ON N-SHEETED COVERING OF CURVES AND SEMIGROUPS WHICH CANNOT BE REALIZED AS WEIERSTRASS SEMIGROUPS

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Abstract. We discuss a Hurwitz’s problem concerning semigroups via certain N-sheeted covering of curves.

1. Introduction and statement of results

Throughout, let $K$ be an algebraically closed field. By a curve over $K$ we mean a projective, irreducible, non-singular algebraic curve defined over $K$. In Weierstrass point theory one associates a numerical semigroup $H(P)$ to any point $P$ of a curve over $K$. This semigroup is called the Weierstrass semigroup at $P$ and is the same for all but finitely many points. These finitely many points, where exceptional values of the semigroup occur, are called the Weierstrass points of the curve. They carry a lot of information about the curve. A sub-semigroup $H$ of $(\mathbb{N}, +)$ is called numerical if its genus, namely $g = g(H) := \#(\mathbb{N} \setminus H)$ is finite. In 1893, Hurwitz asked about the characterization of numerical semigroups which arise as Weierstrass semigroups. See [9] for further historical information. Long after that, in 1980 Buchweitz [3], [4] showed that not every numerical semigroup can occur as a Weierstrass semigroup. In fact, this was a consequence of the following necessary condition which is related to the dimension of the $n$-th pluricanonical divisor of a curve.

(BNC) If a numerical semigroup $H$ of genus $g$ is realized as a Weierstrass semigroup, then for each integer $n \geq 2$, the number $(2n - 1)(g - 1)$ is an upper bound for the cardinality of the set $G_n$ of sums of $n$ elements of the set of gaps of $H$, namely $G := \mathbb{N} \setminus H = \{\ell_1 < \ldots < \ell_g\}$.

As a matter of fact, Buchweitz showed that for every integer $n \geq 2$ there exist semigroups which do not satisfy the (BNC). However, as was noticed by Oliveira [14, Thm. 1.5] and Oliveira-Stöhr [15, Thm. 1.1], the (BNC) cannot be applied to symmetric ($\ell_g = 2g - 1$) and quasi-symmetric ($\ell_g = 2g - 2$) semigroups because in the first case, $\#G_n = (2n - 1)(g - 1)$, and in the second case $\#G_n = (2n - 1)(g - 1) - (n - 2)$.

Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a $N$-sheeted covering of curves of genus $g$ and $\gamma$ respectively. The goal of this paper is to examine the (BNC) at a totally ramified point $P$ of $\pi$. In order
do that we characterize \( \pi \) via linear series arising from the Weierstrass semigroups \( H(P) \) at \( P \) as was carried out in [20] for the case \( N = 2 \). This characterization was used by Stöhr (loc. cit.) to show the existence of symmetric semigroups that cannot be realized as a Weierstrass semigroups. A similar result for quasi-symmetric was obtained by Oliveira-Stöhr here they used coverings of degree \( N = 3 \).

We introduce the following terminology. A (numerical) semigroup \( H \) is called \((N, \gamma)\)-hyperelliptic if the following conditions are satisfied:

(a) \( H \) has \( \gamma \) positive multiplies of \( N \) in the interval \([N, 2\gamma N]\);
(b) the \( \gamma \)-th element of \( H \) is \( 2N\gamma \);
(c) \((2\gamma + 1)N \in H \).

A curve \( X \) is called \((N, \gamma)\)-hyperelliptic if there exists a curve \( Y \) of genus \( \gamma \), a morphism \( \pi : X \to Y \) of degree \( N \) and a point \( P \in X \) which is totally totally ramified for \( \pi \).

We now state the main result of this paper. Let \( A, u, N \) and \( \gamma \) be non-negative integers. Set

\[
(1.1) \quad \rho_1(A, N, \gamma) = \frac{A(N-1)N}{2} + N\gamma - N + 1,
\]

\[
(1.2) \quad \rho_2(N, \gamma) = N(2N-1)\gamma - (N-1)(N+2),
\]

\[
(1.3) \quad \rho_3(N, \gamma) = (2N-1)(N\gamma + N - 1),
\]

\[
(1.4) \quad \rho_4(A, u, N, \gamma) = \frac{(N-u-1)}{2}[(A-\gamma-1)(N+u) - 2(N\gamma + N - 1)] + \rho_3(N, \gamma).
\]

**Theorem A.** Consider the following statements:

(i) \( X \) is \((N, \gamma)\)-hyperelliptic.
(ii) There exists \( P \in X \) such that \( H(P) \) is a \((N, \gamma)\)-hyperelliptic semigroup.
(iii) There exists \( P \in X \) and an integer \( A \) such that the linear system \(|AP|\) is base-point-free of dimension \( A - \gamma \).

Let \( N \) be prime and \( A \geq 2\gamma + 1 \).

(A1) Suppose \( N \neq \text{char}(K) \). If \( g > \rho_1(2\gamma, N, \gamma) = N^2\gamma - N + 1 \) or \( g > \rho_2(N, \gamma) \), then

\( (i) \Rightarrow (ii) \);

(A2) If \( g > \rho_3(N, \gamma) \), then \( (ii) \Rightarrow (i) \);

(A3) If \( g > \rho_1(A, N, \gamma) \), then \( (ii) \Rightarrow (iii) \);

(A4) Let \( u = u(A) \) be the biggest integer less than or equal to \( \frac{N\gamma + N - 1}{A - \gamma - 1} \). If the following conditions

\[
(1.5) \quad A \neq 0 \pmod{t} \text{ for } t \leq \frac{AN}{A - \gamma}, \ t \neq N \text{ and }
\]

\[
(1.6) \quad g > \rho_4(A, u, N, \gamma)
\]
hold, then (iii) \( \Rightarrow \) (i).

