Dynamical obstruction to the existence of continuous sub-actions for interval maps with regularly varying property

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November 12, 2019

Abstract

For transformations with regularly varying property, we identify a class of moduli of continuity related to the local behavior of the dynamics near a fixed point, and we prove that this class is not compatible with the existence of continuous sub-actions. The dynamical obstruction is given merely by a local property. As a natural complement, we also deal with the question of the existence of continuous sub-actions focusing on a particular dynamic setting. Applications of both results include interval maps that are expanding outside a neutral fixed point, as Manneville-Pomeau and Farey maps.

Keywords: ergodic optimization, sub-actions, modulus of continuity, non-uniformly expanding dynamics, intermittent maps.

Mathematical subject classification: 26A12, 37E05, 49L25, 49N15.

1 Introduction

For a given dynamical system and a class of real-valued functions, the existence of sub-actions becomes an important tool in the study of the so-called optimizing measures in the theory of ergodic optimization. For a brief general exposure to the sub-actions, we consider the most classical and simple dynamic situation, given by a continuous map $T : X \to X$ acting on a compact metric space X, and a continuous function $f : X \to \mathbb{R}$ (called *potential*). Let $M(X, T)$ denote the set of T-invariant

*Supported by CNPq grant 304792/2017-9.
†Supported by FONDECYT 11130341 and BCH-CONICYT postdoctoral fellowship 74170014.

Borel probability measures on X. As usual the maximum ergodic average is defined as

$$m(f,T) := \max_{\mu \in M(X,T)} \int f \, d\mu.$$

A function $u : X \to \mathbb{R}$ is said to be a *sub-action* for f if it satisfies the cohomological inequality

$$f + u - u \circ T \leq m(f,T). \tag{1}$$

The study of measures μ in $M(X,T)$ that maximize (or minimize) the average $\int_X f d\mu$ gave rise to the ergodic optimization. The existence of sub-actions for a potential f provides relevant information on the set of associated maximizing measures (see [Jen06, Jen18, Gar17] and references therein).

The existence of continuous sub-actions is guaranteed, for instance, when the map is uniformly expanding and the potentials have Hölder modulus of continuity (see [CLT01] for the context of expanding transformations of the circle). For related studies on the existence of sub-actions, see [LT03, LT05, LRR07, GLT09], and see also [Sou03, BraF07, Bra08, Mor09] for results in one-dimensional dynamics.

For transitive expanding dynamics, generic continuous potentials do not admit bounded measurable sub-actions (see [BJ02, Theorem C] and for details [Gar17, Appendix]). Surprisingly there are few cases in the literature about specific examples of non-existence of continuous sub-actions. An example is provided by Morris [Mor07, Proposition 2] in the context of shift spaces.

Our first result highlights a dynamical obstruction to the existence of continuous sub-actions. It seems that Morris [Mor09] was the first to notice this kind of phenomenon for Manneville-Pomeau maps and Hölder modulus of continuity. We show here that his observation holds for a large family of interval maps with a regularly varying property and for general moduli of continuity. Precisely, we identify an associated class of moduli of continuity whose members do not always admit continuous sub-actions (see Theorem 1). In this general approach, the non-existence of continuous sub-actions for certain potentials is an exclusive consequence of a local property of the dynamics, without direct intervention neither the regularity nor the behavior of the dynamics outside a neighborhood of the neutral fixed point in analysis.

As a complement of study, the second result is addressed to the natural question of the existence of sub-actions in this comprehensive scenario. One of the main contributions here is to extend the study of sub-actions to a much larger class of potentials. Although the Lipschitz and Hölder classes are among the most studied regularity classes in ergodic optimization, as might be expected there are more general frameworks considered in this theory, especially inspired by significant potentials in thermodynamic formalism, such as, for example, potentials of summable variation. Furthermore, when results with respect to the existence of sub-actions (as our Theorem 2) reveal potentials and sub-actions having different moduli of continuity, one may start to respond as the regularity of the former affects the regularity of the latter.

Our dynamic setting are specific maps with intermittency on a compact interval. We do not necessarily assume the map to be continuous on the whole domain.

As a matter of fact, for the non-existence result, we suppose that the map is merely measurable outside of a neighborhood of a certain indifferent fixed point. This allows to emphasize the completely local character of the obstruction to the existence of continuous sub-actions, since even an analytical behavior of the dynamics outside of this neighborhood would not affect the conclusion. Moreover, with respect to the existence of sub-actions, we state the result for dynamics that have a discontinuous point, and this could be easily generalized for the case of finitely many discontinuities.

The techniques we follow are inspired by Morris [Mor09] for the non-existence result and by Contreras, Lopes and Thieullen [CLT01] for the existence one. For both theorems, technical issues in general scenarios require that the original approaches be suitably adapted. In fact, as a first difference, both methods were initially developed assuming continuity of the dynamics. It is worth noting that, already in a non-continuous dynamic context, the technique of [CLT01] was successful adjusted to address the existence of Hölder sub-actions either for Hölder potentials that are monotonous in a neighborhood of an indifferent fixed point (see [Sou03]) or for Lipschitz potentials in general (see [Bra08]). It is likely that these methods can be extended to even more general situations: although our non-existence result holds for interval dynamics, we are convinced that such an obstruction must occur in a similar way for multidimensional settings.

We precisely state our results in the following subsections.

1.1 A general non-existence result

Let $[0,1]$ be endowed with the standard metric d given by the absolute value on $\mathbb{R}$. Our dynamical setting will be interval maps $T : [0,1] \to [0,1]$, defined for x close enough to 0 as an invertible function of the form $T(x) := x(1 \pm V(x))$, where for some $\sigma > 0$, the continuous and increasing function $V : [0,+\infty) \to (0,1)$ satisfies

$$\lim_{x\to 0} \frac{V(tx)}{V(x)} = t^{\sigma}, \text{ for all } t > 0. \tag{2}$$

The function V is said to be *regularly varying at* 0 *with index* σ. We do not assume any extra condition on T outside this neighborhood of 0, except the fact that one may apply Birkhoff's Ergodic Theorem, which essentially means that T is supposed to be just Borel measurable on the whole interval.

By a *modulus of continuity*, we mean a continuous and non-decreasing function $\omega : [0,+\infty) \to [0,+\infty)$ satisfying $\lim_{\epsilon\to 0} \omega(\epsilon) = \omega(0) = 0$. Let $\mathcal{M}$ denote the family of concave modulus of continuity. For a given $\omega \in \mathcal{M}$, we denote by $\mathscr{C}_\omega$ the space of functions $\varphi : [0,1] \to \mathbb{R}$ with a multiple of ω as modulus of continuity: $|\varphi(x) - \varphi(y)| \le C\omega(d(x,y))$ for some constant $C > 0$, for all $x, y \in [0,1]$.

Theorem 1. *Let $T : [0,1] \to [0,1]$ be a Borel measurable map such that, for x close to 0, T is invertible and has the form $T(x) := x(1 \pm V(x))$, where the continuous and increasing function $V : [0,+\infty) \to (0,1)$ is regularly varying at 0 with index $\sigma > 0$. Suppose that $\omega \in \mathcal{M}$ satisfies*

$$\liminf_{x\to 0} \frac{\omega(x)}{V(x)} > 0. \tag{3}$$

Then there exists a function $f \in \mathscr{C}_\omega$*, with* $m(f,T) = \int f\, d\delta_0 = f(0)$*, that does not admit continuous sub-action.*

The main novelty of this result is to exhibit condition (3) as an obstruction to the existence of continuous sub-actions. This local condition near the indifferent fixed point can be easily checked when dealing with Maneville-Pomeau maps defined on the circle and potentials with Hölder modulus of continuity. Since property (3) is satisfied for more general moduli of continuity (see Corollary 1 and Corollary 2), it is not a surprise that one may extend Morris' result [Mor09, Theorem 2 (b)].

An immediate question is whether the opposite condition, that is, a null limit inferior would be sufficient to ensure existence. As a complement of discussion, in the following subsection, we state a result of existence: by considering certain maps with an indifferent fixed point and a stronger assumption than a null limit inferior, we show that sub-actions do exist and we highlight their associated regularity.

1.2 On the existence of continuous sub-actions

For the study of non-existence of sub-actions, we have just looked at the behavior of the map in a neighborhood of an indifferent fixed point, regardless of its behavior outside this special region, except perhaps that Birkhoff's Ergodic Theorem can be applied. The analysis of the existence of sub-actions involves nevertheless the study of an entire dynamical system and is therefore a global issue. We thus fix a particular class of dynamics with two inverse branches and with a neutral fixed point, for which not only we guarantee the existence of sub-actions for potentials with various moduli of continuity, but we also point out the associated regularity of these sub-actions. Similar arguments are feasible for intermittent dynamics with more inverse branches.

Let $\mathscr{J}$ be the class formed by piecewise two-to-one transformations $T : [0,1] \to [0,1]$ with the following properties. Each T has exactly one discontinuity $c \in (0,1)$ such that $\lim_{x\to c^-} T(x) = 1$ and $\lim_{x\to c^+} T(x) = 0$. Moreover, T takes the form $T(x) := x(1+V(x))$ on $[0,c]$, where for some $\sigma > 0$, the continuous and increasing function $V : [0,+\infty) \to [0,1)$ is regularly varying with index σ (recall (2)). Finally, we assume that there is $\lambda > 1$ such that for all $x,y \in (c,1]$, $d(T(x),T(y)) \geq \lambda d(x,y)$.

