On κ-Sparse Numerical Semigroups

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Numerical Semigroup

H is a *numerical semigroup* if:

1. $H \subset \mathbb{N}_0$;
2. $0 \in H$;
3. $|\,\mathbb{N}_0 \setminus H\,|$ is finite;
4. $m, n \in H \Rightarrow m + n \in H$.

- $g = |\,\mathbb{N}_0 \setminus H\,|$ is called the *genus* of H.
- $\ell_g = \max(\mathbb{Z} \setminus H)$ is called the *Frobenius number* of H. [6, Oliveira, 1991] $\ell_g \leq 2g - 1$.
- If $g > 0$, $\mathrm{Gaps}(H) := \mathbb{N}_0 \setminus H = \{1 = \ell_1 < \cdots < \ell_g\}$ is the set of *gaps* of H.
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$\mathbb{N}_0$ is a numerical semigroup of genus $g = 0$ and $\ell_0 = -1$. $\mathbb{N}_0$ is called the *trivial numerical semigroup*.

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Enumeration of a Numerical Semigroup

Since $H \subset \mathbb{N}_0$ is infinite, there exist a unique strictly increasing bijective map $n_H : \mathbb{N}_0 \longrightarrow H$.
Thus, if $n_i = n_i(H) := n_H(i)$, we have

$$H = \{0 = n_0 < n_1 < n_2 < \cdots\}$$

- $c := \ell_g + 1 = n_{c-g}$ is called the *conductor* of H.
- n_1 is called the *multiplicity* of H.

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Definition

Let $H = \{0 = n_0 < n_1 < \cdots\}$ be a numerical semigroup of genus $g > 0$ with $\mathrm{Gaps}(H) = \{\ell_1 < \cdots < \ell_g\}$. H is called *sparse numerical semigroup* if

$$\ell_i - \ell_{i-1} \leq 2, \quad \forall\, i = 1, \ldots, g,$$

or equivalently, $n_i - n_{i-1} \geq 2, \quad \forall\, i = 1, \ldots, c - g$.
For convenience, the numerical semigroup $\mathbb{N}_0$ is considered sparse.

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Examples of Sparse Numerical Semigroups

Examples

- A semigroup $H = \{0 = n_0 < n_1 < \cdots\}$ is called *Arf numerical semigroup* if

$$n_i + n_j - n_k \in H, \quad \forall\, i \geq j \geq k \geq 1.$$

All Arf numerical semigroup is sparse.

- For each non-negative integer g, let

$$\mathbb{N}_g := \{0\} \cup \{n \in \mathbb{N} : n \geq g + 1\}.$$

$\mathbb{N}_g$ is an numerical semigroup with genus equal to g called *ordinary numerical semigroup.*

- For all non-negative integer g, $\mathbb{N}_g$ is an sparse numerical semigroup and Arf numerical semigroup.

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- The ordered pair (ℓ_{i-1}, ℓ_i) is called *leap of* H, $1 \leq i \leq g$.
- The set of leaps

$$\mathscr{V}(H) := \{(\ell_{i-1}, \ell_i) : 1 \leq i \leq g\}.$$

Note that: $|\mathscr{V}(H)| = g$.

- The set of m*-leaps*

$$\mathscr{V}_m(H) := \{(\ell_{i-1}, \ell_i) : \ell_i - \ell_{i-1} = m,\ 1 \leq i \leq g\}.$$

Let $v_m(H) := |\mathscr{V}_m(H)|$.

This concept was introduced in [2, Contiero-Moreira-Veloso, 2015] for sparse numerical semigroups of genus $g \geq 2$.

- For convenience, $v_m(\mathbb{N}_0) := 0,\ \forall\ m \in \mathbb{N}$.

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- For convenience, $v_m(\mathbb{N}_0) := 0,\ \forall\ m \in \mathbb{N}$.

The sets of leaps

m-leap

Let H be a numerical semigroup of genus $g > 0$ and $\mathrm{Gaps}(H) = \{1 = \ell_1 < \cdots < \ell_g\}$. Remembering that $\ell_0 = -1$.

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$$\mathscr{V}(H) := \big\{(\ell_{i-1}, \ell_i) : 1 \leq i \leq g\big\}.$$
Note that: $|\,\mathscr{V}(H)\,| = g$.
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A equivalence of Sparse

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Let H be a numerical semigroup of genus g. Then:

1. *H is an sparse numerical semigroup $\Leftrightarrow$ $v_1(H) + v_2(H) = g$.*
2. *If H is an sparse numerical semigroup, then the Frobenius number of H is $\ell_g = v_1(H) + 2v_2(H) - 1$.*
3. *If $g > 0$ and $\ell_g(H) = 2g - K$, for some positive integer K, then H is an sparse numerical semigroup $\Leftrightarrow$ $v_1(H) = K - 1$ and $v_2(H) = g - K + 1$.*

The proof of a necessary condition of the item 1 and the item 3, are distinct from that originally published in [2, Theorem 2.1].

