MINIMAL YET MEASURABLE FOLIATIONS

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ABSTRACT. In this paper we mainly address the problem of disintegration of Lebesgue measure along the central foliation of volume preserving diffeomorphisms isotopic to hyperbolic automorphisms of 3-torus. We prove that atomic disintegration of the Lebesgue measure (ergodic case) along the central foliation has the peculiarity of being mono-atomic (one atom per leaf). This implies the measurability of the central foliation. As a corollary we provide open and non-empty subset of partially hyperbolic diffeomorphisms with minimal yet measurable central foliation.

CONTENTS

1. INTRODUCTION

Let $(M, \mu, \mathcal{B})$ be a probability space, where M a compact metric space, μ a probability measure and $\mathcal{B}$ the borelian σ-algebra. Given a partition $\mathcal{P}$ of M by measurable subsets, the quotient space can be equipped with a natural measure and in general it may be a singular measure space, i.e the measurable subsets have just measure zero or one. For example, this is the case for the

partition of $\mathbb{T}^2$ into orbits of irrational flow (Kronecker flow). By the way, there is an opposite situation which is the case of *measurable partitions*. The word measurable here should be understood as countably generated, as in Definition 2.4. In this case, the measure μ can be disintegrated into probability (conditional) measures. See Section 2 for details. A foliation is a particular case of (not necessarily measurable) partition. It is well worth recall that the quotient space of a foliated space may not be Hausdorff. This is the case of the example of Sullivan [22] where all the leaves are compact however, their size goes to infinity. Indeed, any foliation with compact leaves is measurable. See Proposition 2.5.

In the smooth ergodic theory, one of the natural partitions to be studied is the partition into leaves of invariant foliations by a dynamical system. Some celebrated examples are the stable and unstable foliations of (uniformly) hyperbolic dynamics. The partition into global stable or unstable leaves is non measurable and there are deep results where techniques are developed to find measurable partitions "subordinated" to the global leaves (see for instance [16]).

Non measurability "typically" comes with some non trivial topological properties of the leaves. For instance this is the case of stable and unstable leaves of a transitive Anosov diffeomorphism. These foliations are minimal, i.e every leaf is dense. Although the notion of minimality seems to be in the opposite direction to the measurability, in this paper we find minimal yet measurable foliations.

Our results mainly focus on the comprehension of disintegration of volume measure along the central invariant foliation of partially hyperbolic diffeomorphisms.

Definition 1.1. *A diffeomorphism is called partially hyperbolic if the tangent bundle of the ambient manifold admits an invariant decomposition $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors $v^\sigma \in E_x^\sigma, \sigma \in \{s, c, u\}$ for all $x \in M$ satisfy:*

$$\|D_x f v^s\| < \|D_x f v^c\| < \|D_x f v^u\|$$

*and moreover $\|Df|E^s\| < 1$ and $\|Df^{-1}|E^u\| < 1$. We call f **absolute** partially hyperbolic, if it is partially hyperbolic and for any $x, y, z \in M$*

$$\|D_x f v^s\| < \|D_y f v^c\| < \|D_z f v^u\|$$

where v^s, v^c and v^u belong respectively to E_x^s, E_y^c and E_z^u.

In this paper by partial hyperbolicity, we mean absolute partial hyperbolicity. It is well known (Hirsch-Pugh-Shub [15]) that E^s and E^u are tangent to two invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. In general E^c is not tangent to an invariant foliation, however absolute partially hyperbolic diffeomorphisms defined on $\mathbb{T}^3$ admit invariant central foliation (See [6].)

Theorem A. *There exists an open subset U of volume preserving partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ such that for any $g \in U$ the central foliation is minimal yet measurable (with respect to Lebesgue measure) as partition.*

The proof of the above result is based on a careful study of disintegration of Lebesgue measure along the central foliation of diffeomorphisms in U which were constructed in [17]. We consider the example of the above theorem as a nice application of dynamical systems argument in topologic and geometric measure theory properties of foliations.

In fact we prove a general result on the disintegration of Lebesgue measure along the central foliation of *derived from Anosov* diffeomorphisms on $\mathbb{T}^3$.

Definition 1.2. *We say that $f : \mathbb{T}^3 \to \mathbb{T}^3$ is derived from Anosov or just a DA diffeomorphism if it is partially hyperbolic and its action on the homotopy is a hyperbolic automorphism (no eigenvalue of norm one).*

Theorem B. *Let f be an ergodic volume preserving DA diffeomorphism on $\mathbb{T}^3$. If the volume has atomic disintegration on the center leaves, then the disintegration is mono atomic.*

In principle the disintegration of the Lebesgue measure along central foliation of a DA diffeomorphism can be non-atomic even if it is singular with respect to Lebesgue (see [24]). This makes the study of derived from Anosov (even Anosov, seen as partially hyperbolic systems) diffeomorphisms interesting from the point of view of geometric measure theory.

Another relevant comment is that for a general partially hyperbolic diffeomorphism (not derived from Anosov), atomic disintegration along central foliation does not imply mono atomicity. There exist partially hyperbolic diffeomorphisms where the disintegration of Lebesgue measure is atomic with finite (strictly larger than one) atoms. There are also systems with conditional measures with infinitely many atoms. See section 3 for some examples and discussions.

Let us recall the definition of Lyapunov exponents of f along the invariant bundles E^s, E^c and E^u.

$$\lambda^\tau(x) := \lim_{n \to \infty} \frac{1}{n} \log ||Df^n(x) \cdot v||$$

where $v \in E^\tau$ and $\tau \in \{s, c, u\}$. If f is ergodic, the stable, center and unstable Lyapunov exponents are constant almost everywhere.

