# Honda-Serre-Tate Theory

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For a field $k$, we consider the following category:
objects: abelian varieties over $k$;
morphisms: $\mathrm{Mor}(A,B) := Hom(A,B) \otimes \mathbb{Q}$.

This is called the *category of abelian varieties up to isogeny*, $Isab(k)$, over $k$ because two abelian varieties become isomorphic in $Isab(k)$ if and only if they are isogenous. It is a $\mathbb{Q}$-linear category (i.e., it is additive and the $Hom$-sets are vector spaces over $\mathbb{Q}$) and Poincare-Weil theorem implies that every object in $Isab(k)$ is a direct sum of a finite number of simple objects. In order to describe such a category (up to a nonunique equivalence), it suffices to list the isomorphism classes of simple objects and, for each class, the endomorphism algebra. The theorems of Honda and Tate, which we now explain, allow this to be done in the case $k = \mathbb{F}_q$. For abelian varieties $X$ and $Y$, we use $\mathrm{Hom}^0(X,Y)$ to denote $\mathrm{Hom}(X,Y) \otimes \mathbb{Q}$ it is a finite-dimensional $\mathbb{Q}$-vector space.

Let $X$ be a abelian variety over $\mathbb{F}_q$, and let $\pi_X$ be the Frobenius endomorphism of $X$. Then $\pi$ commutes with all endomorphisms of $X$, and so lies in the centre of $End(X)^0$. If $X$ is simple, then $End(X)^0$ is a division algebra. Therefore, in this case, $\mathbb{Q}(\pi)$ is a field (not merely a product of fields). An isogeny $X \to Y$ of simple abelian varieties defines an isomorphism $End(X)^0 \to End(Y)^0$ carrying $\pi_X$ into $\pi_Y$, and hence mapping $\mathbb{Q}[\pi_X]$ isomorphically onto $\mathbb{Q}[\pi_Y]$.

Define a *Weil $q$-integer* to be an algebraic integer such that, for every embedding $\sigma : \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$, $\mid \sigma\pi \mid = \sqrt{q}$ and let $W(q)$ be the set of Weil $q$-integers in $\mathbb{C}$. Say that two elements $\pi$ and $\pi'$ are conjugate, $\pi \sim \pi'$, if any one of the following (equivalent) conditions holds:

(a) $\pi$ and $\pi'$ have the same minimum polynomial over $\mathbb{Q}$;

(b) there is an isomorphism $\mathbb{Q}(\pi) \to \mathbb{Q}(\pi')$ carrying $\pi$ into $\pi'$;

(c) $\pi$ and $\pi'$ lie in the same orbit under the action of $\mathrm{Gal}(\mathbb{Q}^{al}/\mathbb{Q})$ on $W(q)$.

For any simple abelian variety $X$, the image of $\pi_X$ in $\mathbb{Q}^{al}$ under any homomorphism $\mathbb{Q}[\pi_X] \hookrightarrow \mathbb{Q}^{al}$ is a Weil $q$-integer, well-defined up to conjugacy. The remark above, shows that the conjugacy class of $\pi_X$ depends only on the isogeny class of $X$.

**Theorem 0.1.** *The map $X \mapsto \pi_X$ defines a bijection*

$$\{\textit{simple abelian varieties } /\mathbb{F}_q\}/(\textit{isogeny}) \to \ W(q)/(\textit{conjugacy}).$$

The fact
- that the map is defined follows by Weil [4] ,
- the map is injective by Tate [2] and
- surjective by Honda [1] and Tate [3] .

# References


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[2] J. Tate, Endomorphisms of abelian varieties over finite fields, *Invent. Math.* **2** (1966), 134-144.

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[4] A. Weil, *Courbes algébriques et variétés abéliennes*, Hermann, Paris, (1971).



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