AUBRY SET FOR ASYMPTOTICALLY SUB-ADDITIVE POTENTIALS

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Given a topological dynamical systems (X, T), consider a sequence of continuous potentials $\mathcal{F} := \{f_n : X \to \mathbb{R}\}_{n \geq 1}$ that is asymptotically approached by sub-additive families. In a generalized version of ergodic optimization theory, one is interested in describing the set $\mathcal{M}_{\max}(\mathcal{F})$ of T-invariant probabilities that attain the following maximum value

$$\max\Big\{ \lim_{n\to\infty} \frac{1}{n} \int f_n \, d\mu \, : \, \mu \text{ is } T\text{-invariant probability}\Big\}.$$

For this purpose, we extend the notion of Aubry set, denoted by $\Omega(\mathcal{F})$. Our central result provides sufficient conditions for the Aubry set to be a maximizing set, i. e. μ belongs to $\mathcal{M}_{\max}(\mathcal{F})$ if, and only if, its support lies on $\Omega(\mathcal{F})$. Furthermore, we apply this result to the study of the joint spectral radius in order to show the existence of periodic matrix configurations approaching this value.

Keywords: Atkinson's theorem; Aubry set; ergodic optimization; generalized spectral radius; joint spectral radius; maximizing probabilities; maximizing set.

AMS Subject Classification: 15A18, 15A60, 37A05, 37A50, 37B10.

1. Introduction

Ergodic optimization was presented as a new branch of ergodic theory by Contreras, Lopes and Thieullen in [4], where typical concepts of Aubry-Mather theory were reformulated in a discrete-dynamics context. Given a continuous potential $f : X \to \mathbb{R}$ on the topological dynamical system (X, T), this research area is concerned with the value

$$\beta[f] := \max\Big\{ \int f d\mu \, : \, \mu \text{ is a } T\text{-invariant probability}\Big\}$$

*Supported by CNPq-Brazil grant 306177/2011-0.
†Supported by CNPq-Brazil doctoral scholarship 140675/2014-0.

and with the T-invariant probabilities that attain the above maximum, also called *maximizing probabilities*.

One way to characterize maximizing probabilities consists in showing the existence of a *maximizing set* for f, namely, a closed set K_f of X which satisfies

$$\mu \text{ is maximizing probability for } f \qquad \Leftrightarrow \qquad \operatorname{supp}\mu \subset K_f,$$

where $\operatorname{supp}\mu$ as usual denotes the support of the measure μ. For a topologically transitive hyperbolic dynamical system [4, 8], a natural candidate is given by the *Aubry set*. The nomenclature is borrowed from Aubry-Mather theory and, roughly speaking, this set consists in all those non-wandering points with maximal Birkhoff sums. Moreover, from the perspective of the sub-action approach, the Aubry set is the smallest maximizing set (for details, see [6]).

Our aim in these notes is to generalize the notion of Aubry set for families of asymptotically sub-additive potentials and to show that such a set is an aspirant to maximizing set in this context. We defer the precise definitions and statements to the next section. Some results of ergodic optimization theory have been successfully extended to such a general setting (see, for instance, [3, 11]). Given a sequence of measurable potentials $\mathcal{F} = \{f_k : X \to \mathbb{R}\}_{k\geq 1}$, we always consider in this paper situations in which these potentials satisfy conditions for integrability with respect to invariant probabilities – as, for example, when all f_k's are non-positive or continuous. Hence, the *ergodic maximizing value* of such a sequence may be defined as

$$\beta[\mathcal{F}] := \sup\Big\{ \limsup_{k\to\infty} \frac{1}{k}\int f_k \, d\mu \, : \, \mu \text{ is a } T\text{-invariant probability}\Big\} \in [-\infty, +\infty].$$

For sequences of functions satisfying a sub-additive property, by Kingman's sub-additive ergodic theorem, the above supremum limit is actually a limit and belongs to $[-\infty, +\infty)$.

Sub-additive sequences arise naturally in hyperbolic dynamics, dimension theory and spectral theory. For instance, given an alphabet of $d \times d$ matrices $\Sigma = \{M_1, M_2, \ldots, M_s\}$ and a sub-multiplicative matrix norm $\|\cdot\|$, the *joint spectral radius* is

$$\rho(\Sigma) := \lim_{k\to\infty} \max\big\{\|M_{i_{k-1}} \cdots M_{i_0}\|^{1/k} \, : \, 1 \leq i_j \leq s\big\}.$$

For a dynamical approach, consider the metrizable compact full-shift $\Sigma^{\mathbb{N}}$, provided with the one-sided shift map $\sigma : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$, and the sub-additive family of functions $\{\log\phi_k : \Sigma^{\mathbb{N}} \to \mathbb{R}\}_{k\geq 1}$ defined by $\log\phi_k(M_{i_0}, M_{i_1}, \ldots) := \log\|M_{i_{k-1}} \cdots M_{i_0}\|$. Schreiber's theorem [12] guarantees the existence of a σ-invariant probability $\mu_{\max}$ such that

$$\begin{aligned}\log\rho(\Sigma) &= \max\Big\{ \lim_{k\to\infty} \frac{1}{k}\int \log\phi_k \, d\mu \, : \, \mu \text{ is a } \sigma\text{-invariant probability}\Big\}\\ &= \lim_{k\to\infty} \frac{1}{k}\int \log\phi_k \, d\mu_{\max}.\end{aligned}$$

Our generalized notion of Aubry set allows to show that the joint spectral radius can be approximated, with prescribed precision, by periodic matrix configurations.

The paper is organized as follows. In section 2, we give the framework of ergodic optimization for sequences of asymptotically sub-additive potentials. Moreover, we introduce the Aubry set in this context and give the statement of our central result. The proof of this result is presented in subsections 2.1 and 2.2. During the proof, we obtain an extension of the well-known Atkinson's theorem (see theorem 2.2), which has its own interest. In the last section, we will investigate some consequences of the central result for the study of the joint spectral radius.