In the proof of Theorem A we take advantage of the fact that the sign of central Lyapunov exponent for any $f \in U$ is opposite to the sign of the central Lyapunov exponent of its linearization (see section 3 for definition) f_*. Then, the following result actually guarantees the proof of Theorem A.

Theorem C. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a volume preserving, DA diffeomorphism. Suppose its linearization A has the splitting $T_A M = E^{su} \oplus E^{wu} \oplus E^s$ (su and wu represents strong unstable and weak unstable.) If f has $\lambda^c(x) < 0$ for Lebesgue almost every point $x \in \mathbb{T}^3$, then volume has atomic disintegration on $\mathcal{F}_f^c$, in fact the disintegration is mono atomic.*

1.1. **Comments on the results and the structure of the paper.** The diffeomorphisms in U are constructed isotopic to Anosov linear diffeomorphisms on $\mathbb{T}^3$. By a Hammerlindl [10] result, partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ are leaf conjugated to their linearization one and hence the central foliation of any $g \in U$ is minimal.

As we mentioned before, any $g \in U$ satisfies the hypothesis of Theorem C and consequently the disintegration of Lebesgue measure along the central foliation is mono-atomic. The mono-atomicity implies that the partition into central leaves is equivalent to the partition into single points and hence it is measurable. So, we get minimal and measurable foliation.

And finally, the key point in the proof of Theorem C is to show that under the hypothesis of the theorem, we get atomic disintegration. The technical part of the proof of Theorem C is similar to the arguments of [19].

The paper is organized as following: In sections 2 and 3 we review some preliminary results in abstract measure theory and partially hyperbolic dynamics. In section 4.1 we recall an open set of partially hyperbolic diffeomorphisms which satisfy the hypothesis of Theorem C and give the example claimed in Theorem A. Finally we prove Theorems B and C in Section 5. The proof of Theorem A comes directly from Theorems C and B.

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2. Measurable partitions and disintegration of measures

Let $(M, \mu, \mathcal{B})$ be a probability space, where M is a compact metric space, μ a probability measure and $\mathcal{B}$ the borelian σ-algebra. Given a partition $\mathcal{P}$ of M by measurable sets, we associate the probability space $(\mathcal{P}, \widetilde{\mu}, \widetilde{\mathcal{B}})$ by the following way. Let $\pi : M \to \mathcal{P}$ be the canonical projection, that is, π associates a point x of M with the partition element of $\mathcal{P}$ that contains it. Then we define $\widetilde{\mu} := \pi_* \mu$ and $\widetilde{\mathcal{B}} := \pi_* \mathcal{B}$.

Definition 2.1. *Given a partition $\mathcal{P}$. A family $\{\mu_P\}_{P \in \mathcal{P}}$ is a system of conditional measures for μ (with respect to $\mathcal{P}$) if*

  i) *given $\phi \in C^0(M)$, then $P \mapsto \int \phi \mu_P$ is measurable;*
  ii) *$\mu_P(P) = 1$ $\widetilde{\mu}$-a.e.;*
  iii) *if $\phi \in C^0(M)$, then $\displaystyle\int_M \phi d\mu = \int_{\mathcal{P}} \left(\int_P \phi d\mu_P \right) d\widetilde{\mu}.$*

When it is clear which partition we are referring to, we say that the family $\{\mu_P\}$ *disintegrates* the measure μ.

Proposition 2.2. [7, 18] *If $\{\mu_P\}$ and $\{\nu_P\}$ are conditional measures that disintegrate μ, then $\mu_P = \nu_P$ $\widetilde{\mu}$-a.e.*

Corollary 2.3. *If $T : M \to M$ preserves a probability μ and the partition $\mathcal{P}$, then $T_*\mu_P = \mu_{T(P)}, \widetilde{\mu}$-a.e.*

Proof. It follows from the fact that $\{T_*\mu_P\}_{P \in \mathcal{P}}$ is also a disintegration of μ. □

Definition 2.4. *We say that a partition $\mathcal{P}$ is measurable (or countably generated) with respect to μ if there exist a measurable family $\{A_i\}_{i \in \mathbb{N}}$ and a measurable set F of full measure such that if $B \in \mathcal{P}$, then there exists a sequence $\{B_i\}$, where $B_i \in \{A_i, A_i^c\}$ such that $B \cap F = \bigcap_i B_i \cap F$.*

Proposition 2.5. *Let $(M, \mathcal{B}, \mu)$ a probability space where M is a compact metric space and $\mathcal{B}$ is the Borel sigma-algebra. If $\mathcal{P}$ is a continuous foliation of M by compact measurable sets, then $\mathcal{P}$ is a measurable partition.*

Proof. This proposition has been proved in a preprint by Avila, Viana and Wilkinson. We put the argument for the sake of completeness. Let $\{x_i\}$ be a countable dense subset of M. For each x_i and $n \geq 1$, define $V(x_i, n)$ as the points $z \in M$ such that $\mathcal{P}_z$ intersects the closed ball of radius $\frac{1}{n}$ around x_i. It is easy to see that $V(x_i, n)$ is closed and hence measurable. Here we use the continuity of foliation. By definition $V(x_i, n)$ is a saturated subset, i.e it contains the whole leaves of its points. For any two different leaves $\mathcal{P}_x$ and $\mathcal{P}_y$. Take a large n and x_i such that $\mathcal{P}_x$ intersects the closed ball $B(x_i, \frac{1}{n})$. By compactness of leaves, if n is large enough ($\frac{2}{n}$ is smaller than the distance between the leaves) then $B(x_i, \frac{1}{n}) \cap \mathcal{P}_y = \emptyset$. □

