eigenvector.
(ii) if λ is an eigenvalue and $\lambda \neq r(T)$, then $|\lambda| < r(T)$.
(iii) if $S : X \to X$ is bounded linear operator and $Sx \geq Tx$ on K, then $r(S) \geq r(T)$, while $r(S) > r(T)$ if $Sx > Tx$ for $x \in K, x > 0$.

Definition 3.2. (Deimling[3]) Let X, Y be Banach spaces, $J = (\lambda_0 - \delta, \lambda_0 + \delta)$ a real interval, $\Omega \subset X$ a neighborhood of 0 and $F : J \times \Omega \longrightarrow Y$ such that $F(\lambda, 0) = 0$ for all $\lambda \in J$, then $(\lambda_0, 0)$ will be a **bifurcation point** for $F(\lambda, x)$ if

$$(\lambda_0, 0) \in \overline{\{(\lambda, x) \in J \times \Omega; F(\lambda, x) = 0, x \neq 0\}}.$$

Theorem 3.6. *(Bifurcation Theorem, Griffel[9]) Consider the equation $Au = \eta u$, where A is a compact non-linear operator, Fréchet-differentiable at $u = 0$, such that $A0 = 0$. Then:*
(i) if μ_0 is a bifurcation point of $F(\mu, x) = x - \mu Ax$, then μ_0^{-1} is an eigenvalue of the linear operator $A'(0)$.
(ii) if $\mu_0^{-1}$is an eigenvalue of $A'(0)$ with odd multiplicity, then μ_0 is a bifurcation point of $F(\mu, x)$.

Theorem 3.7. ***(Existence Theorem)*** *Let us consider the operator $T : C[0, L] \to C[0, L]$ described by the equation (6), or*

$$Tu(a) = \int_0^L B(a, \zeta) M(\zeta, u(\zeta), \nu(\zeta)) u(\zeta) d\zeta.$$

If $r(T'(0)) \leq 1$, the only solution of equation (4), that is,

$$\lambda(a) = \int_0^L B(a, \zeta) M(\zeta, \lambda(\zeta), \nu(\zeta)) \lambda(\zeta) d\zeta$$

is the trivial solution. Otherwise, if $r(T'(0)) > 1$ there is at least one non-trivial positive solution for this equation.

Proof: We use the same arguments given in Greenhalgh[7]. Suppose $r(T'(0)) \leq 1$ and the equation (4) has a non-trivial positive solution λ^*, that is,

$$\lambda^*(a) = \int_0^L B(a, \zeta) M(\zeta, \lambda^*(\zeta), \nu(\zeta)) \lambda^*(\zeta) d\zeta.$$

Since $\lambda^* > 0$ and $M(\zeta, \lambda, \nu)$ is strictly monotone decreasing for λ we have that

$$\int_0^L B(a,\zeta)\, M(\zeta, \lambda^*(\zeta), \nu(\zeta))\, \lambda^*(\zeta)\, d\zeta < T'(0)\, \lambda^*(a).$$

Since both side of last equation are continuous on compact, then there exists $\varepsilon > 0$ such that

$$\lambda^*(1+\varepsilon) < T'(0)\, \lambda^*.$$

By finite inducing over n we have that

$$\lambda^*(1+\varepsilon)^n < T'(0)^n\, \lambda^*.$$

So

$$\|\lambda^*(1+\varepsilon)^n\| < \|T'(0)^n\, \lambda^*\| \leq \|T'(0)^n\|\, \|\lambda^*\|,$$

and

$$(1+\varepsilon)^n < \|T'(0)^n\|$$

for every $n = 1, 2, 3, \cdots$. Then $r(T'(0)) > 1$, which is an absurd.
Let us suppose that $r(T'(0)) > 1$. Firstly, we will calculate $T'(\infty)$. For every $u \in K$, since

$$T(tu) = \int_0^L B(a,\zeta)\, e^{-t\int_0^\zeta u(s)ds}\, tu(\zeta)\, d\zeta,$$

where $B(a,\zeta)$ is given by equation (5), we will have

$$\lim_{t\to\infty} \frac{T(tu)}{t} = 0,$$

then $T'(\infty) = 0$. Now, we show that T is strongly asymptotically linear with respect to the cone K,

$$\lim_{R\to\infty} \sup_{\|x\|\geq R, x\in K} \frac{\|Tx - T'(\infty)\, x\|}{\|x\|} = \lim_{R\to\infty} \sup_{\|x\|\geq R, x\in K} \frac{\|Tx\|}{\|x\|}.$$