For a given function V as above, we consider a concave modulus of continuity $\omega \in \mathcal{M}$ satisfying the following assumption:

[A] There exist constants $\gamma > 0$, $\xi_0 > 1$ and $\eta_0 \in (0,1)$ such that

$$\frac{\omega(\xi h)}{V(\xi h)} \geq \xi^\gamma \frac{\omega(h)}{V(h)}, \qquad \forall\, h \in (0,\eta_0),\ \forall\, \xi \in (1,\xi_0].$$

One can easily verify that, for V and ω fulfilling Assumption A,

$$\lim_{h\to 0} \frac{\omega(h)}{V(h)} = 0. \tag{4}$$

The converse statement is not satisfied in general, see Remark 2.

From Assumption A, we define a modulus of continuity $\Omega \in \mathcal{M}$ so that potentials with modulus of continuity ω admit sub-actions with modulus of continuity Ω.

Defining a continuous non-decreasing concave modulus of continuity

For V and ω fulfilling Assumption A, let $\vartheta_0 : [0, \infty) \to [0, \infty)$ be the continuous function defined as

$$\vartheta_0(x) := \begin{cases} \frac{\omega(x)}{V(x)}, & x > 0, \\ 0, & x = 0, \end{cases} \tag{5}$$

and let $\vartheta_1 : [0, \infty) \to [0, \infty)$ be the continuous non-decreasing function given as

$$\vartheta_1(x) = \begin{cases} \max_{0 \le y \le x} \vartheta_0(y), & 0 \le x \le 1, \\ \max_{[0,1]} \vartheta_0, & x \ge 1, \end{cases} \tag{6}$$

Denote then ϑ_1^* the *concave conjugate Legendre transform* of ϑ_1, defined as

$$\vartheta_1^*(x) = \min_{y \in [0,\infty)} [xy - \vartheta_1(y)], \quad \forall\, x \ge 0. \tag{7}$$

By the very definition, ϑ_1^* is concave, non-decreasing and continuous on $(0, \infty)$. To see that ϑ^* is continuous at 0, note that $\vartheta_1^*(0) = -\max_{[0,1]} \vartheta_0$ and $\vartheta_1^*(0) \le \vartheta_1^*(\epsilon) \le \epsilon - \vartheta_1(1) = \epsilon + \vartheta_1^*(0)$. For the continuous concave non-decreasing function

$$\vartheta_2(x) = \min\{\vartheta_1^*(x), \vartheta_1^*(1)\}, \tag{8}$$

a similar reasoning shows that its *concave conjugate Legendre transform*,

$$\vartheta_2^*(x) = \min_{y \in [0,\infty)} [xy - \vartheta_2(y)], \quad \forall\, x \ge 0, \tag{9}$$

is also a continuous concave non-decreasing function. Moreover $\vartheta_0(x) \le \vartheta_1(x) \le \vartheta_2^*(x)$ for all $x \in [0, 1]$. Actually, ϑ_2^* is the *smallest* concave function that lies above ϑ_1 on $[0, 1]$. Note that $\vartheta_2^*(0) = -\vartheta_1^*(1)$.

We have obtained a function $\Omega := \vartheta_2^* + \vartheta_1^*(1)$ that belongs to $\mathcal{M}$.

Theorem 2. *Let $T : [0, 1] \to [0, 1]$ be a map in $\mathscr{J}$ with discontinuity $c \in (0, 1)$ such that $T(x) = x(1 + V(x))$ for all $x \in [0, c]$, where V is regularly varying at 0. Let ω be a modulus of continuity in $\mathcal{M}$ for which Assumption A holds. Then, every $f \in \mathscr{C}_\omega$ admits continuous sub-actions in $\mathscr{C}_\Omega$, where Ω is the modulus defined by the process* (5)– (9).

Assumption A is satisfied whenever we consider a suitable couple formed by a dynamics with a particular behavior near to an indifferent fixed point and a potential with a convenient modulus of continuity (see Subsection 1.4). This includes several cases of of Manneville-Pomeau type maps and Hölder potentials previously studied (see [Sou03], [Bra08, Theorem 4.1] and [Mor09, Theorem 1]). Thanks to the fact that Assumption A charges jointly the dynamical behavior and the regularity of the potential, we can guarantee the existence of continuous sub-actions in a general way, and even better we are able to exhibit the regularity of such sub-actions in general terms as well. At the best of our knowledge, there are no previous works at such a level of generality about the regularity of potentials and sub-actions. Finally, the technique used to prove this result follows as in [CLT01, Proposition 11], and essentially consists in taking into account as candidate to sub-action a function defined analogously to sub-solutions of the Hamilton-Jacobi equation in the Lagrangian theory.

The rest of the paper is organized as follows. In the following subsections, we provide examples of applications of Theorem 1 as well as examples of maps in $\mathscr{J}$ for which Assumption A holds. We gather in Section 2 preliminary results for the Theorem 1 and its proof is presented in Section 3. The proof of Theorem 2 is detailed in Section 4.

1.3 Examples of applications of Theorem 1

A trivial example of elements of $\mathcal{M}$ are the functions $\omega(h) = Ch^\alpha$ with $\alpha \in (0,1]$, which describe α-Hölder continuous functions. The family $\mathcal{M}$ also includes the minimal concave majorants ω_0 of non-decreasing subadditive functions $\omega : [0,+\infty) \to [0,+\infty)$, with $\lim_{h\to 0}\omega(h) = \omega(0) = 0$. Following [Med01] these concave majorants are infinitely differentiable on $(0,+\infty)$. Moreover, if $\omega'(0) < \infty$ then $\omega_0(h) = \omega'(0)h$ on some neighborhood of 0.

Another example of members of $\mathcal{M}$ are the functions $\omega(h) = h\left(\log\left(\frac{1}{h^k}\right) + 1\right)$ (for $k > 0$ and h small enough), which describe locally Hölder continuous functions. A more general class of modulus of continuity in $\mathcal{M}$ is defined as follows: for $0 \le \alpha < 1$ and $\beta \ge 0$ with $\alpha + \beta > 0$, consider $\omega_{\alpha,\beta} : [0,+\infty) \to [0,+\infty)$ given as

$$\omega_{\alpha,\beta}(h) := \begin{cases} h^\alpha(-\log h)^{-\beta}, & 0 < h < h_0, \\ h_0^\alpha(-\log h_0)^{-\beta}, & h \ge h_0, \end{cases} \tag{10}$$

where h_0 is taken small enough so that $\omega_{\alpha,\beta}$ is concave. Note that $\omega_{\alpha,0}$ is reduced to the Hölder continuity, and $\omega_{0,\beta}$ for $\beta > 0$ determines a class that is larger than local Hölder continuity – see property (11).

Remark 1. *Let $\omega_{\alpha,\beta} : [0,+\infty) \to [0,+\infty)$ be the modulus of continuity defined in (10). It is easy to see that for every $\epsilon > \alpha$,*

$$\lim_{h\to 0} \frac{\omega_{\alpha,\beta}(h)}{h^\epsilon} = +\infty. \tag{11}$$

Note that $\mathcal{M}$ includes many functions besides the previous examples for the simple fact that for each pair $\omega_1, \omega_2 \in \mathcal{M}$, we have $\omega_1 \circ \omega_2 \in \mathcal{M}$. However, we are interested in a class of modulus of continuity whose behavior near 0 satisfies condition (3), which is dictated by the dynamics.

Let $V : [0,+\infty) \to (0,1)$ be a continuous and increasing function which is regularly varying at 0 with index $\sigma > 0$. Consider the modulus of continuity $\omega_{\alpha,\beta}$ defined in (10) with $0 \le \alpha < \min\{\sigma, 1\}$ and $\beta \ge 0$ such that $\alpha + \beta > 0$. Thanks to property (11), the condition $\liminf_{x\to 0} \frac{\omega_{\alpha,\beta}(x)}{V(x)} > 0$ holds whenever $\liminf_{x\to 0} \frac{x^\sigma}{V(x)} > 0$. Therefore, we obtain the following corollary.

Corollary 1. *Let $T : [0,1] \to [0,1]$ be a measurable interval map such that, in a neighborhood of the origin, T is invertible and has the form $T(x) = x(1 \pm V(x))$,* *where $V : [0,+\infty) \to (0,1)$ is a continuous, increasing and regularly varying function at 0 with index $\sigma > 0$ that satisfies $\liminf_{x\to 0} \frac{x^\sigma}{V(x)} > 0$. Let $\omega_{\alpha,\beta}(x)$ be defined as in (10). Then, for $\alpha = \sigma$ and $\beta = 0$ or for $0 \le \alpha < \min\{\sigma, 1\}$ and $\beta \ge 0$ with $\alpha + \beta > 0$, there is a function $f \in \mathscr{C}_{\omega_{\alpha,\beta}}$ which does not admit continuous sub-action.*

Examples of this kind of dynamics include Manneville-Pomeau interval map: for a given $s > 0$, $T_s : [0,1] \to [0,1]$ is defined as

$$T_s(x) := x(1 + x^s) \mod 1.$$

Note that $T_s'(x) \ge 1$ for all x with equality only at $x = 0$. Let c be the unique point in $(0,1)$ such that $T_s(c) = 1$ and $T_s|_{[0,c]} : [0,c] \to [0,1]$ is a diffeomorphism. Let us denote $U_s : [0,1] \to [0,c]$ the corresponding inverse branch. Note that $U_s'(x) \le 1$ for all x and U_s is concave, so that $cx \le U_s(x) \le x$. If we write $U_s(x) = x(1 - V(x))$, then $0 \le V(x) \le 1 - c$. Moreover, by using the identity $T_s \circ U_s = \mathrm{Id}$, we have $V(x) = x^s(1 - V(x))^{s+1}$ for all $x \ne 0$. Hence $\lim_{x\to 0} V(x) = 0$,

$$\lim_{x\to 0} \frac{V(tx)}{V(x)} = \lim_{x\to 0} t^s \left(\frac{1 - V(tx)}{1 - V(x)}\right)^{s+1} = t^s \text{ and } \lim_{x\to 0} \frac{x^s}{V(x)} = \lim_{x\to 0} \frac{1}{(1 - V(x))^{s+1}} = 1.$$

It is not difficult to argue that V is increasing. Then Corollary 1 applies to U_s as well.