2. Framework and central result

Let $T : X \to X$ be a continuous function on a compact metric space (X, d). If $\mathscr{B}$ denotes the Borel σ-algebra, we may consider the measurable aspects of the space $(X, \mathscr{B})$, as well as focus on the set of all T-invariant Borel probabilities measures $\mathcal{M}_T$, which is compact with respect to weak* topology and convex.

In this work, we direct our attention to sequences of continuous potentials $\mathcal{F} := \{f_k : X \to \mathbb{R}\}_{k\geq 1}$ that verify the *asymptotically sub-additive property.* Recall that a sequence of continuous potentials $\Phi := \{\phi_k : X \to \mathbb{R}\}_{k\geq 1}$ is *sub-additive* if $\phi_{k+l} \leq \phi_k + \phi_l \circ T^k$, for every $k, l \geq 1$. We say that $\mathcal{F} := \{f_k\}_{k\geq 1}$ is asymptotically sub-additive if, for every $\varepsilon > 0$, there exits a sub-additive sequence of potentials $\Phi^\varepsilon := \{\phi_k^\varepsilon : X \to \mathbb{R}\}_{k\geq 1}$ such that

$$\limsup_{k\to\infty} \frac{1}{k} \big\| f_k - \phi_k^\varepsilon \big\|_\infty \leq \varepsilon.$$

Basic examples of asymptotically sub-additive sequences are almost sub-additive families, that is, any sequence $\mathcal{F} = \big\{f_k\big\}_{k\geq 1}$ for which there exists $C > 0$ such that $\big\{f_k + C\big\}_{k\geq 1}$ is sub-additive. The asymptotically sub-additive property is sufficient to ensure the following important conditions (for a proof, see the appendix in [5]).

C1. The function

$$\mu \in \mathcal{M}_T \longmapsto \lim_{k\to\infty} \frac{1}{k} \int f_k \, d\mu \in \mathbb{R} \cup \{-\infty\}$$

is upper semi-continuous.

C2. For every T-invariant probability μ,

$$\tilde{f}(x) := \lim_{k\to\infty} \frac{f_k(x)}{k} \quad \text{exists } \mu\text{-a.e. } x \in X \quad \text{and} \quad \int \tilde{f} \, d\mu = \lim_{k\to\infty} \frac{1}{k} \int f_k \, d\mu$$

(the above limits may assume the value $-\infty$). Besides, if μ is ergodic, then $\tilde{f}$ is μ-a.e. constant and equals to $\lim_{k\to\infty} \frac{1}{k} \int f_k \, d\mu$.

Notice that, due to **C2**, one can apply the ergodic decomposition theorem to the integrable function $\tilde{f}$, so that

$$\lim_{k\to\infty} \frac{1}{k} \int f_k \, d\mu = \int \tilde{f} \, d\mu = \int_X \Big[\int \tilde{f} \, d\mu_x \Big] \, d\mu(x) = \int_X \Big[\lim_{k\to\infty} \frac{1}{k} \int f_k \, d\mu_x \Big] \, d\mu(x) \tag{2.1}$$

if $\mu = \int_X \mu_x \, d\mu(x)$ is the ergodic decomposition of μ.

The foregoing *ergodic maximizing value*, in this case, gives rise to

$$\beta[\mathcal{F}] := \sup\Big\{ \int \tilde{f} \, d\mu \, : \, \mu \in \mathcal{M}_T \Big\} = \sup\Big\{ \lim_{k\to\infty} \frac{1}{k} \int f_k \, d\mu \, : \, \mu \in \mathcal{M}_T \Big\}.$$

In this context, other characterizations of this constant (see [12, 11, 3]) are

$$\beta[\mathcal{F}] \;=\; \lim_{k\to\infty} \max_{x\in X} \frac{1}{k} f_k(x) \;=\; \sup_{x\in X} \limsup_{k\to\infty} \frac{1}{k} f_k(x) \;=\; \sup_{x\in \mathrm{Reg}\,(\mathcal{F})} \lim_{k\to\infty} \frac{1}{k} f_k(x),$$

where $\mathrm{Reg}\,(\mathcal{F})$ is the set of points $x \in X$ such that the limit $\lim_{k\to\infty} \frac{1}{k} f_k(x)$ exists. (In addition, for the sub-additive case, each of the above supremums is attained by some point in X.)

Due to condition **C1** and the compactness of $\mathcal{M}_T$, there always exists a probability $\mu_{\max} \in \mathcal{M}_T$ such that

$$\lim_{k\to\infty} \frac{1}{k} \int f_k \, d\mu_{\max} = \max\Big\{ \lim_{k\to\infty} \frac{1}{k} \int f_k \, d\mu \, : \, \mu \in \mathcal{M}_T \Big\} = \beta[\mathcal{F}].$$

These T-invariant probabilities that attain the above maximum are called *maximizing probabilities* associated with $\mathcal{F}$ and the set of these measures is denoted by $\mathcal{M}_{\max}(\mathcal{F})$. Notice that, thanks to the ergodic decomposition theorem, $\mathcal{M}_{\max}(\mathcal{F})$ contains at least one ergodic probability.

We remark that the classical notions of ergodic optimization theory may be clearly obtained from the previous concepts in the case of an additive sequence of potentials, i. e. $\mathcal{F} = \big\{ f_k := \sum_{j=0}^{k-1} f_1 \circ T^j \big\}_{k\geq 1}$.

From now on, without being restated each time, we always suppose that $\beta[\mathcal{F}] \in \mathbb{R}$. We propose thus a generalization for the notion of Aubry set.