We have

$$\|Tx\| = \sup_{a\in[0,L]} \left| \int_0^L B(a,\zeta)\, e^{-\int_0^\zeta x(s)ds} x(\zeta)\, d\zeta \right| =$$
$$\sup_{a\in[0,L]} \left| \int_0^L B(a,\zeta)\, \tfrac{d}{d\zeta} \left(-e^{-\int_0^\zeta x(s)ds}\right) d\zeta \right| \leq$$
$$m' \int_0^L \tfrac{d}{d\zeta} \left(-e^{-\int_0^\zeta x(s)ds}\right) d\zeta = m' \left[1 - e^{-\int_0^L x(s)ds}\right],$$

where $m' = \sup_{a,\zeta \in [0,L]} |B(a,\zeta)|$. Then

$$\lim_{R\to\infty} \sup_{\|x\|\geq R, x\in K} \frac{\|Tx\|}{\|x\|} \leq \lim_{R\to\infty} \sup_{\|x\|\geq R, x\in K} \frac{m'\left[1 - e^{-\int_0^L x(s)ds}\right]}{\|x\|} = 0,$$

that is, T is strongly asymptotically linear with respect to the cone K, with the strong asymptotic derivative with respect to the cone K equals $T'(\infty) = 0$.

Let us consider in Theorem 3.4 $\mu_0 = r(T'(0))$. Following Theorem 3.5, $r(T'(0))$ is a simple eigenvalue of $T'(0)$ with eigenvector in $int(K)$ and there is not other eigenvalue of $T'(0)$ with positive eigenvector. Obviously, being $T'(0)$ a positive operator, 1 can not be a positive eigenvalue of $T'(0)$ by above argument. Since T is completely continuous, all conditions of the Theorem 3.4 are satisfy, and we conclude that the equation (4) has a non-trivial solution.

Moreover, let us consider $0 < \overline{\mu} < \frac{1}{r(T'(0))}$ such that there is a $\overline{x} \in K, \overline{x} \neq 0$, with $F(\overline{\mu}, \overline{x}) = \overline{x} - \overline{\mu} T\overline{x}$. Then $T\overline{x} = \frac{1}{\overline{\mu}}\overline{x}$ and it follows that $\frac{1}{\overline{\mu}} \leq r(T'(0))$, and this is not possible. So such $\overline{\mu}$ does not exist.

Since $r(T'(0))$ is a simple eigenvalue of $T'(0)$, we have that $\frac{1}{r(T'(0))}$ is a bifurcation point of $F(\mu, x) = x - \mu Tx$ (Theorem 3.6).

Let us suppose now that there exists $\mu^* > \frac{1}{r(T'(0))}$ being a bifurcation point of $F(\mu, x) = x - \mu Tx$, that is, there exists $(\mu_n, x_n) \to (\mu^*, 0)$ when $n \to \infty$, where $x_n \in K\backslash\{0\}$ and $F(\mu_n, x_n) = x_n - \mu_n T x_n = 0$, that is, $Tx_n = \frac{1}{\mu_n} x_n$. Being T Fréchet differentiable at $u = 0$ in the direction of K, we have

$$Tx_n = T(0) + T'(0)x_n + \omega(0, x_n) = \frac{1}{\mu_n} x_n,$$

where $\lim_{n\to\infty} \frac{\|\omega(0,x_n)\|}{\|x_n\|} = 0$. Since $T(0) = 0$ we have

$$T'(0)\frac{x_n}{\|x_n\|} + \frac{\omega(0, x_n)}{\|x_n\|} = \frac{1}{\mu_n}\frac{x_n}{\|x_n\|}.$$

Being $T'(0)$ is a compact operator and $\left(\frac{x_n}{\|x_n\|}\right)$ is a bounded sequence, we can assume that there is $v \in K$ such that

$$\lim_{n\to\infty} \left\{ T'(0)\frac{x_n}{\|x_n\|} \right\} = v.$$

On the one hand,

$$\lim_{n\to\infty} \left\{ T'(0)\left(\frac{x_n}{\|x_n\|}\right) \right\} = v.$$