Theorem 2.6 (Rokhlin's disintegration [18]). *Let $\mathcal{P}$ be a measurable partition of a compact metric space M and μ a borelian probability. Then there exists a disintegration by conditional measures for μ.*

In general the partition by the leaves of a foliation may be non-measurable. It is for instance the case for the stable and unstable foliations of a linear Anosov diffeomorphism. Therefore, by disintegration of a measure along the leaves of a foliation we mean the disintegration on compact foliated boxes. In principle, the conditional measures depend on the foliated boxes, however, two different foliated boxes induce proportional conditional measures. See [1] for a discussion. We define absolute continuity of foliations as follows:

Definition 2.7. *We say that a foliation $\mathcal{F}$ is absolutely continuous if for any foliated box, the disintegration of volume on the segment leaves have conditional measures equivalent to the Lebesgue measure on the leaf.*

Definition 2.8. *We say that a foliation $\mathcal{F}$ has atomic disintegration with respect to a measure μ if the conditional measures on any foliated box are a sum of Dirac measures.*

Although the disintegration of a measure along a general foliation is defined in compact foliated boxes, it makes sense to say that the foliation $\mathcal{F}$ has a quantity

$k_0 \in \mathbb{N}$ atoms per leaf. The meaning of "per leaf" should always be understood as a generic leaf, i.e. almost every leaf. That means that there is a set A of μ-full measure which intersects a generic leaf on exactly k_0 points. Let's see that this implies atomic disintegration. Definition 2.1 shows that it only make sense to talk about conditional measures from the generic point of view, hence when restricted to a foliated box $\mathfrak{B}$, the set $A \cap \mathfrak{B}$ has μ-full measure on $\mathfrak{B}$, therefore the support of the conditional measure disintegrated on $\mathfrak{B}$ must be contained on the set A. This implies atomic disintegration.

It is well worth to remark that the weight of an atom for a conditional measure naturally depends on the foliated box, but the fact that a point x is atom or not is independent of the foliated box where we disintegrate a measure and by Corollary 2.3 the set of atoms is invariant under the dynamics. For a more detailed discussion about dependence of disintegration to the foliated box see Lemma 3.2 of [1].

3. Partial hyperbolicity and disintegration of volume

Let f be a partially hyperbolic diffeomorphism with

$$TM = E^s \oplus E^c \oplus E^u.$$

The subbundles E^s and E^u integrate into f-invariant foliations, respectively the stable foliation, $\mathcal{F}^s$, and the unstable foliation, $\mathcal{F}^u$. These foliations are absolutely continuous. A set of full volume measure on M must intersect almost every leaf of $\mathcal{F}^s$ (or $\mathcal{F}^u$) in a set of full Lebesgue measure of the leaf. Although the absolute continuity of $\mathcal{F}^s$ and $\mathcal{F}^u$ are mandatory for a (general) C^2 partially hyperbolic diffeomorphism, this is not the case for the center foliation $\mathcal{F}^c$ (it is not even true that there will exist such a foliation, but by [6] for all absolute partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ the center foliation exists). The center foliation might not be absolutely continuous, at least this is, in general, expected to happen for diffeomorphisms which preserves volume (see [20], [14], [9], [21], [2]). For many examples (some of them described below) the center foliation has atomic disintegration. In principle for a general partially hyperbolic diffeomorphism the geometric structure of the support of disintegration measures is not clear.

There exist essentially three known category of partially hyperbolic diffeomorphisms on three-dimensional manifolds (see conjecture of Pujals in [4] and new results of Hammerlindl-Potrie [12]).

Any $A \in SL(3, \mathbb{Z})$ with at least one eigenvalue with norm larger than one, induces a linear partially hyperbolic diffeomorphism on $\mathbb{T}^3$. Conversely for any partially hyperbolic diffeomorphism f on $\mathbb{T}^3$, there exist a unique linear diffeomorphism A, such that A induces the same automorphism as f on the fundamental group $\pi_1(\mathbb{T}^3)$.

Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeomorphism. Consider $f_* : \mathbb{Z}^3 \to \mathbb{Z}^3$ the action of f on the fundamental group of $\mathbb{T}^3$. f_* can be extended to $\mathbb{R}^3$

and the extension is the lift of a unique linear automorphism $A : \mathbb{T}^3 \to \mathbb{T}^3$ which is called the linearization of f. It can be proved that A is a partially hyperbolic automorphism of torus ([5]). A. Hammerlindl proved that f is (central) leaf conjugated to f_*. This means that there exist an homeomorphism $H : \mathbb{T}^3 \to \mathbb{T}^3$ such that H sends the central leaves of f to central leaves of f_* and conjugates the dynamics of the leaf spaces.

We have just defined derived from Anosov (DA diffeomorphism). The other two known classes of partially hyperbolic diffeomorphisms are the skew-product type and perturbations of time-one map of Anosov flows.

3.1. **Disintegration of volume along central foliation.** Let $B := A \times Id$ where $A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then arbitrarily close to $B : \mathbb{T}^3 \to \mathbb{T}^3$ there is an open set of partially hyperbolic diffeomorphisms g such that g is ergodic and there is an equivariant fibration $\pi : \mathbb{T}^3 \to \mathbb{T}^2$ such that the fibers are circles, $\pi \circ g = B \circ \pi$. And g has positive central Lyapunov exponent, hence the central foliation is not absolutely continuous. In Ruelle and Wilkinson's paper [19], we see that there exist $S \subset \mathbb{T}^3$ of full volume and $k \in \mathbb{N}$ such that S meets every leaf in exactly k points. In Shub and Wilkinson's example [21] the fibers of the fibration are invariant under the action of a finite non-trivial group and consequently in their example the number of atoms cannot be one.