Definition 2.1. Given a sequence of continuous potentials $\mathcal{F} = \{f_k\}_{k\geq 1}$, we say that $x \in X$ is an *Aubry point* if, for all $\varepsilon > 0$ and for any integer $L \geq 1$, there exist $y \in B_\varepsilon(x)$ and integers $m > n \geq 0$, with $m - n \geq L$, such that

$$T^n y \in B_\varepsilon(x), \quad T^m y \in B_\varepsilon(x) \quad \text{and} \quad \Big| \big[f_m(y) - f_n(y) \big] - (m-n)\beta[\mathcal{F}] \Big| < \varepsilon,$$

where $B_\varepsilon(x)$ denotes the open ball of center x and radius ε and, by convention, $f_0 \equiv 0$. The collection of such points is the *Aubry set*, being denoted by $\Omega(\mathcal{F})$.

It is a routine exercise to verify that the classical Aubry set definition (see [4, 8]) coincides with the above concept for an additive sequence of potentials.

Lemma 2.1. *For a sequence of continuous potentials $\mathcal{F} = \{f_k\}_{k\geq 1}$ that satisfies both conditions* **C1** *and* **C2***, the Aubry set is a non-empty compact set.*

Proof. The fact that $\Omega(\mathcal{F})$ is a non-empty set will follow from proposition 2.1 and the existence of a maximizing probability. In order to obtain that $\Omega(\mathcal{F})$ is a compact set, it suffices to prove that it is closed. Consider thus a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \Omega(\mathcal{F})$ converging to some point $x \in X$. For all $\varepsilon > 0$, choose an Aubry point $x_i \in B_{\frac{\varepsilon}{2}}(x)$.

Given an integer $L \geq 1$, there exist $y \in B_{\frac{\varepsilon}{2}}(x_i) \subset B_\varepsilon(x)$ and integers $m > n \geq 0$, with $m - n \geq L$, such that

$$T^n y,\, T^m y \in B_{\frac{\varepsilon}{2}}(x_i) \subset B_\varepsilon(x) \quad \text{and} \quad \Big|\big[f_m(y) - f_n(y)\big] - (m-n)\beta[\mathcal{F}]\Big| < \frac{\varepsilon}{2} \leq \varepsilon.$$

Therefore, $x \in \Omega(\mathcal{F})$ and $\Omega(\mathcal{F}) = \overline{\Omega(\mathcal{F})}$. □

The T-invariance of the Aubry set (namely, $\Omega(\mathcal{F}) \subset T^{-1}\Omega(\mathcal{F})$) is in general an open question. It could be obtained from the *co-homological invariance*, i. e.

$$\Omega(\mathcal{F}) \subset \Omega\Big(\big\{f_k + v_k \circ T - v_k + c\big\}_{k\geq 1}\Big) \quad \text{for } v_k : X \to \mathbb{R} \text{ continuous}, \quad c \in \mathbb{R},$$

by considering $v_k = f_k$, $c = 0$ and by noticing that $\Omega\left(\{f_k \circ T\}_{k\geq 1}\right) \subset T^{-1}\Omega(\mathcal{F})$. The co-homological invariance holds for functions v_k's such that, for every $\varepsilon > 0$, there exists $\delta > 0$ for which $\mathsf{d}\big(x, T^k x\big) \leq \delta$ implies $\big|v_k \circ T(x) - v_k(x)\big| \leq \varepsilon$ for all $k > 0$. Indeed, we can always suppose that $\delta(\varepsilon) < \varepsilon$, so that the claimed inclusion follows by applying the definition of an Aubry point for any $\frac{1}{2}\delta(\frac{\varepsilon}{3}) > 0$ and $L \geq 1$. Additive sequences of potentials perturbed by sub-additive sequences of constants are obvious examples of families verifying such a uniform-continuity like regularity. It is however unknown whether the co-homological invariance holds in general.

The statement of our central result is given below.

Theorem 2.1. *Given any almost sub-additive sequence of continuous potentials $\mathcal{F} = \{f_k\}_{k\geq 1}$ such that*

$$\sup_{x\in X} \sup_{p>q\geq 0} \big[f_p(x) - f_q(x) - (p-q)\beta[\mathcal{F}]\big] < \infty$$

(where by convention $f_0 \equiv 0$), the Aubry set $\Omega(\mathcal{F})$ is a maximizing set.

We separate this theorem in two propositions. The first part is a general result.

Proposition 2.1. *For a sequence of continuous potentials $\mathcal{F} = \{f_k\}_{k\geq 1}$ that satisfies both conditions* **C1** *and* **C2***, the Aubry set contains the support of every maximizing probability:*

$$\mu \in \mathcal{M}_{max}(\mathcal{F}) \qquad \Rightarrow \qquad supp\, \mu \subset \Omega(\mathcal{F}).$$

Actually, this part is an immediate consequence of a generalized version that we obtain for the classical Atkinson's theorem [1], which, by its independent interest, consists in another contribution of this paper (for its statement, see theorem 2.2).

The converse implication requires the additional hypotheses.

Proposition 2.2. *For an almost sub-additive sequence of continuous potentials $\mathcal{F} = \{f_k\}_{k\geq 1}$, suppose that $\sup_{x\in X} \sup_{p>q\geq 0} \big[f_p(x) - f_q(x) - (p-q)\beta[\mathcal{F}]\big] < \infty$. Then any T-invariant probability whose support lies on the Aubry set is a maximizing measure:*

$$supp\, \mu \subset \Omega(\mathcal{F}) \qquad \Rightarrow \qquad \mu \in \mathcal{M}_{max}(\mathcal{F}).$$

For the sub-additive case, we remark that the following equality holds

$$\sup_{x\in X}\sup_{p>q\geq 0}\big[f_p(x)-f_q(x)-(p-q)\beta[\mathcal{F}]\big] = \sup_{x\in X}\sup_{k\geq 1}\big[f_k(x)-k\beta[\mathcal{F}]\big].$$

The equivalent hypothesis $\sup_{x\in X}\sup_{k\geq 1}\big[f_k(x)-k\beta[\mathcal{F}]\big]<\infty$ was already introduced to prove the so-called subordination principle in the additive and sub-additive contexts (for details, see [9, 3]).