In Section 2 we study some arithmetical properties of \((N, \gamma)\)-hyperelliptic semigroups. As in the \(\gamma\)-hyperelliptic case (cf. [20]) we find that the multiples of \(N\) contained in the semigroup are completed determined (Lemma 2.3 (i)). We also state sufficient conditions for a numerical semigroup to be of type \((N, \gamma)\) (Corollaries 2.5 and 2.6). If \(N \geq 2\) is an integer, we have a lower bound for the elements \(h\) of the semigroup such that \(gcd(h, N) = 1\) (Lemma 2.1). This result generalizes Lemma 2.2 in [20]. Both Lemma 2.1 and Lemma 2.3 are used to obtain linear series on curves having a point whose Weierstrass semigroup is \((N, \gamma)\)-hyperelliptic.

In Section 3 we prove Theorem A and some results concerning Weierstrass semigroups at totally ramified points. In Remark 3.11 we discuss the sharpness of the bounds on \(g\) used in the results of the paper as well as the necessity of hypothesis (5).

In section 4, using the proof of item (A2), we show how to construct numerical semigroups of type \((N, \gamma)\) that are not realized as Weierstrass semigroups of non-singular curves. We also construct numerical semigroups that do not satisfy Buchweitz’s criterion for \(n = 2\). These examples contain Buchweitz’s semigroup and they may be well known, but we included them here in order to compare them with the semigroups arising from the proof of item (A2).

We have employed the methods used in [20], where the key tool is Castelnuovo’s genus bound for curves in projective space [5], [2, p. 116], [16, Cor. 2.8]. Since the coverings considered here can be of degree bigger than two, we will also used as a key tool the other famous genus bound of Castelnuovo which concerns subfields of the field of rational functions of the curve [6], [19].

2. \((N, \gamma)\)-HYPERELLIPTIC SEMIGROUPS

Let \(H\) be a (numerical) semigroup. The natural number \(g = g(H) := \#(\mathbb{N} \setminus H)\) is called the genus of \(H\). The elements of \(G = G(H)\) are called the gaps of \(H\) and those of \(H\) are called the non-gaps of \(H\). Fix a positive non-gap \(m \in H\). For \(i = 1, \ldots, m - 1\), denote by \(s_i = s_i(H, m)\) the smallest element of \(H\) such that \(s_i \equiv i \pmod{m}\) and define \(e_i = e_i(H, m)\) by the equation

\[(2.1) \quad s_i = m \cdot e_i + i.\]

By the semigroup property, \(e_i\) is the number of gaps \(\ell\) for which \(\ell \equiv i \pmod{m}\). Consequently

\[(2.2) \quad g = \sum_{i=1}^{m-1} e_i.\]
and also
\[
\begin{cases}
e_i + e_j \geq e_{i+j} & \text{if } i + j < m, \\
e_i + e_j \geq e_{i+j-m} - 1 & \text{if } i + j > m.
\end{cases}
\]

Conversely, given numbers \(m, e_1, \ldots, e_{m-1}\) satisfying the above relations one indeed has a semigroup. In particular, \(m = m_1\) and the respective \(e_i\)'s completely determine \(H\) (cf. [12]). Let \(N\) be a positive integer. We associate to \(H\) the number:
\[
\gamma_N := \{\ell \in G(H) : \ell \equiv 0 \pmod{N}\}.
\]

**Lemma 2.1.** Let \(H\) be a numerical semigroup of genus \(g\), \(N \geq 2\) an integer. If \(h \in H\) such that \(gcd(h, N) = 1\), then
\[
h \geq \frac{2g - 2N\gamma_N}{N - 1} + 1.
\]

**Proof.** Set \(\gamma = \gamma_N\) and let \(m = Nn\) be the least positive non-gap of \(H\) which is multiple of \(N\). Then, \(\gamma = \sum_{i=1}^{m-1} e_{ni}\) and there exists \(i \in \{1, \ldots, m-1\}\) so that \(gcd(i, N) = 1\) and \(h \geq s_i\).

**Claim.** For \(k = 1, \ldots, N - 1; \ell = 0, \ldots, n - 1; ki + N\ell \not\equiv 0 \pmod{N}\). Moreover these numbers are pairwise different modulo \(Nn\).

Indeed, if \(ki + N\ell \equiv 0 \pmod{N}\), then \(ki \equiv 0 \pmod{N}\). Since \(gcd(i, N) = 1\) we have \(k \equiv 0 \pmod{N}\), a contradiction. On the other hand, \(ki + N\ell \equiv k_1i + N\ell_1 \pmod{Nn}\) implies \((k - k_1)i \equiv 0 \pmod{N}\) and so \(k - k_1 \equiv 0 \pmod{N}\) which gives \(k = k_1\). Consequently from \(N\ell \equiv N\ell_1 \pmod{Nn}\) we obtain that \(\ell \equiv \ell_1 \pmod{n}\) and so \(\ell = \ell_1\).

For \(k\) and \(\ell\) as in the above claim, write \(ki + N\ell = a_{k\ell}Nn + r_{k\ell} \ (*)\) with \(0 < r_{k\ell} < Nn\). From (9) and induction on \(k\) and \(\ell\) we have
\[
k e_i + e_{N\ell} \geq e_{r_{k\ell}} - a_{k\ell}
\]
where we assume \(e_0 := 0\). Adding up these inequalities, from the Claim and (8) we get
\[
\frac{(N - 1)Nn e_i}{2} + (N - 1)\gamma \geq g - \gamma - \sum_{k,\ell} a_{k\ell}.
\]

Now from (*) and the Claim we have \(\sum_{k,\ell} a_{k\ell} = (N - 1)(i - 1)/2\) and hence the proof follows from the above inequality and (7). \(\square\)

**Remarks 2.2.** (i) The above lemma subsumes the following result due to Jenkins [13]: 
“let \(H\) be a numerical semigroup of genus \(g\) and \(0 < m < n\) non-gaps of \(H\) so that \(gcd(m, n) = 1\); then \(g \leq (m - 1)(n - 1)/2 \)”. Indeed, by using the notation of the lemma, take \(N = n\); then \(\gamma_N = 0\) and Jenkins’ result follows with \(h = m\).