Corollary 2. *Let $s \in (0,1)$ and $T_s(x) = x + x^{1+s}$ for x close enough to 0. Denote U_s the corresponding inverse branch. Let $\omega_{\alpha,\beta}(x)$ be defined as in (10), where either $\alpha \in [0, \min\{s,1\})$ and $\beta \ge 0$ with $\alpha + \beta > 0$ or $\alpha = s$ and $\beta = 0$. Then there are functions $f, g \in \mathscr{C}_{\omega_{\alpha,\beta}}$ which do not admit continuous sub-actions with respect to T_s and U_s, respectively.*

The above corollary is an extension of Morris' result [Mor09], which established that for $T_s(x) = x + x^{1+s} \mod 1$, there is $f \in \mathscr{C}_{\omega_{s,0}}$ that does not admit continuous sub-action.

Another one-parameter family of maps on the interval $[0,1]$ with indifferent fixed point at $x = 0$ is defined as follows: for $\rho \in (0,1]$, let $F_\rho : [0,1] \to [0,1]$ be given as

$$F_\rho(x) = \begin{cases} \frac{x}{(1-x^\rho)^{1/\rho}} & \text{if } 0 \le x \le 2^{-1/\rho} \\ \frac{(1-x^\rho)^{1/\rho}}{x} & \text{if } 2^{-1/\rho} < x \le 1. \end{cases}$$

Note that Farey map corresponds to the special case $\rho = 1$. For any $\rho \in (0,1]$, the first inverse branch has an explicit expression: $G_\rho(x) = \frac{x}{(1+x^\rho)^{1/\rho}}$. Note then that the functions $V(x) = \frac{1}{(1-x^\rho)^{1/\rho}} - 1$ and $W(x) = 1 - \frac{1}{(1+x^\rho)^{1/\rho}}$ are continuous, increasing, regularly varying with index ρ, and satisfy $\lim_{x\to 0} \frac{x^\rho}{V(x)} = \lim_{x\to 0} \frac{x^\rho}{W(x)} = \rho > 0$. Clearly, $F_\rho(x) = x(1 + V(x))$ and $G_\rho(x) = x(1 - W(x))$.

Corollary 3. *For $\rho \in (0,1]$, let $F_\rho(x) = \frac{x}{(1-x^\rho)^{1/\rho}}$ and $G_\rho(x) = \frac{x}{(1+x^\rho)^{1/\rho}}$ for x close to 0. Let $\omega_{\alpha,\beta}(x)$ be defined as in (10), where either $\alpha \in [0, \rho)$ and $\beta \ge 0$ with $\alpha + \beta > 0$ or $\alpha = \rho$ and $\beta = 0$. Then there are functions $f, g \in \mathscr{C}_{\omega_{\alpha,\beta}}$ which do not admit continuous sub-actions with respect to F_ρ and G_ρ, respectively.*

As a final example of application of Theorem 1, let

$$T(x) = \begin{cases} 0 & \text{if } x = 0 \\ x + \frac{2}{\log 2}\, x^2 |\log x| & \text{if } 0 < x \leq 1/2 \\ 2x - 1 & \text{if } 1/2 < x \leq 1. \end{cases}$$

Note that $V(x) = \frac{2}{\log 2}\, x|\log x|$, $x > 0$, is a regularly varying function with index 1. For $k > 0$, the concave modulus of continuity defined for h sufficiently small as $\omega(h) = h\left(\log\left(\frac{1}{h^k}\right) + 1\right)$ clearly satisfies $\lim_{x \to 0} \frac{\omega(x)}{V(x)} = \frac{2k}{\log 2} > 0$. Recalling that such a modulus describes locally Hölder continuous functions, we have the following result.

Corollary 4. *With respect to a dynamics that behaves as $T(x) = x + \frac{2}{\log 2}\, x^2 |\log x|$ for $x > 0$ sufficiently small, there exist locally Hölder continuous functions that do not admit continuous sub-actions.*

1.4 Examples of maps in $\mathscr{J}$ for which Assumption A holds

A prototypical example in $\mathscr{J}$ is the Manneville-Pomeau interval map $T_s(x) := x(1 + x^s) \mod 1$, with $s \in (0, 1)$. Consider the class of modulus of continuity $\omega_{\alpha,\beta}$ as in (10). For $s < \alpha < 1$, Condition A follows immediately with $\gamma = \alpha - s$: for h sufficiently small,

$$\frac{\omega_{\alpha,\beta}(\xi h)}{(\xi h)^s} \geq \xi^{\alpha - s} \frac{h^\alpha (-\log h)^{-\beta}}{h^s} = \xi^{\alpha - s} \frac{\omega_{\alpha,\beta}(h)}{h^s}.$$

Another interesting family of interval maps in $\mathscr{J}$ is given by $H_\rho : [0, 1] \to [0, 1]$, for $\rho \in (0, 1]$, defined as

$$H_\rho(x) = \begin{cases} \frac{x}{(1 - x^\rho)^{1/\rho}} & \text{if } 0 \leq x \leq 2^{-1/\rho}, \\ \frac{2^{1/\rho} x - 1}{2^{1/\rho} - 1} & \text{if } 2^{-1/\rho} < x \leq 1. \end{cases}$$

The function $V(h) = \frac{1}{(1 - h^\rho)^{1/\rho}} - 1$ is continuous, increasing, regularly varying with index ρ. For $\rho < \alpha < 1$, we have that $\omega_{\alpha,\beta}$ and V satisfy Condition A, since

$$\frac{\omega_{\alpha,\beta}(\xi h)}{V(\xi h)} \frac{V(h)}{\omega_{\alpha,\beta}(h)} = \xi^\alpha \left(\frac{\log(\xi h)}{\log h} \right)^{-\beta} \frac{V(h)}{V(\xi h)}$$

implies that $\lim_{h \to 0} \frac{\omega_{\alpha,\beta}(\xi h)}{V(\xi h)} \frac{V(h)}{\omega_{\alpha,\beta}(h)} = \xi^\alpha \lim_{h \to 0} \frac{V(h)}{V(\xi h)} = \xi^{\alpha - \rho}$. As another example, following [Hol05], consider a family defined for $0 < \tau < 1$ and $\theta > 0$ as

$$T_{\tau,\theta}(x) = \begin{cases} x + \frac{2^\tau}{(\log 2)^{\theta + 1}} x^{1 + \tau} |\log x|^{\theta + 1} & \text{if } 0 \leq x \leq 1/2, \\ 2x - 1 & \text{if } 1/2 < x \leq 1. \end{cases}$$

In this case, the function $V_{\tau,\theta}(h) = \frac{2^\tau}{(\log 2)^{\theta + 1}} h^\tau |\log h|^{\theta + 1}$ is regularly varying with index τ. Condition A is satisfied, for instance, with the modulus of continuity $\omega_k(h) = h\left(\log\left(\frac{1}{h^k}\right) + 1\right)$ for $k \geq 1$ and h sufficiently small. Indeed, one has

$$\frac{\omega_k(\xi h)}{V_{\tau,\theta}(\xi h)} \frac{V_{\tau,\theta}(h)}{\omega_k(h)} = \xi^{1 - \tau} \left| \frac{\log h}{\log(\xi h)} \right|^{\theta + 1} \frac{1 - k \log(\xi h)}{1 - k \log h},$$

so that $\lim_{h\to 0} \frac{\omega_k(\xi h)}{V_{\tau,\theta}(\xi h)} \frac{V_{\tau,\theta}(h)}{\omega_k(h)} = \xi^{1-\tau} \lim_{h\to 0} \frac{1-k\log(\xi h)}{1-k\log h} = \xi^{1-\tau}$.

Remark 2. *[Assumption A is more restricted than the limit* (4)*.] For $\theta > 0$ and $k \geq 1$, consider $T_{1,\theta}$ and ω_k as above. It is easy to see that*

$$\frac{\omega_k(h)}{V_{1,\theta}(h)} \to 0 \qquad \textit{as } h \to 0.$$

However, from $\frac{\omega_k(\xi h)}{V_{1,\theta}(\xi h)} \frac{V_{1,\theta}(h)}{\omega_k(h)} = \left|\frac{\log h}{\log(\xi h)}\right|^{\theta+1} \frac{1-k\log(\xi h)}{1-k\log h}$, *we get*

$$\lim_{h\to 0} \frac{\omega_k(\xi h)}{V_{1,\theta}(\xi h)} \frac{V_{1,\theta}(h)}{\omega_k(h)} = 1.$$

Hence, property (4) *is satisfied, however Condition A fails.*

2 Preliminaries for the proof of Theorem 1

2.1 Basic facts about modulus of continuity

Recall that d stands for the usual distance on $\mathbb{R}$ given by the absolute value and $\mathcal{M}$ denotes the family of concave modulus of continuity. Note that, given a non-identically null $\omega \in \mathcal{M}$, then $([0,1], \omega\circ d)$ is a metric space. Indeed, the subadditivity of ω follows from its concavity and thus, since ω is non-decreasing, we obtain the triangle inequality: $\omega(d(x,y)) \leq \omega(d(x,z)) + \omega(d(z,y))$, for all x, y, z. In particular, a function $\varphi : [0,1] \to \mathbb{R}$ with modulus of continuity $\omega \in \mathcal{M}$ is nothing else than a Lipschitz function with respect to the metric $\omega \circ d$.