In the next subsections, we provide the proofs of the preceding propositions.

2.1. *Proof of Proposition 2.1*

We follow here the main ideas of the proof of the analogous result in the additive case (see [4, 8]) to show the first implication of our central theorem: $\mu\in\mathcal{M}_{\max}(\mathcal{F})\Rightarrow \operatorname{supp}\mu\subset\Omega(\mathcal{F})$.

Let then $x\in\operatorname{supp}\mu$, where $\mu\in\mathcal{M}_{\max}(\mathcal{F})$. Given $\varepsilon>0$ and $L\geq 1$, we have to ensure the existence of a point $y\in B_\varepsilon(x)$ and integers $m>n\geq 0$, with $m-n\geq L$, such that

$$T^n y\in B_\varepsilon(x),\quad T^m y\in B_\varepsilon(x)\quad\text{and}\quad \Big|\big[f_m(y)-f_n(y)\big]-(m-n)\beta[\mathcal{F}]\Big|<\varepsilon.$$

Notice it is enough to show that, if $\lim_{k\to\infty}\frac{1}{k}\int f_k d\mu=\beta[\mathcal{F}]$, then, for some integers $m>n\geq 0$ with $m-n\geq L$, the set

$$\Big\{y\in B_\varepsilon(x)\cap T^{-n}\big(B_\varepsilon(x)\big)\cap T^{-m}\big(B_\varepsilon(x)\big):\big|\big[f_m(y)-f_n(y)\big]-(m-n)\beta[\mathcal{F}]\big|<\varepsilon\Big\}$$

has positive μ-measure.

This last claim is actually a corollary of the next generalized version of Atkinson's theorem. For such a general result, we only assume that a pointwise ergodic theorem holds for the sequence of functions $\mathcal{F}=\big\{f_k\big\}_{k\geq 1}$.

Theorem 2.2 (Generalized Atkinson's theorem). *Let $(X,\mathcal{B},\mu)$ be an arbitrary probability space and let $T:X\to X$ be any measure preserving map. Let then $\big\{f_k:X\to\mathbb{R}\big\}_{k\geq 1}$ be a sequence of measurable functions that satisfies condition* **C2**. *Consider the following assertions:*

(i) $\displaystyle\lim_{k\to\infty}\frac{1}{k}\int f_k\,d\mu=0;$

(ii) given a measurable set B with $\mu(B)>0$, for all $\varepsilon>0$ and $L\geq 1$, there exist $m>n\geq 0$ such that $m-n\geq L$ and

$$\mu\Big(B\cap T^{-n}(B)\cap T^{-m}(B)\cap\big\{y\in X:\big|f_m(y)-f_n(y)\big|<\varepsilon\big\}\Big)>0.$$

Then item (i) implies item (ii).

Atkinson's theorem was initially presented in [1] as a characterization of recurrence of random walks. The proof for the above generalized version is obtained with natural adaptations from the demonstration given in [13] for the original theorem.

Proof. Thanks to (2.1), we may assume without loss of generality that μ is an ergodic probability. We will argue by contradiction. Suppose that item (ii) does not hold, i. e. there exists a measurable set B with $\mu(B) > 0$, $\varepsilon > 0$ and $L \geq 1$ such that, for all $m > n \geq 0$ with $m - n \geq L$,

$$\mu\Big(B \cap T^{-n}(B) \cap T^{-m}(B) \cap \big\{y \in X : \big|f_m(y) - f_n(y)\big| < \varepsilon\big\}\Big) = 0. \tag{2.2}$$

For μ-a.e. $y \in B$, consider $\tau_B(k, y) := \sum_{j=0}^{k-1} \tau_B\big(T_B^j y\big)$, where $\tau_B : B \to \mathbb{N}$ is the first return map on the set B and $T_B y := T^{\tau_B(y)} y$. For every $k \geq 1$ and μ-a.e. $y \in B$, denote $f_B(k, y) := f_{\tau_B(k,y)}(y)$. Thanks to Poincaré's recurrence theorem, the set $\hat{B} := \cap_{k,l\geq 1} \cup_{m>n\geq k,\, m-n\geq l} \big(B \cap T^{-n}(B) \cap T^{-m}(B)\big)$ has the same positive measure as B. Notice that $\big\{y \in \hat{B} : \big|f_B(pL, y) - f_B(qL, y)\big| < \varepsilon$ for some $p > q \geq 1\big\}$ is a subset of $\cup_{m>n\geq 0,\, m-n\geq L}\big\{y \in B \cap T^{-n}(B) \cap T^{-m}(B) : \big|f_m(y) - f_n(y)\big| < \varepsilon\big\}$, which by (2.2) has zero measure. Therefore, for all $p > q \geq 1$,

$$\big|f_B(pL, y) - f_B(qL, y)\big| \geq \varepsilon \quad \mu\text{-a.e. } y \in B.$$

From this fact, an easy counting argument shows that there exist at most $N_r := \left\lceil \frac{2r}{\varepsilon} + 1 \right\rceil$ distinct values of $\big\{f_B(kL, y)\big\}_{k\geq 1}$ in the interval $[-r, r]$. Hence, one can inductively introduce sequences of positive integers $\{p_j\}_{j\geq 0}$ and $\{r_j\}_{j\geq 0}$ (with $r_0 = 1$) given by

$$p_j = \min\big\{k : |f_B(kL, y)| > r_j\big\} \quad \text{and}$$
$$r_{j+1} = 1 + \sup\big\{|f_B(kL, y)| : k \leq N_{r_j} + 1\big\}.$$

The sequences $\{r_j\}_{j\geq 0}$ and $\{p_j\}_{j\geq 0}$ are both increasing with $p_{j+1} > N_{r_j} + 1 \geq p_j$. Moreover,

$$r_j > \frac{(N_{r_j} - 2)\varepsilon}{2} \qquad \text{and} \qquad \frac{N_{r_j} - 2}{p_j} \geq 1 - \frac{3}{p_j}.$$