(ii) The lower bound of Lemma 2.1 is the number of ramified points minus one of an \(N\)-sheeted covering of curves of genus \(g\) and \(\gamma_N\) respectively (defined over a field of characteristic \(p \nmid N\)), where all the ramified points are totally ramified points.
The next lemma will help us to understand the structure of the semigroups of type \((N, \gamma)\).

**Lemma 2.3.** Let \(H\) be a numerical semigroup.

(i) Suppose that \(H\) fulfills conditions (a) and (c) of the definition of \((N, \gamma)\)-hyperellipticity. Set \(F := \{(2\gamma + i)N : i \in \mathbb{N}\}\). Then:

\[
\text{(i.1)} \quad F \subseteq H, \quad 2N\gamma \in H, \\
\text{(i.2)} \quad \gamma = \gamma_N.
\]

(ii) Conversely, let \(N > 0\) be an integer. Then, \(H\) fulfills condition (a) and (c) mentioned above with \(N\) and \(\gamma_N\), and \(2N\gamma_N \in H\).

**Proof.**

(i) If \(\gamma = 0\) then \(N \in H\) and so we have (i.1) and (i.2). Let \(\gamma \geq 1\) and denote by \(f_1 < \ldots < f_\gamma\) the \(\gamma\) positive multiples of \(N\) non-gaps of \(H\) in \([N, 2N\gamma]\). So \(f_1 > N\).

Suppose that \(F \not\subseteq H\) and let \((2\gamma+i)N\) be the least element of \(F \cap G(H)\). By the semigroup property of \(H\) we have \((2\gamma + i)N - f_j \in G(H)\) for \(j = 1, \ldots, \gamma\). Then, by the selection of \((2\gamma + i)N\) we have that \(\{(2\gamma + i)N - f_j : j = 1, \ldots, \gamma\}\) are all the gaps \(\ell\) of \(H\) such that \(\ell \equiv 0 \mod \(N)\) and \(\ell \leq 2N\gamma\). The least of these gaps satisfies \((2\gamma + i)N - f_\gamma \geq iN \geq 2N\) due to condition (c) of Definition 2. Consequently \(f_1 = N\) which is a contradiction.

Statement (i.2) follows from (i.1) since the gaps \(\ell\) for which \(\ell \equiv 0 \mod \(N)\) belong to the interval \([N, 2N\gamma]\). It remains to proof that \(2N\gamma \in H\). Suppose that \(f_\gamma < 2N\gamma\). Then we get \(\gamma + 1\) gaps multiples of \(N\) namely, \(2N\gamma - f_\gamma, \ldots, 2N\gamma - f_1, 2N\gamma\) which is a contradiction due to (i.2).

(ii) By the definition of \(\gamma = \gamma_N\) (see (10)), there exist at least \(\gamma\) positive non-gaps - all of them being multiples of \(N\) - in the interval \([N, 2N\gamma]\). Denote by \(f_1 < \ldots < f_\gamma\) such non-gaps. Let \(\ell\) be the biggest gap of \(H\) so that \(\ell \equiv 0 \mod \(N)\). We claim that \(\ell < f_\gamma\), because on the contrary case we would have - as in the previous proof with \(\ell\) instead of \(2N\gamma - (\gamma + 1)\) gaps which is a contradiction with the definition of \(\gamma\). This implies that \(2N\gamma, (2\gamma + 1)N, \ldots\) are non-gaps and we are done. \(\square\)

**Corollary 2.4.** Let \(H\) be a numerical semigroup, \(\gamma\) a non-negative integer, \(M, N, r\) positive integers so that \(2(\gamma + r)M > (2\gamma + r)N\). Then \(H\) cannot be both of type \((N, \gamma)\) and of type \((M, \gamma + r)\).

**Proof.** Suppose \(H\) is both of type \((N, \gamma)\) and of type \((M, \gamma + r)\). From the previous lemma and since \(H\) is of type \((M, \gamma + r)\) we have

\[
2(\gamma + r)M = m_{\gamma+r} \leq (2\gamma + r)N.
\]

\(\square\)

Using Lemma 2.3 (ii) and Lemma 2.1 we have the following criteria for the type \((N, \gamma_N)\) of numerical semigroups.
Corollary 2.5. Let $H$ be a numerical semigroup and $N > 0$ an integer. Suppose that every $h \in H$ such that $h \not\equiv 0 \pmod{N}$ satisfies $h \geq 2N\gamma_N + 1$. Then, $H$ is of type $(N, \gamma_N)$.

Corollary 2.6. Let $H$ be a numerical semigroup of genus $g$ and $N$ prime. If $g > \rho_1(2\gamma_N, N, \gamma) = N^2\gamma_N - N + 1$, then $H$ is of type $(N, \gamma_N)$.

We also have:

Corollary 2.7. Let $H$ be a semigroup of type $(N, \gamma)$ with $N$ prime. Let $A \geq \gamma + 1$ be an integer and $g$ the genus of $H$. If $g > \rho_1(A, N, \gamma)$ (see (1)), then

$$\gcd(m_1, \ldots, m_{A-\gamma}) = N.$$  

3. Proof of Theorem A.

We study certain $N$-sheeted coverings

$$\pi : X \to \tilde{X}$$

of curves. To fix notation, let $X$ and $\tilde{X}$ be curves of genus $g$ and $\gamma$, respectively. We assume that there is a point $P \in X$ such that $\pi$ is totally ramified at $P$, i.e., $X$ will be a curve of type $(N, \gamma)$. We are mainly interested in relating the Weierstrass semigroups at $P$ and $\tilde{P} := \pi(P)$. Since $P$ is totally ramified, $\tilde{m}_iN \in H(P)$ for $\tilde{m}_i \in H(\tilde{P})$. Moreover, since $\tilde{m}_{\gamma+j} = 2\gamma + j$ for $j \in \mathbb{N}$, we have the following statements:

(I) $\gamma_N = \gamma_N(P) := \#\{\ell \in G(P) : \ell \equiv 0 \pmod{N}\} \leq \gamma$.