We will use the following property.

Lemma 1. *Let $\omega \in \mathcal{M}$. For any positive constant χ, we have*

$$\frac{\chi}{1+\chi}\omega(h) \leq \omega(\chi h) \leq (\chi+1)\omega(h).$$

Proof. Since ω is subadditive, we have for all positive integer $n \geq 1$, $\omega(nh) \leq n\omega(h)$. For a positive constant χ, by monotonicity of ω, we see that $\omega(\chi h) \leq \omega(\lceil\chi\rceil h) \leq \lceil\chi\rceil\omega(h) \leq (\chi+1)\omega(h)$, where $\lceil\cdot\rceil$ denotes the ceiling function. Then, we also obtain $\omega(\chi h) \geq \frac{1}{\frac{1}{\chi}+1}\omega(h) = \frac{\chi}{1+\chi}\omega(h)$. □

2.2 Local behavior near a fixed point

Given $\sigma > 0$, a measurable function $V : [0,+\infty) \to (0,+\infty)$ is said to be *regularly varying at* 0 *with index* σ if condition (2) holds. A regularly varying function can be represented in the form $V(x) = x^\sigma \mathcal{V}(x)$, where the function $\mathcal{V}$ satisfies $\lim_{x\to 0} \frac{\mathcal{V}(tx)}{\mathcal{V}(x)} = 1$, for all $t > 0$. Similarly a measurable function $V : [0,+\infty) \to (0,+\infty)$ is *regularly varying at* ∞ *with index* $\sigma \in \mathbb{R}$ if the function $x \mapsto V(\frac{1}{x})$ is regularly varying at 0. For properties of regularly varying functions, we refer to [Sen76] and [Aar97]. See also [Kar33] for details concerning the original literature.

Recall that near to origin the dynamics is supposed invertible and defined as $T(x) = x(1 \pm V(x))$. Let $(w_n)_{n=0}^{+\infty} \subset [0,1]$ be a sequence of points obtained by choosing w_0 close enough to 0 and by defining $w_{n+1} = T^{\mp 1}(w_n)$, $n \geq 0$. In clear terms, for $x \mapsto x(1+V(x))$ we take pre-images, and for $x \mapsto x(1-V(x))$ we consider future iterates. Note that in both cases $w_n \to 0$ as $n \to \infty$. A sequence of iteration times will also play a central role in our construction. More precisely, let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers such that for some $\gamma \in (0,1)$,

$$\lim_{k\to\infty} \frac{n_k}{n_{k+1}} = \gamma. \tag{12}$$

The study of the behavior close to 0 can be done in a similar way for both $x \mapsto x(1 + V(x))$ and $x \mapsto x(1 - V(x))$. From now on in this subsection, we look at the case $T(x) = x(1-V(x))$. We will point out in the end similarities and particularities to the other case

We write $\alpha_j \sim \beta_j$ whenever $\frac{\alpha_j}{\beta_j} \to 1$ as $j \to \infty$. The next lemma summarizes the main properties concerning the asymptotic behavior of the sequences $(w_n = T(w_{n-1}))$ and (n_k).

Lemma 2. *The following properties hold*

(i)

$$w_n \sim \frac{1}{\sigma^{1/\sigma} b(n)}, \qquad \textit{where } b^{-1}(x) := \frac{1}{V(\frac{1}{x})}; \tag{13}$$

(ii)

$$d(w_n, w_{n+1}) \sim \frac{1}{\sigma^{1+1/\sigma}} \frac{1}{nb(n)}; \tag{14}$$

(iii)

$$\frac{n_k}{n_{k+1}} \sim \gamma^{1+1/\sigma} \frac{b(n_{k+1})}{b(n_k)}. \tag{15}$$

Proof. To verify Part (iii), we first note that $\frac{b^{-1}(tx)}{b^{-1}(x)} = \frac{V(1/x)}{V(1/tx)} \to \frac{1}{(1/t)^\sigma} = t^\sigma$ as $x \to \infty$, which means that b^{-1} is regularly varying at ∞ with index σ. Hence, its inverse, the increasing function b, is regularly varying at ∞ with index $1/\sigma$ (for details, see [Sen76]). Thus, since the sequence of positive integers $(n_k)_{k\geq 1}$ satisfies $\lim_{k\to\infty} \frac{n_k}{n_{k+1}} = \gamma$, then $(b(n_k))_{k\geq 1}$ verifies $\lim_{k\to\infty} \frac{b(n_k)}{b(n_{k+1})} = \gamma^{1/\sigma}$. Indeed, for every $\varepsilon > 0$

$$\limsup_k \frac{b(n_k)}{b(n_{k+1})} \leq \limsup_k \frac{b(n_{k+1}(\gamma+\varepsilon))}{b(n_{k+1})} = (\gamma+\varepsilon)^{1/\sigma},$$

and

$$(\gamma-\varepsilon)^{1/\sigma} = \liminf_k \frac{b(n_{k+1}(\gamma-\varepsilon))}{b(n_{k+1})} \geq \liminf_k \frac{b(n_k)}{b(n_{k+1})}.$$

Then, $\frac{n_k b(n_k)}{n_{k+1} b(n_{k+1})} \to \gamma^{1+1/\sigma}$ as $k \to \infty$.

Part (i) follows from [Aar97, Lemma 4.8.6] which is deduced using that

$$b^{-1}\Big(\frac{1}{w_n}\Big) \sim n\sigma. \tag{16}$$

The asymptotic equivalence (16) implies that $V(w_n) = 1/b^{-1}\big(\frac{1}{w_n}\big) \sim \frac{1}{n\sigma}$, so it follows that $d(w_n, w_{n+1}) = w_n V(w_n) \sim \frac{1}{\sigma^{1+1/\sigma}} \frac{1}{nb(n)}$ and therefore Part (ii) holds. □

Remark 3. *Since b is a continuous and increasing function and since we consider the standard metric on $\mathbb{R}$, by the asymptotic equivalence (14), there exists a constant $C_0 > 1$ such that for every $i \le j$,*

$$(j-i)C_0^{-1}\frac{1}{\sigma^{1+1/\sigma}}\frac{1}{j\,b(j)} \le d(w_i, w_j) \le (j-i)C_0\frac{1}{\sigma^{1+1/\sigma}}\frac{1}{i\,b(i)}. \tag{17}$$

The next lemma provides us estimates on the cardinality of future iterates that stay within suitable intervals.

Lemma 3. *Let us consider $(w_{n_k})_{k=1}^{+\infty}$ a subsequence of $(w_n)_{n=0}^{+\infty}$, where $(n_k)_{k\ge1}$ is an increasing sequence satisfying (12) and $T^{n_k-n_{k-1}}(w_{n_{k-1}}) = w_{n_k}$. For $k \ge 1$, denote*

$$R_k := \frac{1}{3C_0^3}\frac{n_{k-1}b(n_{k-1})}{n_k b(n_k)} d(w_{n_k}, w_{n_{k-1}}).$$

Then, for $z \in [w_{n_k} + R_k, w_{n_{k-1}}]$ and k large enough,

$$\begin{aligned}\#\big\{0 \le j < n_k - n_{k-1} : R_k \le d(T^j(z), w_{n_k}) \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}})\big\} \ge \\ \ge C_1 n_{k-1}b(n_{k-1})d(w_{n_k}, w_{n_{k-1}}),\end{aligned}$$

where $C_1 := \frac{1}{4}(C_0^{-1} - C_0^{-2})\sigma^{1+1/\sigma} > 0$. In particular, there is $C_2 > 0$ such that, for k sufficiently large,

$$\#\big\{0 \le j < n_k - n_{k-1} : R_k \le d(T^j(w_{n_{k-1}}), w_{n_k}) \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}})\big\} \ge \frac{C_2}{V(w_{n_k})}.$$

Proof. Let $\ell \ge 1$ be such that $w_{n_{k-1}+\ell} < z \le w_{n_{k-1}+(\ell-1)}$. Note that a nonnegative integer j such that

$$R_k \le d(w_{n_{k-1}+\ell+j}, w_{n_k}) \quad \text{and} \quad d(w_{n_{k-1}+(\ell-1)+j}, w_{n_k}) \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}}) \tag{18}$$

belongs to $\big\{j : R_k \le d(T^j(z), w_{n_k}) \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}})\big\}$. Moreover, thanks to (17), any $j \ge 0$ such that

$$\begin{aligned}&R_k \le (n_k - n_{k-1} - \ell - j)C_0^{-1}\frac{1}{\sigma^{1+1/\sigma}}\frac{1}{n_k b(n_k)} \quad \text{and} \\ &\qquad (n_k - n_{k-1} - (\ell-1) - j)C_0\frac{1}{\sigma^{1+1/\sigma}}\frac{1}{n_{k-1}b(n_{k-1})} \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}})\end{aligned} \tag{19}$$

satisfies (18). Denoting $\kappa := n_k - n_{k-1} - \ell$, there are exactly

$$\lfloor \kappa - C_0\sigma^{1+1/\sigma}n_k b(n_k)R_k \rfloor - \lceil \kappa + 1 - \frac{1}{3}C_0^{-1}\sigma^{1+1/\sigma}n_{k-1}b(n_{k-1})d(w_{n_k}, w_{n_{k-1}})\rceil + 1$$

nonnegative integers j that fulfill (19). Therefore, we have

$$\begin{aligned}\#\{j : R_k \le d(T^j(z), w_{n_k}) \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}})\} \ge \\ &\ge \frac{1}{3}C_0^{-1}\sigma^{1+1/\sigma}n_{k-1}b(n_{k-1})d(w_{n_k}, w_{n_{k-1}}) - C_0\sigma^{1+1/\sigma}n_k b(n_k)R_k - 2 \\ &= \frac{1}{3}(C_0^{-1} - C_0^{-2})\sigma^{1+1/\sigma}n_{k-1}b(n_{k-1})d(w_{n_k}, w_{n_{k-1}}) - 2.\end{aligned}$$