Therefore, we obtain the following inequalities for μ-a.e. $y \in B$

$$\liminf_{j\to\infty} \frac{|f_B(p_j L, y)|}{p_j L} \geq \liminf_{j\to\infty} \frac{r_j}{p_j L} \geq \liminf_{j\to\infty} \frac{(N_{r_j} - 2)\varepsilon}{2p_j L} \geq \frac{\varepsilon}{2L} - \lim_{j\to\infty} \frac{3\varepsilon}{2p_j L} = \frac{\varepsilon}{2L}.$$

On the other hand, condition **C2** and Birkhoff's ergodic theorem applied to $\mathbb{1}_B$ ensure that, for μ-a.e. $y \in B$,

$$\begin{aligned}
\frac{1}{\mu(B)}\left|\lim_{k\to\infty} \frac{1}{k} \int f_k\, d\mu\right| &= \frac{1}{\mu(B)}\left|\lim_{k\to\infty} \frac{f_k(y)}{k}\right| = \Big(\int \mathbb{1}_B\, d\mu\Big)^{-1}\left|\lim_{k\to\infty} \frac{f_{\tau_B(k,y)}(y)}{\tau_B(k, y)}\right| \\
&= \lim_{k\to\infty} \frac{\tau_B(k, y)}{\sum_{j=0}^{\tau_B(k,y)-1} \mathbb{1}_B \circ T^j y} \left|\lim_{k\to\infty} \frac{f_B(k, y)}{\tau_B(k, y)}\right| \\
&= \lim_{k\to\infty} \frac{\big|f_B(k, y)\big|}{k} \geq \liminf_{j\to\infty} \frac{\big|f_B(p_j L, y)\big|}{p_j L} \geq \frac{\varepsilon}{2L}.
\end{aligned}$$

Hence, $\big|\lim_{k\to\infty} \frac{1}{k} \int f_k d\mu\big| \geq \frac{\varepsilon}{2L}\mu(B) > 0$, which contradicts item (i). □

2.2. *Proof of Proposition 2.2*

Let $\mathcal{F} = \{f_k\}_{k\geq 1}$ be an asymptotically sub-additive sequence of continuous potentials such that

$$R := \sup_{x\in X} \sup_{p>q\geq 0} \big[f_p(x) - f_q(x) - (p-q)\beta[\mathcal{F}]\big] < \infty.$$

We would like to show the converse implication: $\operatorname{supp}\mu \subset \Omega(\mathcal{F}) \Rightarrow \mu \in \mathcal{M}_{\max}(\mathcal{F})$.

In order to do that, we will need an auxiliary tool, which should be seen as a generalization of the usual concept of sub-action associated with additive potentials in ergodic optimization (see [4, 6, 8]).

Definition 2.2. A sequence of measurable functions $\mathcal{U} = \{u_k : X \to \mathbb{R}\}_{k\geq 1}$ is a *corrector* for $\mathcal{F} = \{f_k : X \to \mathbb{R}\}_{k\geq 1}$ if

(i) $f_k(x) - u_k(x) \leq k\beta[\mathcal{F}], \quad \forall\, x \in X, \quad \forall\, k \geq 1;$

(ii) $\displaystyle\lim_{k\to\infty} \frac{1}{k}\int u_k\, d\mu = 0, \quad \forall\, \mu \in \mathcal{M}_T \text{ with } \operatorname{supp}\mu \subset \Omega(\mathcal{F}).$

It is an easy task to show from these conditions that the *corrected sequence* $\mathcal{F}-\mathcal{U} := \{f_k - u_k\}_{k\geq 1}$ verifies both $\beta[\mathcal{F}] = \beta\big[\mathcal{F}-\mathcal{U}\big]$ and $\mathcal{M}_{\max}(\mathcal{F}) \subset \mathcal{M}_{\max}(\mathcal{F}-\mathcal{U})$. Conversely, $\mu \in \mathcal{M}_{\max}(\mathcal{F}-\mathcal{U})$ with $\operatorname{supp}\mu \subset \Omega(\mathcal{F})$ implies $\mu \in \mathcal{M}_{\max}(\mathcal{F})$. From this fact, to prove proposition 2.2 we show that $\operatorname{supp}\mu \subset \Omega(\mathcal{F}) \Rightarrow \mu \in \mathcal{M}_{\max}(\mathcal{F}-\mathcal{U})$. Note then that, given a corrector $\mathcal{U} = \{u_k\}_{k\geq 1}$ and a positive constant Γ, clearly

$$\operatorname{supp}\mu \subset \bigcap_{k\geq 1} \big(f_k - u_k\big)^{-1}\Big[k\beta[\mathcal{F}] - \Gamma, k\beta[\mathcal{F}]\Big] \quad \Rightarrow \quad \mu \in \mathcal{M}_{\max}(\mathcal{F}-\mathcal{U}). \tag{2.3}$$

The following lemma provides an example of a corrector.

Lemma 2.2. *For an asymptotically sub-additive sequence of continuous potentials $\mathcal{F} = \{f_k\}_{k\geq 1}$, the real-valued measurable functions $\mathcal{U} = \{u_k\}_{k\geq 1}$ defined by*

$$u_k(x) := f_k(x) - k\beta[\mathcal{F}] + R - \lim_{\varepsilon\to 0} \sup_{\substack{p>q\geq 0\\ p-q\geq k+1}} \sup_{T^q z\in B_\varepsilon(x)} \big[f_p(z) - f_q(z) - (p-q)\beta[\mathcal{F}]\big]$$

verify $f_k - u_k \leq k\beta[\mathcal{F}]$ and $\lim_{k\to\infty} \frac{1}{k}\int u_k\, d\mu \leq 0$ for $\mu \in \mathcal{M}_T$ with $\operatorname{supp}\mu \subset \Omega(\mathcal{F})$. Moreover, if $\mathcal{F} = \{f_k\}_{k\geq 1}$ is almost sub-additive, then $\mathcal{U}$ is a corrector for $\mathcal{F}$.