(II) $m_\gamma = m_\gamma(P) \leq 2\gamma N \in H(P)$.

Note that equality in (II) implies equality in (I), and, $H(P)$ is of type $(N, \gamma)$ if and only if equality in (II) holds. Moreover, if $h \in H(P)$ so that $\gcd(h, N) = 1$, from Lemma 2.1 and (I) we have

(3.1) $$h \geq \frac{2g - 2N\gamma_N}{N - 1} + 1 \geq \frac{2g - 2N\gamma}{N - 1} + 1.$$  

Hence we have the following generalization of [20, Lemma 3.1].

Lemma 3.1. Assume the above notation and suppose $g > \rho_1(2\gamma, N, \gamma) = N^2\gamma - N + 1$. Then, every $h \in H(P)$ such that

$$h \leq \frac{g + N(N - 2)\gamma}{N - 1}$$

satisfies $\gcd(h, N) > 1$.

The following result - due to Castelnuovo - will be used, among other things, to prove the implication (ii) $\Rightarrow$ (i) of Theorem A regardless of the characteristic of the base field and to construct examples in order to show that in some cases the bounds of our results are sharp.
Lemma 3.2 ([6], [19]. Let $X$ be a curve of genus $g$ and $K_1, K_2$ be subfields of $K(X)$ with compositum $K(X)$. If $n_i$ is the degree of $K(X)$ over $K_i$ and $g_i$ is the genus of $K_i$ for $i = 1, 2$, then

$$g \leq (n_1 - 1)(n_2 - 1) + n_1g_1 + n_2g_2.$$  

For $N$ prime, we have the uniqueness of $\pi$ above provided $g$ is large enough, and we also have a criterion to decide when a point is totally ramified:

Corollary 3.3. Let $X$ be a curve of genus $g$, $N$ prime and $\gamma$ a non-negative integer.

(i) If

$$g > \rho_5(N, \gamma) := 2N\gamma + (N - 1)^2,$$

then $X$ admits at most one $N$-sheeted covering of a curve of genus $\gamma$.

(ii) Let $P \in X$, $\tilde{X}$ be a curve of genus $\gamma$ and, $\pi$ an $N$-sheeted covering map from $X$ to $\tilde{X}$. Then, $P$ is totally ramified for $\pi$ provided there exists $h \in H(P)$ such that

$$N - 1)h < g - N\gamma + N - 1.$$  

Proof. (i) If $K(X)$ have two different fields $K_1$ and $K_2$ both of genus $\gamma$, then by Lemma 3.2 we have $g \leq \rho_5(N, \gamma)$.

(ii) Let $f \in K(X)$ with $\text{div}_\infty(f) = hP$ and $K'$ be the compositum of $K(f)$ and $K(\tilde{X})$. Using $N$ prime and the hypothesis on $h$, from Lemma 3.2 it follows that $K' = K(\tilde{X})$. Then, there exists $\tilde{f} \in K(\tilde{X})$ so that $f = \tilde{f} \circ \pi$. Consequently, the ramification number of $\pi$ at $P$ is $N$ and so $P$ is totally ramified for $\pi$.

Next we look for conditions to have equality in (I) or (II).

Lemma 3.4. Let $X$, $\tilde{X}$, $\pi$, $P$, $\tilde{P}$ and $N$ be as above. If either $p = \text{char}(K) \nmid N$, or $N$ is prime and $g > \rho_2(N, \gamma)$ (see (2)), then

$$\gamma_N = \gamma.$$  

Proof. It will be enough to show that $nN \in H(P) \Rightarrow n \in H(\tilde{P})$.

Case 1: $p \nmid N$. Let $z$ be a local parameter at $P$ so that $z^N$ is also a local parameter at $\tilde{P}$. Let $\Psi$ (resp. $\tilde{\Psi}$) denote the immersion of $K(X)$ (resp. $K(\tilde{X})$) into the field of Puiseux series at $P$ (resp. $\tilde{P}$) $F_1 = K((z))$ (resp. $F_2 = K((z^N))$). Since $\Psi|_{K(\tilde{X})} = \tilde{\Psi}$ we have that $Tr_{F_1|F_2} \circ \Psi = \tilde{\Psi} \circ Tr_{K(X)|K(\tilde{X})}$ ($\ast$) (Tr means trace). Let $f \in K(X)$ with $\text{div}_\infty(f) = nN$. Write $f = \sum_{i=-n}^\infty c_i z^{i}$. Then, by considering the base $\{1, z, \ldots, z^{N-1}\}$ of $F_1 | F_2$, we have that $Tr_{F_1|F_2}(f) = \sum_{i=-n}^\infty NC_i N z^{inN}$. Consequently, from ($\ast$) it follows that the order of $\tilde{f} := Tr_{K(X)|K(\tilde{X})}(f)$ at $\tilde{P}$ is $n$ and, since $f$ has no other pole, $\text{div}_\infty(\tilde{f}) = n\tilde{P}$ and we are done.

Case 2: $g > \rho_2(N, \gamma)$. From the proof of Corollary 3.3 (ii) we have that $f = \tilde{f} \circ \pi$ for some $\tilde{f} \in K(\tilde{X})$ whenever $\text{div}_\infty(f) = hP$ with $h$ satisfying (12). Now, from the hypothesis on $g$ we can applied the above statement for $h \in H(P)$ with $h \leq 2N\gamma - N$.  

$\square$
From (I), (II), (11) and the lemma above we obtain:

**Corollary 3.5.** Assume the hypothesis of Lemma 3.4.

(i) \( H(P) \) satisfies conditions (a), (c) of Definition 2 (with \( N \) and \( \gamma \)) and \( 2N\gamma \in H(P) \).