Note that, from Remark 3 and Lemma 2, as $k \to \infty$

$$\sigma^{1+1/\sigma}n_{k-1}b(n_{k-1})d(w_{n_k}, w_{n_{k-1}}) \ge C_0^{-1}n_k\Big(1 - \frac{n_{k-1}}{n_k}\Big)\frac{n_{k-1}b(n_{k-1})}{n_k b(n_k)} \to \infty.$$

Hence, ignoring at most finitely many initial terms of (n_k) if necessary, we obtain

$$\#\{j : R_k \le d(T^j(z), w_{n_k}) \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}})\} \ge C_1 n_{k-1}b(n_{k-1})d(w_{n_k}, w_{n_{k-1}}).$$

In particular, for $z = w_{n_{k-1}}$, from (17) we have

$$\begin{aligned}d(w_{n_k}, w_{n_{k-1}})^\sigma \#\{j : R_k \le d(T^j(w_{n_{k-1}}), w_{n_k}) \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}})\} \ge \\ &\ge C_1 d(w_{n_k}, w_{n_{k-1}})^{\sigma+1}n_{k-1}b(n_{k-1}) \\ &\ge C_1\Big[(n_k - n_{k-1})C_0^{-1}\frac{1}{\sigma^{1+1/\sigma}}\frac{1}{n_k b(n_k)}\Big]^{\sigma+1}n_{k-1}b(n_{k-1}) \\ &= \frac{C_1}{C_0^{\sigma+1}\sigma^{(\sigma+1)^2/\sigma}}\Big(1 - \frac{n_{k-1}}{n_k}\Big)^{\sigma+1}\frac{n_{k-1}b(n_{k-1})}{n_k b(n_k)}\frac{n_k}{b(n_k)^\sigma}.\end{aligned}$$

Note now that, from (13) and (16),

$$\frac{n}{b(n)^\sigma} \sim \sigma n w_n^\sigma \sim \frac{w_n^\sigma}{V(w_n)}.$$

Denote thus $C_1' := \frac{1}{2}\frac{C_1}{C_0^{\sigma+1}\sigma^{(\sigma+1)^2/\sigma}}(1-\gamma)^{\sigma+1}\gamma^{1+1/\sigma} > 0$. Following the previous estimate and the above asymptotic equivalence, from (12) and (15), for k large enough,

$$\#\{j : R_k \le d(T^j(w_{n_{k-1}}), w_{n_k}) \le \frac{1}{3}d(w_{n_{k-1}}, w_{n_k})\} \ge \frac{C_1'}{V(w_{n_k})}\frac{w_{n_k}^\sigma}{d(w_{n_k}, w_{n_{k-1}})^\sigma}.$$

Note now that, from Remark 3 and Lemma 2, for k sufficiently large,

$$d(w_{n_k}, w_{n_{k-1}}) \le \left(1 - \frac{n_{k-1}}{n_k}\right)C_0\frac{1}{\sigma}\frac{n_k b(n_k)}{n_{k-1}b(n_{k-1})}\frac{1}{\sigma^{1/\sigma}b(n_k)} \le 2(1-\gamma)C_0\frac{1}{\sigma}\frac{1}{\gamma^{1+1/\sigma}}w_{n_k}.$$

We obtain thus a constant $C_1'' > 0$ such that $\frac{w_{n_k}^\sigma}{d(w_{n_k}, w_{n_{k-1}})^\sigma} \ge C_1''$ whenever k is large enough, which completes the proof with $C_2 := C_1'C_1''$. □

Comments on local behavior near to origin for $x \mapsto x(1+V(x))$. In this case, we deal with a sequence of past iterates ($w_n = T(w_{n+1})$), where $T(x) = x(1+V(x))$ in a neighborhood of 0. It is not a surprise that asymptotic equivalences are exactly the same as in the statement of Lemma 2. One may show easily such a fact with minor adjustments in the proof and an appropriate version of [Aar97, Lemma 4.8.6], which can be obtained repeating almost verbatim original arguments. The statement of Lemma 3 for this case obviously requires contextual changes since the sequences are now related by $T^{n_k-n_{k-1}}(w_{n_k}) = w_{n_{k-1}}$. If one follows the same lines of proof, one will conclude that for $z \in [w_{n_k}, w_{n_{k-1}} - R_k]$ and k large enough,

$$\#\big\{0 \le j < n_k - n_{k-1} : R_k \le d(T^j(z), w_{n_{k-1}}) \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}})\big\} \ge \\ \ge C_1 n_{k-1} b(n_{k-1}) d(w_{n_k}, w_{n_{k-1}}),$$

and in particular for k sufficiently large,

$$\#\big\{0 \le j < n_k - n_{k-1} : R_k \le d(T^j(w_{n_k}), w_{n_{k-1}}) \le \frac{1}{3}d(w_{n_k}, w_{n_{k-1}})\big\} \ge \frac{C_2}{V(w_{n_k})}. \tag{20}$$

3 Proof of Theorem 1

We will present in details the proof of Theorem 1 when $T(x) = x(1 - V(x))$ for x close to 0. In the end, we will comment on the small changes of arguments required to prove the theorem in the case $x \mapsto x(1+V(x))$.

We shall define the potential f as a *signed distance* function with respect to a convenient part of a future orbit of a point sufficiently close to the indifferent fixed point. We shall choose its maximum negative *length* large enough to ensure that each Birkhoff sum can always be decomposed into nonpositive subsums, which implies that the maximum ergodic average of f is null. From property (3), we will show that certain Birkhoff sums are uniformly bounded from below by a positive constant, which prevents the cohomological inequality (1) to hold for any continuous candidate to a sub-action. This is the strategy that we will follow.

Let $(w_{n_k})_{k=1}^{+\infty}$ be a subsequence of future iterates $(w_n = T^n(w_0))_{n=0}^{+\infty}$, where $w_0 \in (0,1)$ is a point close enough to 0 and $(n_k)_{k\ge1}$ is an increasing sequence such that $\lim_{k\to+\infty} \frac{n_k}{n_{k+1}} = \gamma$ for some $\gamma \in (0,1)$.

Define then

$$S := \{w_{n_k}\}_{k=1}^{+\infty} \cup \{0\}.$$

For every $k > 1$, set

$$I_k = \Big(\frac{1}{5}(3w_{n_k} + 2w_{n_{k+1}}), \frac{1}{5}(3w_{n_k} + 2w_{n_{k-1}})\Big) \qquad \text{and}$$
$$J_k = \Big(\frac{1}{3}(w_{n_k} + 2w_{n_{k+1}}), \frac{1}{3}(2w_{n_k} + w_{n_{k+1}})\Big),$$

and denote $Y := (w_{n_1}, 1] \cup \bigcup_k J_k$. Since $\{Y, I_k\ (k > 1)\}$ is an open cover of $((0,1], \omega \circ d)$, we may consider a partition of unity subordinate to it (see Figure 1).

Precisely, let $\{\varphi_Y, \varphi_k : ((0,1], \omega \circ d) \to [0,1] \ (k > 1)\}$ be a family of Lipschitz continuous functions such that $\varphi_Y + \sum_k \varphi_k = 1$, with $\mathrm{Supp}(\varphi_Y) \subset Y$ and $\mathrm{Supp}(\varphi_k) \subset I_k$. In particular, ω is a modulus of continuity of φ_Y and of φ_k $(k > 1)$.

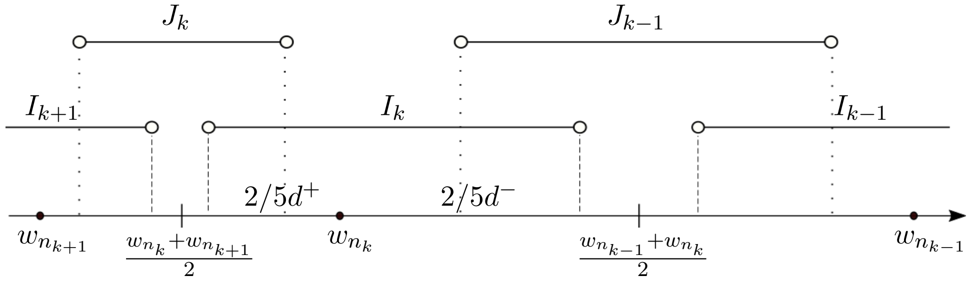

Figure 1: $d^- := d(w_{n_k}, w_{n_{k-1}})$, $d^+ := d(w_{n_k}, w_{n_{k+1}})$

For $\xi > 0$, define

$$\Phi(x) := \begin{cases} \varphi_k(x), & x \in I_k, \ k = 1 \mod 3 \\ -\xi\varphi_k(x), & x \in I_k, \ k = 2 \mod 3 \\ 0, & \text{otherwise}, \end{cases}$$

and consider $f : [0,1] \to \mathbb{R}$ given as

$$f(x) := \Phi(x)\,\omega(d(x,S)). \tag{21}$$

This function clearly vanishes on S. Moreover, f has ω as modulus of continuity. We will show that, for ξ large enough, f does not admit a continuous sub-action.