Proof. First, we focus on the measurable functions

$$h_k(x) := \lim_{\varepsilon\to 0} \sup_{\substack{p>q\geq 0\\ p-q\geq k}} \sup_{T^q z\in B_\varepsilon(x)} \big[f_p(z) - f_q(z) - (p-q)\beta[\mathcal{F}]\big].$$

It is immediate that $R \geq h_k \geq f_k - k\beta[\mathcal{F}]$, for all $k \geq 1$. In particular,

$$f_k - u_k = k\beta[\mathcal{F}] + h_{k+1} - R \leq k\beta[\mathcal{F}], \quad \forall\, k \geq 1.$$

Furthermore, by the definition of an Aubry point, it is easy to see that $h_k(x) \geq 0$ for all $k \in \mathbb{N}$ and for all $x \in \Omega(\mathcal{F})$. Thus, since $h_k \leq R$ everywhere on X, for any $\mu \in \mathcal{M}_T$ such that $\operatorname{supp} \mu \subset \Omega(\mathcal{F})$, we have $\lim_{k\to\infty} \frac{1}{k} \int h_k \, d\mu = 0$, which yields

$$\lim_{k\to\infty} \frac{1}{k} \int u_k \, d\mu = \lim_{k\to\infty} \frac{1}{k} \int f_k \, d\mu - \beta[\mathcal{F}] \leq 0.$$

For the opposite inequality, we suppose that $\mathcal{F} = \{f_k\}_{k\geq 1}$ is almost sub-additive. The following claim is enough to guarantee that $\lim_{k\to\infty} \frac{1}{k} \int u_k \, d\mu \geq 0$ for $\mu \in \mathcal{M}_T$.

Claim. For all $k \geq 1$ and for all $x \in X$, $f_k(x) - k\beta[\mathcal{F}] + R \geq h_{k+1}(x) - C - 2$, where $C > 0$ is the constant given by the almost sub-additive property.

Given $k \geq 1$, there exists $\varepsilon \in (0,1)$ such that $f_k(B_\varepsilon(z)) \in B_1\big(f_k(z)\big)$ for all $z \in X$. Consider integers $m > n \geq 0$, with $m - n \geq k+1$, and a point $y \in T^{-n}\big(B_\varepsilon(x)\big)$ such that

$$f_m(y) - f_n(y) - (m-n)\beta[\mathcal{F}] > \sup_{\substack{p>q\geq 0\\ p-q\geq k+1}} \sup_{T^q z \in B_\varepsilon(x)} \big[f_p(z) - f_q(z) - (p-q)\beta[\mathcal{F}]\big] - 1.$$

Note now that

$$\begin{aligned} f_k(x) - k\beta[\mathcal{F}] + R &\geq f_k(T^n y) - 1 - k\beta[\mathcal{F}] + R \\ &\geq f_{k+n}(y) - f_n(y) - C - 1 - k\beta[\mathcal{F}] + \\ &\quad + f_m(y) - f_{k+n}(y) - (m-k-n)\beta[\mathcal{F}] \\ &= f_m(y) - f_n(y) - (m-n)\beta[\mathcal{F}] - C - 1 \\ &> \sup_{\substack{p>q\geq 0\\ p-q\geq k+1}} \sup_{T^q z \in B_\varepsilon(x)} \big[f_p(z) - f_q(z) - (p-q)\beta[\mathcal{F}]\big] - C - 2 \\ &\geq h_{k+1}(x) - C - 2, \end{aligned}$$

where the first inequality comes from the fact that $T^n y \in B_\varepsilon(x)$, the second one follows from the almost sub-additive property and the definition of R, and the last one reflects that the above supremum decreases to $h_{k+1}(x)$ as ε tends to zero. □

We highlight the following key lemma, which together with (2.3) concludes the proof of proposition 2.2.

Lemma 2.3. *For an almost sub-additive sequence of continuous potentials $\mathcal{F} = \{f_k\}_{k\geq 1}$, the corrector $\mathcal{U} = \{u_k\}_{k\geq 1}$ given by lemma 2.2 satisfies*

$$\Omega(\mathcal{F}) \subset \bigcap_{k\geq 1} \big(f_k - u_k\big)^{-1}\Big[k\beta[\mathcal{F}] - R, k\beta[\mathcal{F}]\Big].$$

Proof. As remarked (in the proof of lemma 2.2), for all $k \geq 1$ and $x \in \Omega(\mathcal{F})$,

$$\lim_{\varepsilon\to 0} \sup_{\substack{p>q\geq 0\\ p-q\geq k+1}} \sup_{T^q z \in B_\varepsilon(x)} \big[f_p(z) - f_q(z) - (p-q)\beta[\mathcal{F}]\big] \geq 0,$$

hence $k\beta[\mathcal{F}] - R \leq f_k - u_k \leq k\beta[\mathcal{F}]$ everywhere on the Aubry set. □

3. Joint spectral radius

In this section, we intend to show some contributions of the Aubry set for the study of the joint spectral radius. The final aim will be to argue that there always exists a periodic matrix configuration that can be used to approximate the joint spectral radius within a given precision (see proposition 3.1).

Initially, we summarize important facts about the joint spectral radius. For a detailed account, consult [2, 7, 10, 11, 14] and references therein. Given a compact set of $d \times d$ matrices $\Sigma \subset \mathbb{R}^{d\times d}$, the aforementioned *joint spectral radius* is

$$\rho(\Sigma) = \lim_{k\to\infty} \max\big\{\|M_{k-1}\cdots M_0\|^{1/k} \,:\, M_j \in \Sigma\big\}.$$

In particular, this definition is independent of the chosen sub-multiplicative norm.