(ii) Suppose \( N \) is prime. If either \( N \neq p \) and \( g > \rho_1(2\gamma, N, \gamma) = N^2\gamma - N + 1 \) or \( g > \rho_2(N, \gamma) \), then \( H(P) \) is of type \( (N, \gamma) \).

**Remark 3.6.** Let \( \pi : X \to \tilde{X} \) be an \( N \)-sheeted covering of curves of genus \( g \) and \( \gamma \) respectively. Assume \( g > \rho(\gamma, N) \) and hence, in particular that \( \pi \) is “strongly branched” (cf. [1]). When \( \pi \) is a “maximal strongly branched” (e.g. \( N \) prime) we still have the result in Lemma 3.4 [1, Lemma 4].

To deal with the “geometry” of Theorem A we need the other Castelnuovo genus bound lemma:

**Lemma 3.7** ([5], [2, p. 116], [16, Cor. 2.8]) Let \( X \) be a curve of genus \( g \) that admits a birational morphism onto a non-degenerate curve of degree \( d \) in \( \mathbb{P}^r(K) \). Then

\[
g \leq c(d, r) := \frac{m(m-1)}{2} (r-1) + m\varepsilon
\]

where \( m \) is the biggest integer \( \leq (d-1)/(r-1) \) and \( \varepsilon = d - 1 - m(r-1) \).

**Lemma 3.8.** Let \( X \) be a curve of genus \( g \), \( N \) a prime and \( \gamma \geq 0 \) an integer. Let \( A \geq 2\gamma + 1 \) be an integer satisfying the hypotheses (5) and (6) of item (A4) (Theorem A). If \( X \) admits a base-point-free linear system \( g_{AN}^{A-\gamma} \), then \( X \) is an \( N \)-sheeted covering of a curve of genus \( \gamma \).

**Proof.** Let \( \pi : X \to \mathbb{P}^{A-\gamma}(K) \) be the morphism defined by \( g_{AN}^{A-\gamma} \).

**Claim:** \( \pi \) is not birational.

If by way of contradiction \( \pi \) is birational, we can applied the lemma above to obtain \( g \leq c(AN, A\gamma) = \rho_4(A, u, N, \gamma) \).

Let \( t \) be the degree of \( \pi \) and \( \tilde{X} \) the normalization of \( \pi(X) \). Then the induced morphism \( \pi : X \to \tilde{X} \) is a covering map of degree \( t \) and \( \tilde{X} \) admits a base-point-free linear system \( \tilde{g}_{AN}^{A-\gamma} \). In particular we have \( t \leq AN/(A - \gamma) \) and the hypothesis (5) implies \( t = N \).

Now, by the Clifford’s theorem we have that \( \tilde{g}_{A}^{A-\gamma} \) is nonspecial, and consequently by the Riemann-Roch theorem the genus of \( \tilde{X} \) is \( \gamma \) and the proof is complete. \( \square \)

**Proof of Theorem A. (A1):** Corollary 3.5.

(A2): Since \( \rho_3(N, \gamma) > \rho_1(2\gamma + 2, N, \gamma) \) from Corollary 2.7 we have \( D := \text{gcd}(m_1(P), \ldots, m_{\gamma+2}) = N \). In particular \( m_{\gamma+2} = (2\gamma + 2)N \). Now, we can apply the Claim in the proof above with \( A = 2\gamma + 2 \) and \( \rho_4(2\gamma + 2, N - 1, N, \gamma) = \rho_3(N, \gamma) \) to conclude that the degree \( t \) of the rational map obtained from the liner system \( m_{\gamma+2}P \) is bigger than 1. Due to the fact that \( t|D \) and \( N \) prime, we conclude that \( t = N \), and by a
similar argument to the above proof (last lines) we see that the covered curve has genus \( \gamma \) and we are done.

**(A3):** By Corollary 2.7 we have \( m_{A-\gamma}(P) = AN \) and it follows the proof.

**(A4):** The above lemma shows that \( X \) is an \( N \)-covering of a curve of genus \( \gamma \). Since the covering is given by \( |ANP| \), we have that \( P \) is a totally ramified point of \( \pi \) and the proof is complete.

**Corollary 3.9.** (i) Let \( \pi : X \to \tilde{X} \) be an \( N \)-sheeted covering of curves of genus \( g \) and \( \gamma \) respectively. Suppose \( N \) prime and \( g > \rho_3(N, \gamma) \). Then \( P \) is totally ramified for \( \pi \) if and only if \( H(P) \) is a semigroup of type \((N, \gamma)\).

(ii) Let \( H \) be a Weierstrass semigroup of genus \( g > \rho_3(N, \gamma) \). Then \( H \) is of type \((N, \gamma)\) if and only if there exists an integer \( A \in [2\gamma + 2, 2\gamma + 2 + \frac{\gamma}{N-1}] \) satisfying (5) and such that \( m_{A-\gamma} = AN \).

**Proof.** (i) Proof of item (A2) of Theorem A.

(ii) For the numbers \( A \) in that interval we have \( \rho_1(A, N, \gamma) \leq \frac{N(2N+1)\gamma}{2} + (N-1)^2 < \rho_3(N, \gamma) \) and hence the “if” part of the statement is just item (A3). To prove the “only if” part notice that \( u(A) = N - 1 \) and hence \( \rho_4(A, u(A), N, \gamma) = \rho_3(N, \gamma) \). Now the hypotheses on \( A \) assure that the degree of the map obtained from \( |ANP| \) is \( N \), and since this number is the \( g.c.d \) of the non-gaps \( m_1, \ldots, m_{A-\gamma} \), it follows the proof.

**Remark 3.10.** Remark 3.11 (ii), (iii) below show that neither the bound on \( g \) nor the hypothesis (5) of the above corollary (part (ii)) can be dropped. It would be interesting to have an arithmetical proof of Corollary 3.9 (ii) (i.e. without the assumption that \( H \) is a Weierstrass semigroup), because any counter example to the above question would be a numerical semigroup that cannot be realized as Weierstrass semigroup.