We have $T^{m_k}(w_{n_{k-1}}) = w_{n_k}$, where $m_k := n_k - n_{k-1}$, and

$$S_{m_k} f(w_{n_{k-1}}) = \sum_{j=0}^{m_k-1} f\left(T^j\left(w_{n_{k-1}}\right)\right) = \sum_{j=0}^{m_k-1} \Phi(w_{n_{k-1}+j})\,\omega(d(w_{n_{k-1}+j}, S)).$$

Recall the definition of R_k in the statement of Lemma 3. Note that, for k large enough, $[w_{n_k}, w_{n_k} + R_k) \subset \left[w_{n_k}, \frac{1}{3}(2w_{n_k} + w_{n_{k-1}})\right) \subset I_k$. Besides, by construction $\varphi_k \equiv 1$ on $\left[\frac{1}{3}(2w_{n_k} + w_{n_{k+1}}), \frac{1}{3}(2w_{n_k} + w_{n_{k-1}})\right]$. Therefore, if $k = 1 \mod 3$ is sufficiently large, from Lemma 3 we get

$$\begin{aligned} S_{m_k} f(w_{n_{k-1}}) &\geq \#\left\{j : R_k \leq d(w_{n_{k-1}+j}, w_{n_k}) \leq \frac{1}{3} d(w_{n_k}, w_{n_{k-1}})\right\}\omega(R_k) \\ &\geq \frac{C_2}{V(w_{n_k})}\omega(R_k). \end{aligned}$$

We will show that for k sufficiently large, $\frac{\omega(R_k)}{V(w_{n_k})}$ is bounded from below by a positive constant. As a matter of fact, by the definition of R_k and (15),

$$\lim_{k\to\infty} \frac{R_k}{d(w_{n_k}, w_{n_{k-1}})} = \frac{1}{3}\frac{\gamma^{1+1/\sigma}}{C_0^3}.$$

For $C_3 := \frac{1}{4}\frac{\gamma^{1+1/\sigma}}{C_0^3} > 0$, using the monotonicity of ω and Lemma 1, we have that for a sufficiently large k,

$$\omega(R_k) \geq \frac{C_3}{1+C_3}\omega(d(w_{n_k}, w_{n_{k-1}})).$$

Moreover, from Remark 3 and Lemma 2, we see that for k sufficiently large,

$$d(w_{n_k}, w_{n_{k-1}}) \geq C_0^{-1}\frac{1}{\sigma}\left(1-\frac{n_{k-1}}{n_k}\right)\frac{1}{\sigma^{1/\sigma}b(n_k)} \geq \frac{1}{2}C_0^{-1}\frac{1}{\sigma}(1-\gamma)w_{n_k}.$$

Then, for $C_4 := \frac{1}{2}C_0^{-1}\frac{1}{\sigma}(1-\gamma) > 0$, we obtain

$$\frac{\omega(R_k)}{V(w_{n_k})} \geq \frac{C_3}{1+C_3}\frac{C_4}{1+C_4}\frac{\omega(w_{n_k})}{V(w_{n_k})}.$$

Therefore, thanks to hypothesis (3), we conclude that there exists a constant $C_5 > 0$ such that, for $k = 1 \mod 3$ large enough,

$$S_{m_k}f(w_{n_{k-1}}) > C_5.$$

We will show in Subsection 3.1 that $m(f,T) = 0$ for ξ large enough. Let us assume this fact for a moment and argue that the inequality

$$f \leq u \circ T - u$$

is impossible for every continuous function $u : [0,1] \to \mathbb{R}$. Suppose the opposite happens. Then, if $k = 1 \mod 3$ is sufficiently large, we have shown that

$$\begin{aligned} u(w_{n_k}) = u\left(T^{m_k}\left(w_{n_{k-1}}\right)\right) &\geq S_{m_k}f(w_{n_{k-1}}) + u(w_{n_{k-1}}) \\ &> C_5 + u(w_{n_{k-1}}). \end{aligned}$$

Since u is continuous at 0, by letting $k \to +\infty$, we get a contradiction.

3.1 A condition for $m(f,T) = 0$

It remains to argue that, for ξ large enough, $m(f,T) = 0$. Since $f(0) = 0$ and δ_0 is T-invariant, clearly $m(f,T) \geq \int f d\delta_0 = f(0) = 0$. If ξ is sufficiently large, by choosing a suitable constant $\gamma \in (0,1)$ and an appropriate initial point w_0 close enough to 0, we will show that for each x there is a positive integer $n(x)$ such that $S_{n(x)}f(x) \leq 0$. From Birkhoff's Ergodic Theorem, we conclude that $m(f,T) \leq 0$, which completes the proof. Indeed, given any regular point $x \in [0,1]$, that is, a point for which the averages $\frac{1}{n}S_nf(x)$ converge, it is enough to pass to the limit along the subsequence of iterates $n_i = n(x_0) + n(x_1) + \ldots + n(x_i)$, where $x_0 = x$ and $x_{i+1} = f \circ T^{n(x_i)}(x_i)$, to see that such a limit is nonpositive.

We first choose $\gamma \in (0,1)$ satisfying

$$\gamma^{1+1/\sigma} > \frac{6}{7}. \tag{22}$$

Note now that, replacing w_0 by w_{n_0} with n_0 large enough, we may assume that the constant C_0 in Remark 3 is as close as we want to 1. Thus, we suppose henceforth that

$$1 < C_0^2 \leq \frac{7}{6}\gamma^{1+1/\sigma}. \tag{23}$$

Furthermore, thanks to (15), if n_0 is sufficiently large, we may also assume that

$$\frac{1}{2}\gamma^{1+1/\sigma} \leq \frac{n_k b(n_k)}{n_{k+1} b(n_{k+1})} \quad \forall\, k \geq 0. \tag{24}$$

If $x \in [0,1] \backslash \bigcup_{k=1 \mod 3} I_k$, just take $n(x) = 1$, since $f(x) \leq 0$. Suppose then $x \in I_k$ for some $k = 1 \mod 3$. Define

$$p(x) := \min\{p \geq 1 : T^p(x) \notin I_k\}.$$

Note that

$$S_{p(x)} f(x) \leq \#\{j \geq 0 : T^j(x) \in I_k\}\, \omega\Big(\frac{2}{5}\max\{d(w_{n_{k+1}}, w_{n_k}), d(w_{n_k}, w_{n_{k-1}})\}\Big).$$

Let us estimate the cardinality in the right term. Denote

$$L_k := \Big\lceil \frac{3}{7} C_0 \sigma^{1+1/\sigma} n_k b(n_k) d(w_{n_k}, w_{n_{k-1}})\Big\rceil.$$

From Remark 3, we have $d(w_{n_k}, w_{n_k - L_k}) \geq L_k C_0^{-1} \frac{1}{\sigma^{1+1/\sigma}} \frac{1}{n_k b(n_k)} > \frac{2}{5} d(w_{n_k}, w_{n_{k-1}})$, which means that $w_{n_k - L_k}$ is greater than the right endpoint of I_k. Thanks to (22), (23) and (24),

$$\frac{3}{7} C_0 \sigma^{1+1/\sigma} n_{k+1} b(n_{k+1}) d(w_{n_{k+1}}, w_{n_k}) \leq \frac{3}{7} C_0^2 \frac{n_{k+1} b(n_{k+1})}{n_k b(n_k)} (n_{k+1} - n_k) \leq n_{k+1} - n_k,$$

so that $L_{k+1} \leq n_{k+1} - n_k$. Hence, a similar reasoning shows that $w_{n_k + L_{k+1}}$ is smaller than the left endpoint of I_k. Therefore, by the monotonicity of T, we obtain

$$\begin{aligned} \#\{j : T^j(x) \in I_k\} &\leq (L_k - 1) + (L_{k+1} - 1) \\ &\leq \frac{3}{7} C_0 \sigma^{1+1/\sigma} n_{k+1} b(n_{k+1}) d(w_{n_{k+1}}, w_{n_{k-1}}). \end{aligned}$$

We have shown that

$$S_{p(x)} f(x) \leq \frac{3}{7} C_0 \sigma^{1+1/\sigma} n_{k+1} b(n_{k+1}) d(w_{n_{k+1}}, w_{n_{k-1}})\, \omega\big(d(w_{n_{k+1}}, w_{n_{k-1}})\big). \tag{25}$$

Now, for $y \in \big[w_{n_{k+1}} + R_{k+1}, \frac{1}{5}(3 w_{n_k} + 2 w_{n_{k+1}})\big]$, denote

$$q(y) := \min\{q \geq 1 : d(T^q(y), w_{n_{k+1}}) < R_{k+1}\}.$$

Clearly,

$$S_{q(y)} f(y) \leq -\xi\, \#\big\{j \geq 0 : R_{k+1} \leq d(T^j(y), w_{n_{k+1}}) \leq \frac{1}{3} d(w_{n_{k+1}}, w_{n_k})\big\}\, \omega(R_{k+1}).$$

Thanks to Lemma 3, we obtain that

$$S_{q(y)}f(y) \le -\xi\, C_1 n_k b(n_k) d(w_{n_k}, w_{n_{k+1}})\omega(R_{k+1}). \tag{26}$$