The *irreducibility* for a set of matrices states that only the trivial subspaces $\{\vec{0}\}$ and $\mathbb{R}^d$ are invariant under all matrices in such a set. If Σ is not irreducible, one can simultaneously block-triangularize matrices in Σ, in the sense that there exists a similarity transformation Ξ for which

$$\Xi M \Xi^{-1} = \begin{bmatrix} M^{11} & M^{12} & \cdots & M^{1t} \\ 0 & M^{22} & \cdots & M^{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M^{tt} \end{bmatrix}, \qquad \forall\, M \in \Sigma.$$

For $1 \le j \le t$, consider the compact sets of matrices $\Sigma_i := \{M^{ii} \,:\, M \in \Sigma\}$, where each of them are irreducible or $\{0\}$. It follows that $\rho(\Sigma) = \max\big\{\rho(\Sigma_i) \,:\, 1 \le i \le t\big\}$. Actually, this equality allows us to always assume that Σ is an irreducible set. It is a known fact that $\rho(\Sigma) > 0$ in this case (see, for instance, lemma 2.2 of [7]).

Irreducibility also guarantees the existence of an *extremal norm*, i. e. a sub-multiplicative matrix norm $\|\cdot\|_e$ which verifies

$$\|M_{k-1}\cdots M_0\|_e \le \rho(\Sigma)^k, \qquad \forall\, M_j \in \Sigma,\ \forall\, k \ge 1.$$

As a matter of fact, it is well known the existence of a vector norm $|\cdot|_B$, called *Barabanov norm*, such that not only $|M\vec{v}|_B \le \rho(\Sigma)|\vec{v}|_B$ for all $\vec{v} \in \mathbb{R}^d$ and $M \in \Sigma$, but also, for any $\vec{v} \in \mathbb{R}^d$, there exists $\bar{M} \in \Sigma$ with $|\bar{M}\vec{v}|_B = \rho(\Sigma)|\vec{v}|_B$. In particular, the induced operator norm for $|\cdot|_B$ is an extremal norm.

From now on, we will focus on a dynamical approach for the joint spectral radius. Consider the topological dynamical system given by the compact full-shift $\Sigma^{\mathbb{N}}$ and the one-sided shift map $\sigma : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$. We fix on $\Sigma^{\mathbb{N}}$ the metric compatible with the product topology

$$\mathsf{d}_{\Sigma^{\mathbb{N}}}\big((M_0, M_1, \ldots), (M_0', M_1', \ldots)\big) := \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{\mathsf{d}_\Sigma(M_j, M_j')}{1 + \mathsf{d}_\Sigma(M_j, M_j')},$$

where d_Σ denotes the restricted Euclidean metric on Σ. For the sequence of continuous potentials $\mathcal{F}_{\|\cdot\|} = \{\log\phi_k : \Sigma \to \mathbb{R}\}_{k\ge1}$ defined by $\log\phi_k(M_0, M_1, \ldots) :=$

$\log \|M_{k-1} \cdots M_0\|$, we have already pointed out that Schreiber's theorem ensures

$$\beta[\mathcal{F}_{\|\cdot\|}] = \max\left\{ \lim_{k\to\infty} \frac{1}{k} \int \log \phi_k \, d\mu \,:\, \mu \in \mathcal{M}_\sigma \right\} = \log \rho(\Sigma), \tag{3.1}$$

whenever $\|\cdot\|$ is sub-multiplicative. Recall that, given a sub-multiplicative $\|\cdot\|_S$, the sequence $\mathcal{F}_{\|\cdot\|_S} = \{\log \phi_k^S\}_{k\geq 1}$ has the sub-additive property. Due to the equivalence of norms in finite dimensional vector spaces, a non-sub-multiplicative norm $\|\cdot\|$ induces an almost sub-additive sequence $\mathcal{F}_{\|\cdot\|} = \{\log \phi_k\}_{k\geq 1}$. In particular, the constant $\beta[\mathcal{F}_{\|\cdot\|}]$ is independent of the chosen norm, and equation (3.1) can be extended to an arbitrary norm, with the obvious generalization of $\rho(\Sigma)$.

By the previous discussion, let Σ be an irreducible compact set of matrices. Thus, $\log \rho(\Sigma) \in \mathbb{R}$. Consider $\mathcal{F}_{\|\cdot\|_e}$ the corresponding sequence of continuous potentials associated with some extremal norm $\|\cdot\|_e$. In particular, such a sequence is sub-additive and satisfies

$$\sup_{(M_0,M_1,\ldots)\in\Sigma^{\mathbb{N}}} \sup_{k\geq 1} \left[\log \|M_{k-1} \cdots M_0\|_e - k\beta[\mathcal{F}_{\|\cdot\|_e}] \right] \leq 0.$$

Given another matrix norm $\|\cdot\|$, let $C_{\|\cdot\|} > 1$ be a constant for which $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent, i. e. $C_{\|\cdot\|}^{-1} \|\cdot\|_e \leq \|\cdot\| \leq C_{\|\cdot\|} \|\cdot\|_e$. From this relation is immediate that

$$\sup_{(M_0,M_1,\ldots)\in\Sigma^{\mathbb{N}}} \sup_{p>q\geq 0} \left[\log \|M_{p-1} \cdots M_0\| - \log \|M_{q-1} \cdots M_0\| - (p-q)\beta[\mathcal{F}_{\|\cdot\|}] \right] \leq$$
$$\leq \sup_{(M_0,M_1,\ldots)\in\Sigma^{\mathbb{N}}} \sup_{p-q\geq 1} \left[\log C_{\|\cdot\|}^2 \|M_{p-1} \cdots M_q\|_e - (p-q)\beta[\mathcal{F}_{\|\cdot\|_e}] \right] \leq 2 \log C_{\|\cdot\|}.$$

Thus, all sufficient conditions of the central result are fulfilled, and we have the following complement for Schreiber's theorem.