**Remarks 3.11.** Let \( N, \gamma \) be a prime and a non-negative integer respectively and suppose \( p \nmid 2N \).

(i) The following example has respect to Lemma 2.1, Corollary 2.7, Lemma 3.1, Corollary 3.5 (ii) and item (A1) of Theorem A. Let \( g > 0 \) be an integer such that \( g - N\gamma \equiv 0 \pmod{(N-1)} \) and \( L := \frac{2g-2\gamma N}{N-1} + 1 \) is coprime with \( 2N \). Define \( i_1 := 2\gamma + 1 \) and \( i_2 := \frac{g-2(N-1)\gamma}{N-1} \) (hence \( i_1 + 2i_2 = L \)). For \( j = 1, \ldots, i_1, k = 1, \ldots, i_2 \), choose \( a_j, b_k \) pairwise distinct elements of \( K \). Now consider the curve \( X \) defined by the equation

\[
y^{2N} = \prod_{j=1}^{i_1} (x - a_j) \prod_{k=1}^{i_2} (x - b_k)^2
\]

Then, by the Riemann-Hurwitz relation we have that the genus of \( X \) is \( g \). Moreover, \( X \) is a \( N \)-sheeted covering of the hyperelliptic curve \( \tilde{X} \) of genus \( \gamma \) whose field of rational functions is \( K(x, z) \), where \( z = y^{N}/\prod_{k=1}^{i_2} (x - b_k) \). Since \( \gcd(L, 2N) = 1 \), there exist just one point \( P \in X \) over \( x = \infty \) and consequently \( X \) is a curve of type \((N, \gamma)\) over \( \tilde{X} \).
Claim: \( H(P) = H := (2N, L, (2\gamma + 1)N) \).

This claim shows that the result in Lemma 2.1 is the best possible. Considering \( g = \rho_4(A, N, \gamma) \), we have \( L = AN - 1 \) and hence \( m_{A-\gamma}(H) \leq AN - 1 \) provided \( A \geq 2\gamma \). Hence the claim also shows the sharpness of the bound on \( g \) of Corollary 2.7. By specializing \( A = 2\gamma \) we also see the sharpness of the bound on \( g \) of Lemma 3.1 and Corollary 3.5 (ii) (case \( p \nmid N \)) respectively.

With respect to item (A1), it is not difficult to see that in the above curve \( X \) (with \( A = 2\gamma \)), all the Weierstrass semigroups at totally ramified points are not of type \((N; \gamma)\) (the case \( N = 2 \) is in [20, Remark 3.9]. However, we cannot say the same about the other Weierstrass semigroups of \( X \).

**Proof of the Claim.** Since the genus of \( H \) is at most \( g \), it will be enough to show that \( H \subseteq H(P) \). This is true because, \( \text{div}_\infty(x) = 2NP \), \( \text{div}_\infty(y) = LP \), and \( (2\gamma + 1)N \in H(P) \) due to the fact that \( P \) is totally ramified over \( \bar{X} \) which has genus \( \gamma \). □

(ii) This example is related to the bound on the genus in Lemma 3.8 and item (A4) of Theorem A. Set \( i_1 := 2N\gamma + 2N - 1 \). The curve \( X \) defined by the equation

\[
y^{2N} = \prod_{j=1}^{i_1} (x - a_j),
\]

where the \( a_j \)s are pairwise distinct elements of \( K \), has the following properties:

1. its genus is \( g = N(2N - 1)\gamma + (N - 1)(2N - 1) = \rho_3(N, \gamma) \);
2. the Weierstrass semigroup at the unique point \( P \) over \( x = \infty \) is generated by \( 2N \) and \( i_1 \);
3. \( m_{A-\gamma}(P) = AN \) provided \( 2\gamma \leq A < 4\gamma + 4 - \frac{2}{N} \);
4. it cannot be an \( N \)-covering of a curve of genus \( \gamma \).

Consequently, the upper bound \( \rho_3(N, \gamma) \) for the genus in both Lemma 3.8 and item (A4) is necessary for \( 2\gamma + 1 \leq A \leq 4\gamma + 4 - \frac{2}{N} \). In particular, if \( N-1 \) is the biggest integer \( \leq \frac{N\gamma + N - 1}{A - \gamma - 1} \), \( \rho_3(N, \gamma) \) is sharp. In the other cases, we do not know the sharpness of \( \rho_4(A, u, N, \gamma) \).

**Proof of properties (1)-(4).** (1) follows from Riemann-Hurwitz relation. To prove (2), we notice that \( \text{div}(x) = 2NP \) and \( \text{div}(y) = i_1P \) and so \( H(P) \supseteq (2N, i_1) \). Since the last semigroup also has genus \( g \), we have (2). To prove (3) we notice that in the interval \([1, AN]\) the number of multiples of \( N \) non-gaps of \( H(P) \) is \( \frac{A}{2} \) (or \( (A + 1)/2 \)); it has \( \frac{A}{2} - \gamma \) (or \( (A - 1)/2 - \gamma \)) non-gaps which are congruent to \( 2N - 1 \) module \( N \), and its other non-gaps are bigger than \( AN \) (here we use \( A < 4\gamma + 4 - \frac{2}{N} \)). Finally, if \( X \) is an \( N \)-sheeted covering of a curve of genus \( \gamma \), by Castelnuovo’s lemma (Lemma 3.2) the genus \( g \) would be at most \( (2N\gamma + 2N - 2)(N - 1) + N\gamma < \rho_3(N, \gamma) \), a contradiction.

