1. Introduction

The concept of numerical semigroup is related to the problem of determining nonnegative integers that can be expressed in the form \( \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_r a_r \) for a given set \( \{a_1, a_2, \ldots, a_r\} \) of positive integers and for arbitrary nonnegative integers \( \lambda_1, \lambda_2, \ldots, \lambda_r \). This problem had been considered by several mathematicians like Frobenius (1849 - 1917) and Sylvester (1814 - 1897) at the end of the 19th century (see [9]).

During the second half of the twentieth century, interest in the study of numerical semigroups resurfaced because of their applications in algebraic geometry. For instance, some terminology from algebraic geometry has been exported to theory of numerical semigroups as the multiplicity, the genus or the embedding dimension.

Among the most studied families of numerical semigroups, we can mention those of the symmetric numerical semigroups and of the pseudo-symmetric numerical semigroups. Both have connections with commutative ring theory. In fact, let \( S \) be a numerical semigroup, \( \mathbb{F} \) a field, and \( \mathbb{F}[[t]] \) the ring of formal power series over \( \mathbb{F} \). It is well-known that \( \mathbb{F}[[S]] = \left\{ \sum_{s \in S} a_s t^s | a_s \in \mathbb{F} \right\} \) is a subring of \( \mathbb{F}[[t]] \), called the ring of semigroup associated to \( S \) (see, for instance, [1]). Properties over the numerical semigroup \( S \) are translated to the ring associated to \( S \). Actually, it is well-known that if the semigroup is symmetric, the ring associated to it is a Gorenstein ring (see [1, 2]) and if the semigroup is pseudo-symmetric, the ring is a Kunz ring (see [7]).

Irreducible numerical semigroups were introduced in [11]. Its study is clearly motivated from the semigroup theory point of view. However, the class of irreducible numerical semigroups is the union of classes of symmetric and pseudo-symmetric numerical semigroups. In fact, symmetric (resp. pseudo-symmetric) numerical semigroups are exactly the irreducible numerical semigroups with odd (resp. even) Frobenius number.

In this monograph, we present the concept of numerical semigroup as a motivation to classify the submonoids of the additive monoid of natural numbers. Each one of these submonoids is isomorphic to a unique numerical semigroup. Next, we present one of the several definitions of symmetric and pseudo-symmetric numerical semigroups, relating both concepts with the so-called almost-symmetric numerical semigroups. We also present the main properties of these semigroups, as well as for the irreducible numerical semigroups. Finally, we study an algorithm (see [13]) to find minimal decompositions of a non-irreducible numerical semigroup as an intersection of irreducible numerical semigroups.
2. Monoids and numerical semigroups

Before we present the numerical semigroups, let us consider a more general class of objects, known as monoids. In this section we are following the paper [3] and the Chapter 1 of [12].

A **monoid** is a triple $(M, *, 1_M)$ with $M$ a set, $*: M \times M \to M$ an **associative** operation on $M$ and $1_M \in M$ an **identity element**, that is, $1_M$ is such that $1_M * a = a = a * 1_M$, for all $a \in M$.

**Example 2.0.1.** The set $\mathbb{N}$ of non-negative integers is a monoid with the operation of addition (identity element 0) or multiplication (identity element 1).

**Example 2.0.2.** Given a set $S$, the set $\mathcal{P}(S)$ of all subsets of $S$ form a monoid under intersection operation (identity element is $S$ itself) or under union operation (identity element is the empty set).

**Example 2.0.3.** The set of all finite strings over some fixed alphabet $\Sigma$ forms a monoid with string concatenation as the operation. The empty string serves as the identity element. This monoid is denoted $\Sigma^*$ and is called the free monoid over $\Sigma$.

A subset $N$ of a monoid $M$ is called a **submonoid** of $M$ if $1_M \in N$ and $a * b \in N$, for all $a, b \in N$. In particular, $M$ and $\{1_M\}$ are submonoids of $M$, called the **trivial** submonoids of $M$.

The intersection of submonoids of a monoid is again one of its submonoids. Given a monoid $M$ and a subset $A$ of $M$, the smallest submonoid of $M$ containing $A$ is

$$\langle A \rangle = \{a_1^{r_1} \cdots a_n^{r_n} | n \in \mathbb{N}, r_1, \ldots, r_n \in \mathbb{N}, a_1, \ldots, a_n \in A\},$$

which is called the submonoid of $M$ generated by $A$. In this case, the subset $A$ is said a **system of generators** of $M$. A monoid $M$ is **finitely generated** if there exists a finite system of generators of $M$.

Given two monoids $X$ and $Y$, a function $f : X \to Y$ is a **monoid homomorphism** if

$$f(a + b) = f(a) + f(b),$$

for all $a, b \in X$ and

$$f(1_X) = 1_Y.$$

We say that $f$ is a **monomorphism**, an **epimorphism**, or an **isomorphism** if $f$ is injective, surjective or bijective, respectively. Two monoids $X$ and $Y$ are said to be isomorphic if there exists an isomorphism between them.