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For all numerical semigroup H genus g, we have $v_m(H) = 0, \quad \forall\, m \geq \max\{3, g+1\}$, and

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Let κ be a positive integer. A numerical semigroup H of genus g is called *κ-sparse numerical semigroup* if

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Examples of Sparse Numerical Semigroups

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- $\mathbb{N}_0$ is a κ-sparse numerical semigroup, for all positive integer κ.
- The sparse numerical semigroups are the 2-sparse numerical semigroups.
- The pure 2-sparse numerical semigroups are the sparse numerical semigroups with genus positive.
- For each non-negative integer g, the ordinary numerical semigroup $\mathbb{N}_g$ is a κ-sparse numerical semigroup, $\forall\ \kappa \geq 2$.

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Let κ be an integer such that $\kappa \geq 2$. Let $H = \{0 = n_0 < n_1 < \cdots\}$ be a numerical semigroup of genus $g > 0$ and $\mathrm{Gaps}(H) = \{1 = \ell_1 < \cdots < \ell_g\}$, with conductor c. The following statements are equivalent:

1. *H is a κ-sparse numerical semigroup;*
2. $\ell_i - \ell_{i-1} \leq \kappa, \quad \forall i = 1, \ldots, g;$
3. *if $c \geq g + \kappa - 1$, then* $n_{i+\kappa-2} - n_{i-1} \geq \kappa, \quad \forall i = 1, \ldots, c - g - \kappa + 2;$
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Equivalences of κ-sparse condition

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For each positive integer κ, let $\mathscr{S}_\kappa$ be the collection of κ-sparse numerical semigroups and let $\mathscr{S}_\kappa^*$ be the collection of pure κ-sparse numerical semigroups.

Theorem

Let $\mathscr{S}$ be the collection of all numerical semigroups.

1. *The sequence $(\mathscr{S}_\kappa)_{\kappa\in\mathbb{N}}$ is an strictly ascending chain.*
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2. The sets of leaps
   - A equivalence of Sparse
3. κ-Sparse Numerical Semigroup
   - Equivalences of κ-sparse condition
4. **Structure of κ-Sparse Numerical Semigroups**
   - Frobenius variety
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6. References

$\mathscr{S}_\kappa$ is a Frobenius variety

Definition

A *Frobenius variety* is a nonempty set $\mathscr{V}$ of numerical semigroups fulfilling the following conditions:

1. for all $H_1, H_2 \in \mathscr{V}$, $H_1 \cap H_2 \in \mathscr{V}$;
2. for all $H \in \mathscr{V}$ with genus $g > 0$, $H \cup \{\ell_g(H)\} \in \mathscr{V}$.

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The order number of semigroup

Let $H = \{0 = n_0 < n_1 < \cdots\}$ be a numerical semigroup. For each $\ell \in \mathbb{N}_0$, consider the set

$$A(\ell+1) := \{(i,j) \in \mathbb{N}_0^2 : n_i + n_j = n_\ell\}.$$

Let $\nu_\ell := \big|A(\ell+1)\big|$. In [5], the authors defined, for $\ell \in \mathbb{N}_0$, the ℓ-th *order bound* of H as

$$d_\ell(H) := \min\{\nu_m : m \geq \ell\}$$

and, in [4], it is defined the *order number* of H as

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Acute and near-acute semigroups

Given a non-ordinary numerical semigroup H with conductor c, we can write

$$H = [c_{m+1}, d_{m+1}] \cup [c_m, d_m] \cup \cdots \cup [c_1, d_1] \cup [c_0, \infty), \quad (1)$$

for some $m \in \mathbb{N}$, where $c_0 = c$, $c_{m+1} = d_{m+1} = 0$, c_i's and d_i's are positive integers such that $d_1 + 1 < c_0$,
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The order of certain families of κ-sparse numerical semigroups

Example

For an integer $g \geq \kappa \geq 2$,

$$H = \{0\} \cup [g, g+\kappa-2] \cup [g+\kappa, \infty) \in \mathscr{S}_\kappa^*$$

and $\mathrm{Gaps}(H) = \{1, \ldots, g-1, g+\kappa-1\}$. We have H is acute and $\mathrm{o}(H) = g+\kappa$.

Proposition

Let κ be an integer such that $\kappa \geq 2$. Let $H \in \mathscr{S}_\kappa$ with genus $g > 1$. If $\ell_g - \ell_{g-1} = \kappa$, then H is acute and $\mathrm{o}(H) = 2\ell_g - g - \kappa + 2$.

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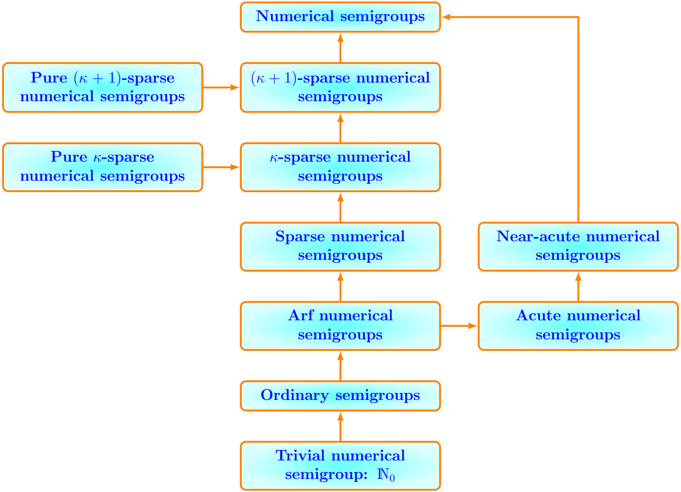
Diagram of certain numerical semigroup classes and inclusions



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