For the perturbation of a time-one map of the geodesic flow for a closed negatively curved surfaces (which is an Anosov flow), it was shown by A. Avila, M. Viana and A. Wilkinson [1] that $\mathcal{F}^c$ has atomic disintegration or it is absolutely continuous. We emphasize that the key property for the diffeomorphisms near to time-one map of Anosov flow is that they are partially hyperbolic and moreover all center leaves are fixed by the dynamics. This implies that in the atomic case, the disintegrated measures do have countably (infinite) many atoms. Indeed, if f is such a partially hyperbolic diffeomorphism $f(\mathcal{F}^c(x)) = \mathcal{F}^c(x)$ and for any a atom, $f^n(a), n \in \mathbb{Z}$ are atoms of the disintegration of volume along unbounded leaves of $\mathcal{F}^c$.

For a large class of skew-product diffeomorphisms, Avila-Viana and Wilkinson announced that they can prove an analogous result, i.e atomicity versus absolute continuity.

It is interesting to emphasize that (conservative) derived from Anosov (DA) diffeomorphisms on $\mathbb{T}^3$ show a feature that is not, so far, shared with any other known partially hyperbolic diffeomorphisms on dimension three, it admits all three disintegration of volume on the center leaf, namely: Lebesgue, atomic, and, by a recent result of R. Varão [24], they can also have a disintegration which is neither Lebesgue nor atomic.

More precisely, R. Varão [24] showed that there exist Anosov diffeomorphisms with non-absolutely continuous center foliation which does not have atomic disintegration.

Here we show a new behavior for DA diffeomorphisms (which are not Anosov) on $\mathbb{T}^3$, and that is the existence of atomic disintegration (Theorem C). This behavior can be verified for an open class of diffeomorphisms found by Ponce-Tahzibi in [17].

We mention that the examples of non-absolutely continuous weak foliation of Anosov diffeomorphisms was known by Saghin-Xia [20] and A. Gogolev [9]. Baraviera-Bonatti [2] also exhibited non-absolutely continuous central foliation for partially hyperbolic diffeomorphisms close to Anosov geodesic flows. We are introducing examples of non-Anosov DA diffeomorphisms with non-absolutely continuous central foliation. In fact we prove that the Lebesgue measure has atomic disintegration along central foliation. The novelty in our example is that, the sign of central Lyapunov exponent of a partially hyperbolic diffeomorphism is opposite to the sign of central Lyapunov exponent of its linearization. It is not known whether the disintegration of the Lebesgue measure can be atomic in the case of Anosov diffeomorphisms.

4. "Pathological" Examples of Derived from Anosov Diffeomorphisms

4.1. **"Pathological" example.** As we remarked before, in Theorem C, one of the hypothesis is that the center Lyapunov exponent of the diffeomorphism f and of its linearization A have opposite sign. In paper [17], the authors found an open set of partially hyperbolic diffeomorphisms isotopic to linear Anosov diffeomorphisms satisfying the required hypothesis of Theorem C. This opposite behavior in the asymptotic growth (manifested by the sign of Lyapunov exponent) which is a local issue contrasts with the compatible behavior in the large scale between f and its linearization and makes the example pathological.

Let us explain what we mean by the similar behavior (see [11], Corollary 2.2): From now on we work on the universal covering and lift f and A to $\mathbb{R}^3$. For each $k \in \mathbb{Z}$ and $C > 1$ there is an $M > 0$ such that for all $x, y \in \mathbb{R}^3$,

$$\|x - y\| > M \Rightarrow \frac{1}{C} < \frac{\|\tilde{f}^k(x) - \tilde{f}^k(y)\|}{\|A^k(x) - A^k(y)\|} < C. \tag{4.1}$$

Let us briefly recall the construction of [17]: Start with the family of linear Anosov diffeomorphisms $f_k : \mathbb{T}^3 \to \mathbb{T}^3$ induced by the integer matrices:

$$A_k = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & -1 & k \end{pmatrix}, k \in \mathbb{N}.$$

This family of Anosov diffeomorphisms has two important characteristics that justify our choice. Denote by $\lambda_k^s, \lambda_k^c, \lambda_k^u$ the three Lyapunov exponents of A_k with $\lambda_k^s < \lambda_k^c < \lambda_k^u$ and, for each k, denote by E_k^s, E_k^c, E_k^u the stable, central and unstable fiber bundles with respect to A_k. Then an easy calculation shows that

$$\lambda_k^s \to -\infty, \lambda_k^c \to 0^-, \lambda_k^u \to \infty,$$

as $k \to \infty$. Moreover, E_k^s, E_k^c, E_k^u converge to the canonical basis.

Using a Baraviera-Bonatti [2] perturbation method, for large k the authors managed to construct small perturbation of f_k and obtain partially hyperbolic diffeomorphisms g_k such that the central Lyapunov exponent of g_k is positive.

By taking the family g_k^{-1} we obtain partially hyperbolic diffeomorphisms with negative center exponent and isotopic to Anosov diffeomorphism with weak expanding subbundle. In fact any $f := g_k^{-1}$ satisfy the desired properties. Moreover, there is an open subset U around f such that any $g \in U$ share the same property, i.e the central Lyapunov exponent of g is negative but its linearization has expanding central bundle.

This family of diffeomorphisms fulfills the hypothesis required in Theorems C.