We claim that, whenever ξ is sufficiently large, for $n(x) := p(x) + q(T^{p(x)}(x))$ one has $S_{n(x)}f(x) \le 0$. Thanks to (25) and (26), it is enough to prove that

$$\sup_k \frac{n_{k+1}b(n_{k+1})d(w_{n_{k+1}}, w_{n_{k-1}})\omega\big(d(w_{n_{k+1}}, w_{n_{k-1}})\big)}{n_k b(n_k) d(w_{n_k}, w_{n_{k+1}})\omega(R_{k+1})} < \infty.$$

Recalling the asymptotic equivalence (15), we just have to show that both suprema

$$\sup_k \frac{d(w_{n_{k+1}}, w_{n_{k-1}})}{d(w_{n_k}, w_{n_{k+1}})} \quad \text{and} \quad \sup_k \frac{\omega\big(d(w_{n_{k+1}}, w_{n_{k-1}})\big)}{\omega(R_{k+1})}$$

are finite. With respect to the first one, from (17) it is immediate that

$$\begin{aligned} \frac{d(w_{n_k}, w_{n_{k-1}})}{d(w_{n_{k+1}}, w_{n_k})} &\le \frac{C_0(n_k - n_{k-1})1/\big[\sigma^{1+1/\sigma} n_{k-1} b(n_{k-1})\big]}{C_0^{-1}(n_{k+1} - n_k)1/\big[\sigma^{1+1/\sigma} n_{k+1} b(n_{k+1})\big]} \\ &= C_0^2 \frac{1 - \frac{n_{k-1}}{n_k}}{\frac{n_{k+1}}{n_k} - 1} \frac{n_{k+1}b(n_{k+1})}{n_k b(n_k)} \frac{n_k b(n_k)}{n_{k-1}b(n_{k-1})}, \end{aligned} \tag{27}$$

which ensures $\frac{d(w_{n_{k+1}}, w_{n_{k-1}})}{d(w_{n_k}, w_{n_{k+1}})} = 1 + \frac{d(w_{n_k}, w_{n_{k-1}})}{d(w_{n_{k+1}}, w_{n_k})}$ is bounded from above. With respect to the second one, note first that, thanks to (27),

$$\frac{d(w_{n_{k+1}}, w_{n_{k-1}})}{R_{k+1}} = 3C_0^3 \frac{n_{k+1}b(n_{k+1})}{n_k b(n_k)} \frac{d(w_{n_{k+1}}, w_{n_{k-1}})}{d(w_{n_k}, w_{n_{k+1}})}$$

is bounded from above. Hence, there exists a positive constant C_6 such that $d(w_{n_{k+1}}, w_{n_{k-1}}) \le C_6 R_{k+1}$. By the monotonicity of ω and Lemma 1, we obtain

$$\frac{\omega\big(d(w_{n_{k+1}}, w_{n_{k-1}})\big)}{\omega(R_{k+1})} \le C_6 + 1 < \infty.$$

The proof is complete.

Comments on the proof of Theorem 1 for $x \mapsto x(1+V(x))$. We consider now a subsequence (w_{n_k}) that fulfills $w_{n_{k-1}} = T^{n_k - n_{k-1}}(w_{n_k})$, where $T(x) = x(1+V(x))$ in a neighborhood of 0. Note that orbits are moving monotonically away from the origin, that is, they are moving to the right instead of to the left as in the previous case. This merely produces a, let us say, *reflexive effect* on our arguments, exchanging the roles of indices $k = 1 \mod 3$ and $k = 2 \mod 3$. In practical terms, we define Φ for this case as

$$\Phi(x) := \begin{cases} -\xi\varphi_k(x), & x \in I_k,\ k = 1 \mod 3 \\ \varphi_k(x), & x \in I_k,\ k = 2 \mod 3 \\ 0, & \text{otherwise.} \end{cases}$$

Introducing f as in (21) and supposing by a moment that $m(f,T)=0$, we apply the same strategy to show that f does not admit continuous sub-action. In fact, for $k=2 \mod 3$ sufficiently large, using (20) one estimates the number of iterates that remain in the interval $[\frac{1}{3}(2w_{n_k}+w_{n_{k+1}}), w_{n_k}-R_{k+1}]$ to conclude that $S_{m_{k+1}}f(w_{n_{k+1}})$ is bounded from below by a positive constant and thus to reach a contradiction. In order to show that, for the same choice of parameters (22), (23), and (24), $m(f,T)=0$ whenever ξ is sufficiently large, suitable adjustments are required to obtain that for $x \in I_k$ with $k=2 \mod 3$, there is $n(x)$ such that $S_{n(x)}f(x) \leq 0$. Similarly to the previous case, the key observation is that such a Birkhoff's sum may be bounded from above by the difference of two terms, the first one takes into account the iterates that remain in I_k, the second one considers iterates that remain in $[\frac{1}{3}(2w_{n_{k-1}}+w_{n_k}), w_{n_{k-1}}-R_k]$, and their ratio is uniformly bounded.

4 Proof of Theorem 2

In the following results we will assume the hypotheses of Theorem 2. In particular, we keep in mind all the constants of Assumption A.

Lemma 4. *There are constants $\varrho_T > 0$ and $C_7 \in (0, \min\{\xi_0, \eta_0^{-1}\}-1]$ such that for all $x, y \in [0,1]$, with $d(x,y) < \varrho_T$, we have*

$$d(T(x),T(y)) \geq d(x,y)\big(1 + C_7 V(d(x,y))\big). \tag{28}$$

Proof. Let $x, y \in [0,c]$ with $x < y$. Since V and T are increasing, note that

$$d(T(x),T(y)) = d(x,y) + d(x,y)\,V(y) + x\,(V(y)-V(x)) \geq d(x,y)\,\big(1+V(d(x,y))\big).$$

Consider now $x, y \in (c,1]$. Since $V([0,1]) \subset [0,1)$, we clearly have

$$d(T(x),T(y)) \geq \lambda\, d(x,y)\; \geq d(x,y)\,\big(1+(\lambda-1)\,V(d(x,y))\big).$$

Fix $\varrho > 0$ such that, for $x \in [c-\varrho/2, c)$ and $y \in (c, c+\varrho/2]$ it follows that $d(T(x),T(y)) \geq 1/2$. We choose $\varrho_T \in (0,\varrho)$ such that $V\big(\frac{1}{2}h\big) \geq \frac{1}{2^{\sigma+1}}V(h)$ for all $h \in [0,\varrho_T]$. Then for $c-\varrho_T/2 \leq x < c < y \leq c+\varrho_T/2$,

$$\begin{aligned} d(T(x),T(y)) \geq 1 - d(T(x),T(y)) &= \lim_{t\to c^-} d(T(t),T(x)) + \lim_{t\to c^+} d(T(y),T(t)) \\ &\geq \lim_{t\to c^-} d(t,x)\,\big(1+V(d(t,x))\big) + \lim_{t\to c^+} d(y,t)\,\big(1+(\lambda-1)\,V(d(y,t))\big) \\ &= d(x,y) + d(c,x)\,V(d(c,x)) + (\lambda-1)\,d(y,c)\,V(d(y,c)). \end{aligned}$$

Suppose that $d(c,x) \geq d(y,c)$, then $2\,d(c,x) \geq d(x,y)$ and

$$d(T(x),T(y)) \geq d(x,y) + \frac{1}{2}\,d(x,y)\,V\Big(\frac{1}{2}d(x,y)\Big) \geq d(x,y) + \frac{1}{2^{\sigma+2}}\,d(x,y)\,V(d(x,y)).$$

Similarly, if $d(c,y) \geq d(x,c)$, then $2\,d(c,y) \geq d(x,y)$ and

$$d(T(x),T(y)) \geq d(x,y) + \frac{(\lambda-1)}{2^{\sigma+2}}\,d(x,y)\,V(d(x,y)).$$

Take $C_7 := \min\left\{\frac{1}{2^{\sigma+2}}, \frac{\lambda-1}{2^{\sigma+2}}, \xi_0 - 1, \frac{1}{\eta_0} - 1\right\}.$ □

Proposition 3. *There are constants* $\varrho_{T,\omega} > 0$ *and* $C_8 > 0$ *such that, given a sequence* $\{x_k\}_{k\geq 0}$ *in* $[0,1]$, *with* $T(x_{k+1}) = x_k$ *for* $k \geq 0$, *and a point* $y_0 \in [0,1]$ *with* $d(x_0, y_0) < \varrho_{T,\omega}$, *there is* $\{y_k\}_{k\geq 1} \subset [0,1]$, *with* $T(y_{k+1}) = y_k$ *for* $k \geq 0$, *satisfying*

$$\Omega\big(d(x_k, y_k)\big) + C_8 \sum_{j=1}^{k} \omega\big(d(x_j, y_j)\big) \leq \Omega\big(d(x_0, y_0)\big) \qquad \forall\, k \geq 1. \tag{29}$$

Proof. Let $\varrho_{T,\omega} = \min\{\varrho_T, \eta_0\}$, where ϱ_T is as in the statement of Lemma 4. For $x_0, x_1, y_0 \in [0,1]$ with $T(x_1) = x_0$ and $d(x_0, y_0) < \varrho_{T,\omega}$, we can choose $y_1 \in T^{-1}(y_0)$ with $d(x_1, y_1) \leq d(x_0, y_0) < \varrho_{T,\omega}$. Then from Lemma 4,

$$d(x_0, y_0) = d(T(x_1), T(y_1)) \geq d(x_1, y_1)\,\big(1 + C_7\, V(d(x_1, y_1))\big).$$

Since Ω is non-decreasing, we have $\Omega\big(d(x_0, y_0)\big) \geq \Omega\big(d(x_1, y_1)\,\big(1 + C_7\, V(d(x_1, y_1))\big)\big)$. For $h = d(x_1, y_1)$, we can write

$$\Omega\big(h(1 + C_7\, V(h))\big) = \Omega\big((1 - V(h))\, h + V(h)\,(1 + C_7)\, h\big).$$