(iii) Here, we show that the arithmetical conditions (5) cannot be dropped if we suppose

\[ g > \max \left\{ \frac{1}{2}AN[A(N - 2) + 2\gamma + 3], A(N - 1)(N - 2) + (3N - 2)\gamma + 3(N - 1) \right\}. \]
Since $A \geq 2\gamma + 1$ we have $\frac{AN}{A-\gamma} \leq 2N - 1$ and then one has to check (5) among the
integers of the set $[2, 2N - 1] \setminus N$. Let $t$ be an integer of the above set such that $t \mid A$ and
set $r := \frac{AN}{t} - A + \gamma + 1$. Let $g > 0$ be an integer satisfying the above bound and such
that $i_1 := \frac{2g}{r-1} + 1$ is also an integer. The curve in the previous remark, with $rt$ instead
of $2N$ and the above $i_1$, has genus $g$ and just one point $P$ over $x = \infty$ which satisfies
$m_{A, \gamma}(P) = AN$ (here we use the first part of the bound). But $X$ cannot be an $N$-sheeted
covering because on the contrary by Castelnuovo’s lemma (Lemma 3.2) the genus would
be at most the second part of the above bound.

(iv) Finally, some words about items (A2) and (A3). Since statement (ii) of Theorem A
is stronger than (iii), one might expect to sharpen $\rho_3(N, \gamma)$ (this would be relevant to the
examples in the next section). In order to do that, one might use Castelnuovo’s theory
(cf. e.g. [10, Sect. 3], [7]) or “results” extending this theory to Hilbert functions of points
in projective spaces [8]. Specifically, one could use analogous bounds to $c(d, r)$ in order
to deal with curves of genus $g \leq \rho_3(N, \gamma) = c(AN, A - \gamma)$. The point is that one knows
how must look the curves whose genus attain the mentioned bounds. For instance, one
can applied the above considerations to double covering of curves of genus one or two and
the result is that (A2) is still valid for $g \geq \rho_3(2, \gamma) - 2 (\gamma \in \{1, 2\})$. In general, we think
that item (A2) must be true with a bound of type “$\rho_3(N, \gamma) - N$”. We remark that by
applying the arithmetical properties of semigroups of type $(N, \gamma)$ one can find a “kind of
algorithm to compute Hilbert functions”’. We will intend to describe this in a later paper.
With respect to the sharpness of the bound on $g$ of item (A3), we just want to say that
it depends on the existence of certain Weierstrass semigroups.

4. Hurwitz’s question.

In this section we construct numerical semigroups, with $\ell_g$ given, that cannot be realized
as Weierstrass semigroup. These examples will include symmetric and quasi-symmetric
semigroups generalizing those in [20, Scholium 3.5] and [15, Example 6.5].

4.1. Corollaries of Buchweitz’s criterion; case $n = 2$. Let $X$ be a curve, $H$ a numerical
semigroup both of genus $g$. Denote by $\ell_1 = \ell_1(H), \ldots, \ell_g = \ell_g(H)$ (resp. $G_n = G(H)$)
the gaps (resp. the set of sums of $n$ gaps) of $H$. By the definition of $g$ and by the
semigroup property of $H$, we have $g \leq \ell_g \leq 2g - 1$ ([3], [14]). Semigroups with $\ell_g = g$
are realized for all but finitely many points of $X$ provided that the field of definition $K$
has characteristic $p = 0$ or $p > 2g - 2$ (see e.g. [18]. If $0 < p < 2g - 2$, the situation
can be different (see e.g. [17], [11]). On the other hand, if $\ell_g = 2g - 1$, $H$ is called
symmetric because between the non-gaps and gaps of $H$ we have the following property:
$S(1): "h \in H \Leftrightarrow \ell_g - h \in G(H)"$. As was mention in Section 1, these semigroups satisfy
the (BNC), namely $\#G_n = (2n - 1)(g - 1)$. What can be said if $\ell_g < 2g - 1$ ?. If
$\ell_g = 2g - 2$, $H$ still satisfies property $S(1)$ except for $g - 1$ [14, Prop. 1.2].
In general we have the following properties of type $S(1)$. Suppose $\ell_g \in \{2g-2i+1, 2g-2i\}$ with $i \geq 1$. Considering the pairs $(r, \ell_g - r)$ for $r = 1, \ldots, g - i$, $H$ satisfies the property:

$S(i)$: If $\ell_g$ is odd, $H$ fulfills property $S(1)$ except $2i - 2$ gaps of type: $g - i < h_{i-1} < \ldots < h_1 < \ell_g$, $\ell_g - h_{i-1}, \ldots, \ell_g - h_1$. If $\ell_g$ is even, $H$ fulfills condition $S(1)$ except $2i - 2$ gaps of the above type and the gap $(g - i)$.

Since for $i > 1$ we have gaps different from those arising in $S(1)$, it seems to be difficult to obtain a closed form for $\#G_n$. However, we think that the following must be true provided $\ell_g \geq 2g - 2$:

$$G_n = \{n, n + 1, \ldots, (n - 1)\ell_g\} \cup \{(n - 1)\ell_k + \ell_j : j = 1, \ldots g\}.$$  

From the proof of [15, Thm. 1.1] at least the inclusion “⊇” holds. In particular we have

$$\#G_2 = (\ell_g - 1) + g + \Lambda,$$

where $\Lambda$ is a non-negative integer. Consequently if $\Lambda \geq 2i - 2$ (resp. $2i - 1$) for $\ell_g = 2g - 2i + 1$ (resp. $2g - 2i$), by Buchweitz’s criterion $H$ is not a Weierstrass semigroup.

Now, consider the following sequences of gaps obtained from those of property $S(i)$:

$$2h_1 > \ldots > h_1 + h_{i-1}, \text{ and } 2h_2 > \ldots > 2h_{i-1}.$$  

With the above notation we have

**Lemma 4.1.1.** Let $H$ be a numerical semigroup with $\ell_g = 2g - 2i + 1$ and $i \geq 4$. If $h_1 + h_{i-1} > 2h_2$ and $2\ell_g - h_u - h_v \in G(H)$ for $(u, v) = (1, 1), \ldots, (1, i - 1)$ and $(u, v) = (2, 2), \ldots, (2, i - 1)$, then $H$ is not a Weierstrass semigroup.