5. Proof of results

Let f be DA diffeomorphism defined as above, then by [8] we know that f is semi-conjugated to its linearization by a function $h : \mathbb{T}^3 \to \mathbb{T}^3, h \circ f = A \circ h$. It follows from [23] that $\mathcal{F}^c(A) = h(\mathcal{F}^c(f))$. Moreover, there exists a constant $K \in \mathbb{R}$ such that if $\tilde{h} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the lift of h to $\mathbb{R}^3$ we have $\|\tilde{h}(x) - x\| \leq K$ for all $x \in \mathbb{R}^3$.

Definition 5.1. *A foliation $\mathcal{F}$ defined on a manifold M is quasi-isometric if the lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ to the universal cover of M has the following property: There exist positive constants Q, Q' such that for all x, y in a common leaf of $\widetilde{\mathcal{F}}$ we have*

$$d_{\widetilde{\mathcal{F}}}(x, y) \leq Q\|x - y\| + Q',$$

where $d_{\widetilde{\mathcal{F}}}$ denotes the Riemannian metric on $\widetilde{\mathcal{F}}$ and $\|x - y\|$ is the distance on the universal cover.

Remark 5.2. In this paper the leaves of foliations under consideration are C^1 and tangent to a continuous subbundle and consequently after a change of constants we can assume $Q' = 0$ in the above definition.

For absolute partially hyperbolic diffeomorphisms on $\mathbb{T}^3$ the stable, unstable and central foliations are quasi isometric in the universal covering $\mathbb{R}^3$ [6, 10]. Firstly we prove Theorem B.

Proof of Theorem B. Let $h : \mathbb{T}^3 \to \mathbb{T}^3$ be the semi-conjugacy between f and its linearization A, hence $h \circ f = A \circ h$. We can assume that A has two eigenvalues larger than one, otherwise we work with f^{-1}.

Let $\{R_i\}$ be a Markov partition for A, and define $\hat{R}_i := h^{-1}(R_i)$. We claim that

$$Vol\left(\bigcup_i int\ \hat{R}_i\right) = 1. \tag{5.1}$$

Indeed, first look at the center direction of A. For simplicity we consider the center direction as a vertical foliation. This means that the rectangle R_i has two types of boundaries, the one coming from the extremes of the center foliation and the lateral ones. We call $\partial_c R_i$ the boundary coming from these extremes of the center foliation, i.e

$$\partial_c R_i = \bigcup_{x \in R_i} \partial(\mathcal{F}_x^c \cap R_i).$$

Since h takes center leaves to center leaves, we conclude that the respective boundary for the $\hat{R}_i$ sets is $\partial_c \hat{R}_i = h^{-1}(\partial_c R_i)$, and since $\bigcup_i \partial_c R_i$ is an A-invariant set, $\bigcup_i \partial_c \hat{R}_i$ is a forward f-invariant set. By ergodicity of f it follows that $\bigcup_i \partial_c \hat{R}_i$ has zero or full measure. Since the volume of the interior cannot be zero, then the volume of $\bigcup_i \partial_c \hat{R}_i$ cannot be one. Therefore it has zero measure. Since f is dynamicaly coherent and h sends each of the invariant foliations (center-stable and center-unstable) to the respective invariant foliations of A, that implies that the lateral boundaries of the $\hat{R}_i$ are inside a finite number of center-stable and center-unstable leaves of f, hence it also has zero volume. We have avoided the boundary points as there may exist ambiguity to which element of partition these points belong.

By (5.1) we can consider the partition $\hat{\mathcal{P}} = \{\mathcal{F}_R^c(x) : x \in \hat{R}_i$ for some $i\}$ where $\mathcal{F}_R^c(x)$ denotes the connected component of $\mathcal{F}_f^c(x) \cap \hat{R}(x)$ which contains x in its interior. Thus we can consider the Rokhlin disintegration of volume on the partition $\hat{\mathcal{P}}$. Denote this system of measures by $\{m_x\}$, so that each m_x is supported in $\mathcal{F}_f^c(x)$.

Lemma 5.3. *There is a natural number $\alpha_0 \in \mathbb{N}$, such that for almost every point, $\mathcal{F}_R^c(x)$ contains exactly α_0 atoms.*

Proof. The semi-conjugacy h sends center leaves of f to center leaves of A. Also, the points of the interior of the $\hat{R}_i$ satisfy that $f(\mathcal{F}_R^c(x)) \supset \mathcal{F}_R^c(f(x))$, which just comes from the Markov property of the rectangles R_i. We claim that

$$f_* m_x \leq m_{f(x)}$$

on $\mathcal{F}_R^c(f(x))$. In fact, since f preserves volume if we normalize the measures $f_* m_x$ on $f(\mathcal{F}_R^c(x)) \cap \mathcal{F}_R^c(f(x))$ they become a disintegration of volume on $f(\mathcal{F}_R^c(x)) \cap \mathcal{F}_R^c(f(x))$, but $m_{f(x)}$ are also a disintegration of volume on $f(\mathcal{F}_R^c(x)) \cap \mathcal{F}_R^c(f(x))$. Hence, $f_* m_x = f_* m_x(f(\mathcal{F}_R^c(x)) \cap \mathcal{F}_R^c(f(x)))m_{f(x)}$. And because the normalization constant $f_* m_x(f(\mathcal{F}_R^c(x)) \cap \mathcal{F}_R^c(f(x)))$ is smaller or equal to one we get the above inequality.