As $\Omega = \vartheta_2^* + \vartheta_1^*(1)$ is concave, we see that

$$\begin{aligned}\Omega\big(h\,(1 + C_7\, V(h))\big) &\geq (1 - V(h))\,\Omega(h) + V(h)\,\Omega\big((1 + C_7)\, h\big) \\ &= \Omega(h) + V(h)\,\Big(\vartheta_2^*((1 + C_7)\, h) - \vartheta_2^*(h)\Big).\end{aligned}$$

Recalling that $\vartheta_2^* \geq \vartheta_0$, we have

$$\begin{aligned}\Omega\big(h\,(1 + C_7\, V(h))\big) &\geq \Omega(h) + V(h)\,\vartheta_2^*(h)\,\Big(\frac{\vartheta_2^*\big((1 + C_7)\, h\big)}{\vartheta_2^*(h)} - 1\Big) \\ &\geq \Omega(h) + \omega(h)\,\Big(\frac{\vartheta_2^*\big((1 + C_7)\, h\big)}{\vartheta_2^*(h)} - 1\Big).\end{aligned}$$

We claim that $\frac{\vartheta_2^*((1+C_7)\, h)}{\vartheta_2^*(h)} \geq (1+C_7)^\gamma$. As a matter of fact, following Assumption A, for $1 + C_7 \leq \xi_0$, since $h = d(x_1, y_1) < \varrho_{T,\omega} \leq \eta_0$,

$$\frac{\vartheta_0((1 + C_7)\, h)}{\vartheta_0(h)} \geq (1 + C_7)^\gamma, \qquad \text{and thus } \frac{\vartheta_1((1 + C_7)\, h)}{\vartheta_1(h)} \geq \ (1 + C_7)^\gamma.$$

Write $\xi = 1 + C_7$ and recall that the transform Legendre is order reversing, then

$$\vartheta_2\Big(\frac{h}{\xi}\Big) = \vartheta_1^*\Big(\frac{h}{\xi}\Big) = (\vartheta_1(\xi\, h))^* \leq \ (\xi^\gamma \vartheta_1(h))^* = \xi^\gamma \vartheta_1^*\Big(\frac{h}{\xi}\Big) = \xi^\gamma \vartheta_2\Big(\frac{h}{\xi^\gamma}\Big).$$

Applying again the concave conjugate, we get

$$\vartheta_2^*(\xi h) = \Big(\vartheta_2\Big(\frac{h}{\xi}\Big)\Big)^* \geq \Big(\xi^\gamma \vartheta_2\Big(\frac{h}{\xi^\gamma}\Big)\Big)^* = \xi^\gamma \vartheta_2^*(h).$$

Therefore, for $C_8 := (1 + C_7)^\gamma - 1$, we have shown that, for $x_0, x_1,\ y_0 \in [0,1]$ with $T(x_1) = x_0$ and $d(x_0, y_0) < \varrho_{T,\omega}$, there is $y_1 \in T^{-1}(y_0)$, with $d(x_1, y_1) \leq d(x_0, y_0) < \varrho_{T,\omega}$, such that

$$\Omega\,(d(x_0, y_0)) \geq \Omega(d(x_1, y_1)) + C_8\, \omega(d(x_1, y_1)).$$

Inequality (29) follows straightforward from the above inequality. $\square$

For $\omega \in \mathcal{M}$ and $\varphi \in \mathscr{C}_\omega$, we denote

$$|\varphi|_\omega = \sup_{x\neq y} \frac{|\varphi(x)-\varphi(y)|}{\omega(d(x,y))}.$$

Lemma 5. *Let $g_k(x) := \sup_{T^k(y)=x} S_k\big(f - m(f,T)\big)(y)$, for $k \geq 1$. Then, there is $L = L(\varrho_{T,\omega}) > 0$ such that for every $k \geq 1$,*

$$|g_k(x) - g_k(y)| \leq L\, C_8^{-1}\, |f|_\omega\, \Omega(d(x,y)), \qquad \forall\, x, y \in [0,1] \text{ and}$$
$$|g_k(x)| \leq 2\, L\, C_8^{-1}\, |f|_\omega\, \Omega(1), \qquad \forall\, x \in [0,1],$$

where $\varrho_{T,\omega}$ and C_8 are as in the statement of Proposition 3.

Proof. Without loss of generality, we suppose that $m(f,T) = 0$. Let $x_0, y_0 \in [0,1]$ be such that $d(x_0,y_0) < \varrho_{T,\omega}$. Fix $k \geq 1$ and assume that $g_k(x_0) \geq g_k(y_0)$. Given $\epsilon > 0$, there exists $x_k \in T^{-k}(x_0)$ with $g_k(x_0) - \epsilon < S_k f(x_k)$. We apply the previous proposition and consider $y_k \in T^{-k}(y_0)$ so that

$$\sum_{j=0}^{k-1} \omega\big(d\big(T^j(x_k), T^j(y_k)\big)\big) \leq C_8^{-1}\Big(\Omega\big(d(x_0,y_0)\big) - \Omega\big(d(x_k,y_k)\big)\Big) \leq C_8^{-1}\Omega\big(d(x_0,y_0)\big).$$

Thus,

$$\begin{aligned}|g_k(x_0) - g_k(y_0)| - \epsilon &< S_k f(x_k) - S_k f(y_k)\\ &\leq |f|_\omega \sum_{j=0}^{k-1} \omega\big(d\big(T^j(x_k), T^j(y_k)\big)\big) \leq C_8^{-1}\, |f|_\omega\, \Omega(d(x_0,y_0)).\end{aligned}$$

Therefore, as $\epsilon > 0$ is arbitrary, if $d(x_0,y_0) < \varrho_{T,\omega}$ and $k \geq 1$,

$$|g_k(x_0) - g_k(y_0)| \leq C_8^{-1}\, |f|_\omega\, \Omega(d(x_0,y_0)).$$

For $z \in [0,1]$, define $I_z = (z - \varrho_{T,\omega}/2, z + \varrho_{T,\omega}/2) \cap [0,1]$. There are finitely many points $z_i \in [0,1]$, $1 \leq i \leq L-1$, which are assumed ordered, such that $\{I_{z_i}\}_{i=1}^{L-1}$ is an open cover of $[0,1]$. Hence, given $x + \varrho_{T,\omega} \leq y$ in $[0,1]$, consider indexes $i_x < i_y$ for which $x \in I_{z_{i_x}}$ and $y \in I_{z_{i_y}}$. Note that, as Ω is non-decreasing, the above local property provides

$$\begin{aligned}|g_k(x) - g_k(y)| \leq &|g_k(x) - g_k(z_{i_x})| + \sum_{i_x \leq i < i_y} |g_k(z_i) - g_k(z_{i+1})| + |g_k(z_{i_y}) - g_k(y)|\\ \leq &L\, C_8^{-1}\, |f|_\omega\, \Omega\big(d(x,y)\big).\end{aligned}$$

We have shown that the family $\{g_k\}_{k\geq 1}$ is equicontinuous. To obtain uniform boundness, denote $C_9 = L\, C_8^{-1}\, |f|_\omega\, \Omega(1)$. By contradiction, suppose that for some $\tilde{x} \in [0,1]$ and $k_0 \geq 1$, one has $|g_{k_0}(\tilde{x})| > 2C_9$. By the previous discussion, we would have $|g_{k_0}(\tilde{x}) - g_{k_0}(x)| \leq C_9$ for all $x \in [0,1]$, so that $|g_{k_0}| > C_9$ everywhere. Then there would be a sequence $(\tilde{x}_\ell)_{\ell\geq 1}$ such that $T^{\ell k_0}(\tilde{x}_\ell) = \tilde{x}$ and $S_{\ell k_0} f(\tilde{x}_\ell) > \ell\, C_9$, hence

$$\frac{1}{\ell\, k_0} S_{\ell k_0} f(\tilde{x}_\ell) > \frac{C_9}{k_0} > 0.$$

This contradicts the fact that $m(f,T)=0$. Indeed, it is easy to see that the Borel probabilities $\nu_\ell = \frac{1}{\ell k_0}\big(\delta_{\tilde{x}_\ell} + \delta_{T(\tilde{x}_\ell)} + \ldots + \delta_{T^{\ell k_0 - 1}(\tilde{x}_\ell)}\big)$ have, with respect to the weak-star topology, T-invariant measures as accumulation probabilities as $\ell \to \infty$. Hence, if ν_∞ is any one of these accumulation probabilities, then

$$m(f,T) \geq \int f\, d\nu_\infty = \lim_{j\to\infty} \frac{1}{\ell_j k_0} S_{\ell_j k_0} f(\tilde{x}_{k_{\ell_j}}) \geq \frac{C_9}{k_0}.$$

□

Proof of Theorem 2. Following [CLT01, Proposition 11], denote $g_0 \equiv 0$ and define, for every $x \in [0,1]$,

$$U_f(x) := \sup_{k\geq 0} g_k(x) = \sup\Big\{ S_k\big(f - m(f,T)\big) : k \geq 0 \text{ and } T^k(y) = x \Big\}.$$

Thanks to Lemma 5, U_f is a well-defined real function and actually $U_f \in \mathscr{C}_\Omega$. Furthermore, it follows from definition that the inequality $U_f \circ T \geq U_f + f - m(f,T)$ holds and therefore U_f is a sub-action. □

Acknowledgment: We are indebted to J. T. A. Gomes for his attentive reading of Section 4.

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