From the linearity and continuity of $T'(0)$, we have

$$\nu = \lim_{n\to\infty} T'(0)\frac{x_n}{\|x_n\|} = \lim_{n\to\infty} T'(0)\left(\mu_n \frac{1}{\mu_n}\frac{x_n}{\|x_n\|}\right) = \\ T'(0)\left(\lim_{n\to\infty} \mu_n \frac{1}{\mu_n}\frac{x_n}{\|x_n\|}\right) = T'(0)(\mu^*\nu),$$

that is, $T'(0)(\nu) = \frac{1}{\mu^*}\nu$, and $\frac{1}{\mu^*}$ is an eigenvalue of $T'(0)$ with a positive eigenvector, which is not possible by Theorem 3.5. So such μ^* can not exist. Therefore, we have that $\frac{1}{r(T'(0))}$ is a unique bifurcation point of $F(\mu, x) = x - \mu T x$.

4. Discussion and conclusion

A characterization of the basic reproduction number R_0 was done considering fixed point and monotone operators[11,12], and properties regarding to the positive operators and strongly positive operators on cones[3].

We compare our results with those obtained by Greenhalgh[7] and Lopez and Coutinho[14]. Greenhalgh[7] assumed that the contact rate is strictly positive, which is not necessary in our case. Lopez and Coutinho[14] applied Schauder's theorem, which requires that the application acts on a convex set. The definition of convexity given by Griffell[9] has the following geometric meaning: a convex set must contain any line segment joining any two points belonging to it. For the sake of simplicity, let us consider $L = 1$. It is easy to verify that the set $T = C[0, L]^+ \cap \{\varphi; \|\varphi\| = 1\}$ is not convex: From the functions $x(a) = a$, $y(a) = 4a(1-a)$ belonging to T and $z(a) = \frac{1}{2}x(a) + \left(1 - \frac{1}{2}\right)y(a)$, we obtain $\|z\| = \frac{50}{64}$, which shows that z does not belong to T, even that it belongs to the line segment joining points of T, namely, x and y. Remember that the Schauder's theorem establishes the existence of a fixed point with respect to a continuous operator acting on a closed and convex set, which image is contained in a relatively compact subset of the defined domain. When we consider the set $C[0, L]^+ \cap \{\varphi; \|\varphi\| \le 1\}$, a convex set, we can not disregard the possibility that the null function is the fixed point obtained by applying the Schauder's theorem, due to the fact that for the particular operator considered, the image of the zero function is the zero function itself.

With respect to the generalization of the results obtained using positive core to include non-negative core made by Lopez and Coutinho[14], they defined that a set of positive functions is cone if a function has a finite number of points at which the function is zero plus zero function. However, a cone must be a closed set. Taking again $L = 1$ for the same reason given above, and considering the following sequence

$$f_n(a) = \begin{cases} \frac{1}{n}\left[sen\left(4n\pi a\right)+1\right], & 0 \le a \le \frac{1}{2} \\ \frac{2(n-1)}{n}\left(a-\frac{1}{2}\right)+\frac{1}{n}, & \frac{1}{2} \le a \le 1, \end{cases}$$

which belongs to positive function, this sequence converges to function

$$f(a) = \begin{cases} 0, & 0 \le a \le \frac{1}{2} \\ 2\left(a-\frac{1}{2}\right), & \frac{1}{2} \le a \le 1, \end{cases}$$

which does not belong to the set.

The characterization of R_ν as the spectral radius of an operator allows us to assess vaccination strategies having as goal the eradication of the disease. It is possible to introduce vaccination rate in the form $\nu(a) = \nu\theta(a-a_1)\,\theta(a_2-a)$, where ν is a constant vaccination rate and $[a_1, a_2]$ is the age interval of individuals that are vaccinated, and we determine[19,20]: (i) if the vaccination programme is efficient, that is, yields $R_\nu \le 1$, in which case we have $\lambda_\infty \equiv 0$; (ii) the minimum vaccination effort, ν_m such that $R_\nu = 1$; and (iii) the more appropriate vaccinated age interval $[a_1, a_2]$ to control the infection.

In a companion paper[4] we show uniqueness of the non-trivial solution in order to validate R_0 obtained by applying spectral radius as the basic reproduction number. The unique bifurcation value corresponds to the appearance of non-trivial solution corresponding to the endemic level. We also evaluate the basic reproduction number for some functions describing the contact rate. Due to the difficulty and complexity found in the calculation of the spectral radius, we evaluate the upper and lower limits for R_0.

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