Given any $\delta \geq 0$ consider the set $A_\delta = \{x \in \mathbb{T}^3 \mid m_x(\{x\}) > \delta\}$, that is, the set of atoms with weight at least δ. If $x \in A_\delta$ then

$$\delta < m_x(\{x\}) = f_* m_x(\{f(x)\}) \leq m_{f(x)}(\{f(x)\}).$$

Thus $f(A_\delta) \subset A_\delta$, and by the ergodicity of f we have that $Vol(A_\delta)$ is zero or one, for each $\delta \geq 0$. Note that $Vol(A_0) = 1$ and $Vol(A_1) = 0$. Let δ_0 be the critical point for which $Vol(A_\delta)$ changes value, i.e, $\delta_0 = \sup\{\delta : Vol(A_\delta) = 1\}$. This means that all the atoms have weight δ_0. And due to the assumption of atomic disintegration, the value of δ_0 has to be a strictly positive number. Since m_x is a probability we have an $\alpha_0 := 1/\delta_0$ number of atoms as claimed.   □

Lemma 5.4. *There is only finitely many atoms on almost every center leaf.*

Proof. Suppose we have infinitely many atoms on each center leaf. Let $\beta \in \mathbb{R}$ be a large number (for instance much bigger than KQ where K is the distance between h and the identity map and Q is the quasi isometric constant in the definition 5.1). Since we have a finite number of $\hat{R}_i$, from the previous Lemma, we know that there is a number $\tau \in \mathbb{R}$ for which every center segment of size smaller then β must contain at most τ atoms. But, since on each center leaf there are infinity many atoms, take a segment of leaf big enough so that it contains more then τ atoms. Iterate this segment backwards and it will eventually be smaller than β but containing more than τ atoms. Indeed,

$$h \circ f^{-n} = A^{-n} \circ h$$
$$\|h(f^{-n}(x)) - h(f^{-n}(y))\| = \|A^{-n}(h(x)) - A^{-n}(h(y))\|$$
$$\leq e^{-n\lambda^{wu}(A)}\|h(x) - h(y)\|$$

As h is at a distance K to identity we have

$$\|f^{-n}(x) - f^{-n}(y)\| \leq e^{-n\lambda^{wu}(A)}\|h(x) - h(y)\| + K \leq \frac{\beta}{Q}$$

So, finally by quasi isometric property

$$d_c(f^{-n}(x), f^{-n}(y)) \leq \beta.$$

The above contradiction implies that the number of atoms can not be infinite and now we proceed as in the previous case.

□

Lemma 5.5. *The disintegration of the Lebesgue measure along the central leaves is mono-atomic, i.e there is just one atom per leaf.*

Proof. We have a finite number of atoms on each center leaf and since the center foliation is an oriented foliation we may talk about the first atom. If the function f preserves orientation along the center direction then the set of first atom of all generic leaves is an invariant set with positive measure, therefore it has full measure. If f reverses orientation, then the set of first and last atoms of all generic

leaves is an invariant set with positive measure, therefore it has full measure. This means that almost every leaf has exactly one atom or almost every leaf has exactly two atoms.

Assume that almost every center leaf has exactly two atoms. Since the set of first atoms is invariant by f^2 then f^2 is not ergodic. By a Theorem of Hammerlindl-Ures [13] there exist a homeomorphism $h : \mathbb{T}^3 \to \mathbb{T}^3$ mapping center (stable, unstable) leaves of f^2 onto center (stable, unstable) leaves of A^2 and conjugating f^2 and its linearization $A^2 : \mathbb{T}^3 \to \mathbb{T}^3$, that is

$$A^2 \circ h = h \circ f^2.$$

It is well known that if we require h to be homotopic to identity then such conjugacy is unique. Thus the homeomorphism h also conjugates f and A,

$$A \circ h = h \circ f.$$

Consider A_k the set of points x such that the distance between two atoms on the central leaf passing through x is less than k. for large k the measure of A_k is positive and almost every point of A_k returns to it infinitely many time. Take such $x \in A_k$ and $a, b \in \mathcal{F}^c(x)$ the two atoms of the disintegration. By invariance $f^{n_i}(a), f^{n_i}(b)$ are atoms on $\mathcal{F}^c(f^{n_i}(x))$. Taking $f^{n_i}(x) \in A_k$, on one hand we have that $d(f^{n_i}(a), f^{n_i}(b)) \leq k$. On the other hand

$$d(A^{n_i}(h(a)), A^{n_i}(h(b))) = d(h(f^{n_i}(a)), h(f^{n_i}(b))).$$

As h is injective and the first hand side of the above equation goes to infinity, we get a contradiction.

$\square$

The above Lemma concludes the proof of the Theorem B. $\square$

Problem 1. Is there any ergodic invariant measure μ with disintegration having more than one atom on leaves?

Remark 5.6. We emphasize that the unique issue to generalize Theorem B for any ergodic invariant measure (not just volume) is that we used the fact that the boundary of $\hat{R}_i$ has zero volume.

We note once again that since the work of Ponce-Tahzibi [17] assures that the set of DA diffeomorphisms satisfying the hypothesis of the next theorem is non-empty, we prove that these diffeomorphisms have atomic disintegration.

5.1. **A Glimpse of Pesin Theory.** Before presenting the proof of Theorem C, we recall some basic notions of Pesin theory. Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ a partially hyperbolic diffeomorphism with splitting

$$TM = E^s \oplus E^c \oplus E^u.$$

Call Λ the set of regular points (See [25]) of f, that is, the set of points $x \in \mathbb{T}^3$ for which in particular the Lyapunov exponents are well defined. Then, for each $x \in \Lambda$ we define the Pesin-stable manifold of f at x as the set

$$W^s(x) = \left\{ y : \limsup_{n \to \infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0 \right\}.$$

The Pesin-stable manifold is an immersed sub manifold of $\mathbb{T}^3$. Similarly we define the Pesin-unstable manifold at x, $W^u(x)$, using f^{-1} instead of f in the definition.