Corollary 3.1. *Let Σ be an irreducible compact set of matrices and $\|\cdot\|$ be an arbitrary matrix norm. Then, a σ-invariant probability μ satisfies*

$$\log \rho(\Sigma) = \lim_{k\to\infty} \frac{1}{k} \int \log \|M_{k-1} \cdots M_0\| \, d\mu(M_0, M_1, \ldots)$$

if, and only if, supp μ is contained in $\Omega(\mathcal{F}_{\|\cdot\|})$.

Let $(M_0, M_1, \ldots)$ be an Aubry point. For all $\tilde{\varepsilon} = \varepsilon/\rho(\Sigma) > 0$ and for any integer $L \geq 1$, there exist $(M_0', M_1', \ldots) \in B_{\tilde{\varepsilon}}(M_0, M_1, \ldots)$ and integers $m > n \geq 0$, with $m - n \geq L$, such that

$$(M_n', M_{n+1}', \ldots) \in B_{\tilde{\varepsilon}}(M_0, M_1, \ldots), \quad (M_m', M_{m+1}', \ldots) \in B_{\tilde{\varepsilon}}(M_0, M_1, \ldots)$$
$$\text{and} \quad \left| \log \|M_{m-1}' \cdots M_0'\| - \log \|M_{n-1}' \cdots M_0'\| - (m-n) \log \rho(\Sigma) \right| < \tilde{\varepsilon}.$$

Rewrite the last inequality in the following form

$$\rho(\Sigma)\, e^{-\tilde{\varepsilon}/(m-n)} < \left(\frac{\|M_{m-1}' \cdots M_0'\|}{\|M_{n-1}' \cdots M_0'\|} \right)^{1/(m-n)} < \rho(\Sigma)\, e^{\tilde{\varepsilon}/(m-n)}.$$

Notice then that

$$\rho(\Sigma) - \frac{\varepsilon}{L} \leq \rho(\Sigma)\left(1 - \frac{\tilde{\varepsilon}}{m-n}\right) \leq \rho(\Sigma)\, e^{-\tilde{\varepsilon}/(m-n)} < \left(\frac{\|M'_{m-1}\cdots M'_0\|}{\|M'_{n-1}\cdots M'_0\|}\right)^{1/(m-n)}.$$

Due to definition of an extremal norm, one concludes that

$$\left(\frac{\|M'_{m-1}\cdots M'_0\|}{\|M'_{n-1}\cdots M'_0\|}\right)^{1/(m-n)} \leq \left(C^2_{\|\cdot\|}\|M'_{m-1}\cdots M'_n\|_e\right)^{1/(m-n)} \leq C^{2/L}_{\|\cdot\|}\rho(\Sigma),$$

and, for a sub-multiplicative norm $\|\cdot\|$,

$$\left(\frac{\|M'_{m-1}\cdots M'_0\|}{\|M'_{n-1}\cdots M'_0\|}\right)^{1/(m-n)} \leq \left(C_{\|\cdot\|}\|M'_{m-1}\cdots M'_n\|_e\right)^{1/(m-n)} \leq C^{1/L}_{\|\cdot\|}\rho(\Sigma).$$

We summarize the previous steps in the next result.

Proposition 3.1. *Let Σ be an irreducible compact set of matrices and $\|\cdot\|$ be an arbitrary matrix norm. Then, for all $\varepsilon > 0$ and for any integer $L \geq 1$, there exist integers $m > n \geq 0$, with $m - n \geq L$, and a periodic matrix configuration $(M'_0, M'_1, \ldots, M'_{m-1}, M'_0, \ldots)$ such that*

$$\rho(\Sigma) - \frac{\varepsilon}{L} \leq \left(\frac{\|M'_{m-1}\cdots M'_0\|}{\|M'_{n-1}\cdots M'_0\|}\right)^{1/(m-n)} \leq C^{2/L}_{\|\cdot\|}\rho(\Sigma).$$

If, in addition, $\|\cdot\|$ is sub-multiplicative, then there exist an integer $k \geq L$ and a periodic matrix configuration $(M'_0, M'_1, \ldots, M'_{k-1}, M'_0, \ldots)$ such that

$$\rho(\Sigma) - \frac{\varepsilon}{L} \leq \|M'_{k-1}\cdots M'_0\|^{1/k} \leq C^{1/L}_{\|\cdot\|}\rho(\Sigma).$$

As a last remark, we could consider the *generalized spectral radius*

$$\varrho(\Sigma) := \limsup_{k\to\infty} \max\left\{\varrho(M_{k-1}\cdots M_0)^{1/k} \,:\, M_j \in \Sigma\right\}.$$

where $\varrho(M) := \max\{|\lambda| : \lambda \text{ eigenvalue of } M\}$ is, as usual, the spectral radius of the matrix M. It is well known that both notions of spectral radius for set of matrices coincide. In an attempt to deal directly with this supremum limit, we could introduce the sequence $\mathcal{F}_\varrho = \{\log\psi_k : \Sigma \to \mathbb{R}\}_{k\geq 1}$, where $\log\psi_k(M_0, M_1, \ldots) := \log\varrho(M_{k-1}\cdots M_0)$. The pointwise asymptotic behaviour of the sequence $\mathcal{F}_\varrho$ is similar to the function $\tilde{f}_{\|\cdot\|}$, as it was proved in [10]: for any σ-invariant probability μ,

$$\limsup_{k\to\infty} \frac{1}{k}\log\varrho(M_{k-1}\cdots M_0) = \lim_{k\to\infty}\frac{1}{k}\log\|M_{k-1}\cdots M_0\|$$

for μ-a.e. $(M_0, M_1, \ldots) \in \Sigma^{\mathbb{N}}$. However, in [2] Avila and Bochi provided an explicit example for which condition **C2** fails for such a sequence.

Acknowledgments

The authors thank J. Bochi and P. Thieullen for enlightening conversations. This article consists in the first results of the PhD thesis of the second author under the supervision of the first one.

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