**Proof.** The hypothesis involving $G(H)$ means that $h_u + h_v$ is not the sum of $\ell_g$ with some other gap. Consequently from the sequences of gaps above we have $\Lambda \geq 2i - 2$ and the proof follows. \hfill $\Box$

Using this criterion we can exhibit numerical semigroups with a fixed last gap which cannot be realized as Weierstrass semigroups. The following example with $i = 4$, $g = 16$ is the well known Buchweitz’s semigroup.

**Corollary 4.1.2.** Let $g, i$ be integers so that $g \geq 9i - 20$, $i \geq 4$ and $3g + 5i - 20$ even, say equal to $2h_1$. Then the numerical semigroup whose gaps are

$$\{1, 2, \ldots, g - i, h_1 - (a + 2(i - 3)), \ldots, h_1 - (a + 2), h_1 - a, h_1, 2g - 2i + 1\},$$

where $a = 2i - 5$ is not a Weierstrass semigroup.

In the above examples one can use $a > 2i - 6$ provided that $g \geq 2a - 10 + 5i$ and $3g + 2a + i - 10$ even.
4.2. An application of item (A2) of Theorem A.. First we notice that from the proof of item (A2), if \(N\) is prime and \(H = \{m_0 = 0, m_1, \ldots\}\) is a Weierstrass semigroup of type \((N, \gamma)\) of genus \(g > \rho_3(N, \gamma)\), then the numerical semigroup
\[
\pi(H) = \left\{\frac{m_i}{N} : 1 \leq i \leq \gamma\right\} \cup \{2\gamma + i : i \in \mathbb{N}\}
\]
is also a Weierstrass semigroup. We use this remark to prove an analogue of Corollary 4.1.2. The semigroups of this result are also inspired by the properties \(S(i)\) of the last subsection. Fix a numerical semigroup \(\tilde{H}\) of genus \(\gamma\) such that it is not a Weierstrass semigroup. Let \(N\) be a prime and \(g\) and integer. Write \(g = \lambda N + u\) with \(0 \leq u < N\). Let \(f\) be an integer such that \(f \leq u\) if \(u > 0\) and \(f < N\) otherwise. Set \(N\tilde{H} := \{hN; h \in \tilde{H}\}\). We are only going to consider the case \(2g - f \not\equiv 0 \pmod{N}\) because in the other case we can replace \(g\) by \(g + 1\).

**Corollary 4.2.1.** With the above notation, consider the following sets

1. \(H_1 = \{2g - f - r : r \not\in N\tilde{H}, r \leq g - 1\}\), if \(2u \not\in [N, f + N]\)
2. \(H_2 = H_1 \setminus \{e\}\) otherwise; where \(e\) is the biggest integer \(\leq (2g - f)/2\).

If \(g > \rho_3(N, \gamma)\), then \(H_1\) and \(H_2\) are numerical semigroups of type \((N, \gamma)\) of genus \(g\) whose last gap is \(2g - f\) which are not Weierstrass semigroups.

**Proof.** Set \(H\) for \(H_1\) or \(H_2\). We notice that \(\pi(H) = \tilde{H}\) and so it will be enough to prove the arithmetical statements. By the definition of \(H\) and the hypothesis on \(g\), it follows that \(H\) is a semigroup of type \((N, \gamma)\) such that \(H \supseteq \{2g, 2g + 1, \ldots\}\) and \(\ell_g(H) = 2g - f\) (here we use \(2g - f \not\equiv 0 \pmod{N}\)). Consider \(H = H_1\) and let
\[
U = \#\{h \in H : h \leq 2g, h = 2g \text{ or } h \equiv 0 \pmod{N}\}, \\
V = \#\{h \in H : h < 2g, h \not\equiv 0 \pmod{N}\}.
\]
Then
\[
U = \begin{cases} 
2\lambda - \gamma & \text{if } u = 0; \\
2\lambda + 1 - \gamma & \text{if } 0 < 2u < N; \\
2\lambda + 2 - \gamma & \text{if } 2u > N;
\end{cases}
\]
\[
V = g - 1 - \#\{1 \leq r \leq g - 1 : r \in N\tilde{H}\} - \#\{1 \leq r \leq g - 1 : r \equiv 2u - f \pmod{N}\}.
\]
Since \(2u - f \not\equiv N\), we have
\[
V = \begin{cases} 
g - 1 - (\lambda - 1 - \gamma) - \lambda & \text{if } u = 0; \\
g - 1 - (\lambda - \gamma) - \lambda & \text{if } 2u - f < N; \\
g - 1 - (\lambda - \gamma) - (\lambda + 1) & \text{if } 2u - f > N.
\end{cases}
\]
Then \(U + V = g\) is the number of non-gaps \(\leq 2g\) of \(H_1\) because \(2u \not\in [N, n + f]\). In the other case, from the above computations we get \(U + V = g + 1\) and since we have excluded \(e\) we are done. \(\square\)
The last two corollaries give us examples of numerical semigroups arising from different phenomena. Moreover, notice that the “intersection” of both families of examples is empty. By considering the distribution of the respective gaps sequences, we can think about of these examples as being the “extremal” cases of numerical semigroups that cannot be realized as Weierstrass semigroups. Finally, we would like to know if any numerical semigroup which is not a Weierstrass semigroup but satisfies Buchweitz’s criterion must be of type \((N, \gamma)\) for some \(N\) and \(\gamma\).

ACKNOWLEDGMENTS

Thank you very much to the International Atomic Energy Agency and UNESCO for the hospitality at the International Centre for Theoretical Physics, Trieste; to Profs. Oliveira and Stöhr who showed me how to applied item (A2) of Theorem A to the case of quasi-symmetric semigroups (in fact, their paper [15] suggested many ideas to this work); to the referee for suggesting to improve the earlier version of the paper.

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