It is clear that for a partially hyperbolic diffeomorphism $W^s(x)$ contains the stable leaf $\mathcal{F}^s(x)$. In Theorem C we assume that the central Lyapunov exponent is negative and consequently the Pesin-stable manifolds are two dimensional. By $\mathcal{W}^c(x)$ we denote the intersection of the Pesin stable manifold $W^s(x)$ of x with the center leaf $\mathcal{F}^c_f(x)$ of x. These manifolds depends only measurable on the base point x, as it is proved in the Pesin theory. However, there is a filtration of the set of regular points by Pesin blocks: $\Lambda = \bigcup_{l \in \mathbb{N}} \Lambda_l$ such that each Λ_l is a closed (not necessarily invariant) subset and $x \to \mathcal{W}^c(x)$ varies continuously on each Λ_l (see [3] for more properties and structure of these sets).

A key property used in the proof of Theorem C is the uniform contraction locally around points in a Pesin block:

Lemma 5.7. *(see chapter 7 of [3]) There exists $C > 0, \lambda < 1$ and $r_l > 0$ such that for any $x \in \Lambda_l$ (Pesin block)*

$$d(f^n(x), f^n(y)) \leq C\lambda^n d(x, y)$$

for any $y \in W^s_{r_l}(x)$. Here W^s stands for the Pesin stable manifold and d is the induced distance.

Proof of Theorem C. To begin, we prove that the size of the weak stable manifolds $\mathcal{W}^c(x)$ is uniformly bounded from above for x belonging to the regular set. In particular this enables us to prove that the partition (mod-0) by $\mathcal{W}^c(x)$ is a measurable partition.

Lemma 5.8. *The size of $\{\mathcal{W}^c(x)\}_{\{x \,:\, \lambda^c(x) < 0\}}$ is uniformly bounded from above for $x \in \Lambda$. More precisely, the image of $\mathcal{W}^c(x)$ by h is a unique point.*

Proof. Let $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\tilde{A} : \mathbb{R}^3 \to \mathbb{R}^3$ denote the lifts of f and A respectively and $\tilde{h} : \mathbb{R}^3 \to \mathbb{R}^3$ the lift of the semi-conjugacy h between f and A. Consider $\gamma \subset \widetilde{\mathcal{W}^c}(x)$, where $\widetilde{\mathcal{W}^c}(x)$ is the lift of $\mathcal{W}^c(x)$. Thus, γ is inside the intersection of the center manifold of $\tilde{f}$ and the Pesin-stable manifold of $\tilde{f}$ passing through x.

Let us show that $\tilde{h}$ collapses $\widetilde{\mathcal{W}^c}(x)$ to a unique point. If we prove that, it clearly comes out (from the bounded distance of h to identity that, the size of $\mathcal{W}^c(x)$ is uniformly bounded. Suppose by contradiction that $\tilde{h}(\gamma)$ has more than one point. By semi-conjugacy $\tilde{h}(\tilde{f}^n(\gamma)) = \tilde{A}^n(\tilde{h}(\gamma))$. As $\tilde{h}(\gamma)$ is a subset of weak

unstable foliation of $\tilde{A}$ for large n the size of $\tilde{A}^n(\tilde{h}(\gamma))$ is large. On the other hand, γ is in the Pesin stable manifold of f and consequently for large n, the size of $\tilde{f}^n(\gamma)$ is very small. As $\|\tilde{h} - id\| \leq K$ we conclude that for large n the size of $\tilde{h}(\tilde{f}^n(\gamma))$ can not be very big. This contradiction completes the proof. $\qquad\square$

Corollary 5.9. *The family* $\{\mathcal{W}^c(x)\}_{\{x \,:\, \lambda^c(x) < 0\}}$ *forms a measurable partition.*

This corollary uses the same idea of the proof of the Proposition 2.5. However, that proposition is proved for continuous foliations and we adapt the proof for the Pesin measurable lamination.

First of all we consider a new partition $\{\overline{\mathcal{W}^c(x)}\}$ whose elements are the closure of the elements $\mathcal{W}^c(x)$, that is, $\overline{\mathcal{W}^c(x)}$ is a bounded length center segment with its extremum points. Since h collapses the manifolds $\mathcal{W}^c(x)$ of f into points, two different elements $\mathcal{W}^c(x)$ and $\mathcal{W}^c(y)$ cannot have a common extrema, so that $\{\overline{\mathcal{W}^c(x)}\}$ is indeed a partition with compact elements. Let us prove that it is indeed a measurable partition.

Let $\{x_j\}_{j \in \mathbb{N}}$ be a countable dense set of $M = \mathbb{T}^3$. For each x_j and $k, l \in \mathbb{N}$ we define $C_l(x_j, k)$ to be the union of $\mathcal{W}^c(y), y \in \Lambda_l$ such that $\mathcal{W}^c(y)$ intersects the closed ball $B(x_j, \frac{1}{k})$. By continuity of $\mathcal{W}^c(.)$ on Λ_l we conclude that $C_l(x_j, k)$ is closed and consequently measurable. Indeed, if $y_n \in C_l(x_j, k)$ converges to y then $y \in \Lambda_l$ and moreover $\mathcal{W}^c(y)$ intersects the closure of $B(x_j, \frac{1}{k})$.

Now, we need to separate two weak stable manifolds by means of some $C_l(x_j, k)$. Taking two elements $\mathcal{W}^c(a)$ and $\mathcal{W}^c(b)$ there exists $l \in \mathbb{N}$ such that $a, b \in \Lambda_l$ and it is enough to take small enough k and some x_j such that $C_l(x_j, k)$ contains $\mathcal{W}^c(a)$ and not $\mathcal{W}^c(b)$. It is easy to see that for each $x \in \Lambda$ we have $\mathcal{W}^c(x) = \bigcap_{k,l,j} C_l(x_j, k)^*$ where $C_l(x_j, k)^*$ is either $C_l(x_j, k)$ or $\mathbb{T}^3 \setminus C_l(x_j, k)$.

Now, observe that if y is an extremum point of $\mathcal{W}^c(x)$ then y cannot have negative center Lyapunov exponent, otherwise it would be in the interior of $\mathcal{W}^c(x)$. So, the set of extremum points of the elements $\overline{\mathcal{W}^c(x)}$ is inside the set of points with non negative center Lyapunov exponent, and therefore it has zero measure. Thus, removing such points, the measurability of the partition $\{\overline{\mathcal{W}^c(x)}\}$ implies the measurability of $\{\mathcal{W}^c(x)\}$, concluding the proof of the corollary.

Lemma 5.10. *Disintegration of volume on the measurable partition* $\{\mathcal{W}^c(x)\}$ *is atomic.*

Let $\mathcal{W}^c(x, r)$ denote a ball inside $\mathcal{W}^c(x)$ of radius r. By elementary Pesin theory we know that $\bigcup_{l \geq 1} \Lambda_l$ has full Lebesgue measure where Λ_l are Pesin blocks. Here we use the fact that for $x \in \Lambda_l$ the size of Pesin manifold is bounded from below by $r_l > 0$. For sufficiently large l we have $\mu(\Lambda_l) > 1/2$.

As $\mathcal{W}^c(x)$ is measurable we consider the disintegration of the Lebesgue measure into conditional probability measures μ_x. Since $\mathcal{W}^c$ is f-invariant (as both Pesin maniful and central foliation are invariant.) and f preserves volume, Corollary

2.3 implies

$$\mu_{f(x)} = f_*\mu_x.$$

Let us define

$$A := \bigcup_{\mu_x(\mathcal{W}^c(x) \cap \Lambda_l) \geq 1/2} \mathcal{W}^c(x)$$

and it is clear that $\mu(A) > 0$. Consider the set $B \subset A$ that consists of points that return to A infinitely many times. By Poincaré recurrence $\mu(B) = \mu(A)$. Let $F : B \to B$ be the first return map of f.

Observe that if $x \in B$ and $F(x) = f^n(x)$ then $\mu_{f^n(x)}(\mathcal{W}^c(f^n(x)) \cap \Lambda_l) \geq 1/2$. By invariance of $\mathcal{W}^c$ we have $f^n(\mathcal{W}^c(x)) = \mathcal{W}^c(f^n(x))$. Using the definition of A, $F(y) = f^n(y)$ for any $y \in \mathcal{W}^c(x)$ and consequently $F(\mathcal{W}^c(x)) = \mathcal{W}^c(F(x))$. As F preserves volume, Corollary 2.3 implies $F_*\mu_x = \mu_{F(x)}$.

As the size of $\mathcal{W}^c(x)$ is bounded from above for almost all x, we can take m such that every $\mathcal{W}^c(x)$ can be covered by at most m balls of radius $r_l/2$. So for all $x \in B$ we can choose a ball B_x^c of radius $r_l/2$ such that $\mu_x(B_x^c) \geq \frac{1}{2m}$ and $B_x^c \cap \Lambda_l \neq \emptyset$.

Now let $B_{n,x}^c = F^n(B_{F^{-n}(x)}^c)$ and observe that $\mu_x(B_{n,x}^c) = \mu_{F^{-n}(x)}(B_{F^{-n}(x)}^c) \geq \frac{1}{2m}$. As the size of the stable manifold of $x \in \Lambda_l$ is larger than r_l we conclude that $B_{n,x}^c$ is completely inside the Pesin stable manifold and consequently the diameter of $B_{n,x}^c$ goes to zero, by a uniform rate due to Lemma 5.7.

We have proved that for all $x \in B$ there is a sequence of sets which diameter is uniformly decreasing and these sets have μ_x measure uniformly positive. By taking a subsequence if necessary we can assume that these sets accumulate on a point. Then, any neighborhood of such point has positive measure uniformly bounded from bellow, this implies that this point is an atom. Since the set of atoms is an invariant set with positive volume, the ergodicity of f implies that this set has full volume, hence μ_x is a sum of Dirac measures. The ergodicity comes from Hammerlindl-Ures [13]. Once obtained atomicity we can apply Theorem B to get one atom per leaf. $\qquad\square$

Finally let us use Theorems B and C to complete the proof of Theorem A.

As we mentioned in section 4 there exist an open set U such that any $g \in U$ is volume preserving partially hyperbolic with Anosov linearization g_* satisfying the following property: The central Lyapunov exponent of g is negative but the central bundle of g_* is expanding. so, g satisfies the hypothesis of Theorem B and consequently there is a subset of full Lebesgue measure which intersects almost all center leaves in a unique point. This means that the partition into central leaves is measure theoretically equivalent to the partition into single points of $\mathbb{T}^3$. This implies measurability of the foliation. The minimality of the central foliation comes out from the minimality of central foliation of g_* and the leaf conjugacy between g and g_*, as we commented in Section 1.1.

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