**Example 2.0.4.** The sequence of number of isomorphism classes of monoids with order $n$ is: $0, 1, 2, 7, 35, 228, 2237, 31559, 1668997, \ldots$ (sequence [http://oeis.org/A058129](http://oeis.org/A058129)).

We are interested in studying the submonoids of the additive monoid $\mathbb{N}$. The concept of numerical semigroup is useful for this. A **numerical semigroup** is a submonoid of the additive monoid $\mathbb{N}$ with finite complement in $\mathbb{N}$. Next, we will present some properties of systems of generators of numerical semigroups.

**Lemma 2.0.5.** Let $A$ be a nonempty subset of $\mathbb{N}$. Then $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

**Proof.** Let $d = \gcd(A)$. If $s$ belongs to $A$, then $d|s$. If $\langle A \rangle$ is a numerical semigroup, then $\mathbb{N} \setminus \langle A \rangle$ is finite. Therefore, there exists a positive integer $n$ such that $d|n$ and $d|(n + 1)$. Thus, $d = 1$. 

On the other hand, by definition of \( \langle A \rangle \), it suffices to prove that \( \mathbb{N} \setminus \langle A \rangle \) is finite. In fact, since \( 1 = \gcd(A) \), by Bézout’s identity, there exist integers \( r_1, \ldots, r_n \) and \( a_1, \ldots, a_n \in A \) such that
\[
(2.0.1) \quad r_1 a_1 + \cdots + r_n a_n = 1.
\]
Let \( I = \{ i | r_i < 0 \} \subset \{ 1, \ldots, n \} \). Then, \( 2.0.1 \) can be rewritten as
\[
(2.0.2) \quad \sum_{j \in \{ 0, \ldots, m \} \setminus I} r_j a_j = 1 - \sum_{i \in I} r_i a_i.
\]
Therefore, there exist \( s \in \langle A \rangle \) such that \( s + 1 \) also belongs to \( \langle A \rangle \), namely, \( s = - \sum_{i \in I} r_i a_i \). We affirm that if \( n \geq (s-1)s + (s-1) \), then \( n \in \langle A \rangle \). By Euclidean division, there exist integers \( q \) and \( r \) such that \( n = qs + r \), where \( 0 \leq r < s \). Since, \( n \geq (s-1)s + (s-1) \), we conclude that \( q \geq s - 1 \geq r \).

Therefore,
\[
\begin{align*}
n &= qs + r \\
&= qs - rs + rs + r \\
&= (q-r)s + r(s+1)
\end{align*}
\]
and \( n \in \langle A \rangle \), since \( s, s+1 \in \langle A \rangle \) and \( r, q-r \in \mathbb{N} (q-r \geq 0) \). Thus, \( \langle A \rangle \) is infinity and then \( \mathbb{N} \setminus \langle A \rangle \) is finite.

By following Proposition 2.0.6, numerical semigroups classify, up to isomorphism, the submonoids of the additive monoid \( \mathbb{N} \).

**Proposition 2.0.6.** Let \( M \neq \emptyset \) be a proper submonoid of \( \mathbb{N} \). Then \( M \) is isomorphic to a numerical semigroup.

**Proof.** Let \( d = \gcd(M) \). By Lemma 2.0.5 \( S = \langle \{ \frac{m}{d} | m \in M \} \rangle \) is a numerical semigroup. Consider the function:
\[
f : M \to S \\
m \mapsto f(m) = \frac{m}{d}.
\]
Notice that \( f \) is clearly a monoid isomorphism.

Moreover, we will see that isomorphic numerical semigroups are equal, that is, numerical semigroups form a complete set of invariants for submonoids of \( \mathbb{N} \).

If \( A \) and \( B \) are subsets of a monoid \( M \), we define \( A + B = \{ a + b \in M | a \in A, b \in B \} \). We also define \( M^* = M \setminus \{ 0 \} \).

**Lemma 2.0.7.** Let \( M \) be a submonoid of \( \mathbb{N} \). Then \( M^* \setminus (M^* + M^*) \) is a system of generators of \( M \). Moreover, every system of generators of \( S \) contains \( M^* \setminus (M^* + M^*) \).

**Proof.** Let \( m \) be an element of \( M^* \). If \( m \notin M^* \setminus (M^* + M^*) \), then there exist \( a, b \in M^* \) such that \( m = a + b \). We can repeat this procedure for \( a \) and \( b \), and after a finite number of steps \( (a, b < m) \) we will find \( m_1, \ldots, m_n \in M^* \setminus (M^* + M^*) \) such that \( m = m_1 + \cdots + m_n \). Therefore, \( M^* \setminus (M^* + M^*) \) is a system of generators of \( M \).

Now, let \( A \) be a system of generators of \( M \). Consider \( s \in M^* \setminus (M^* + M^*) \). Then, there exist \( n \in \mathbb{N}^*, \lambda_1, \ldots, \lambda_n \in \mathbb{N} \) and \( a_1, \ldots, a_n \in A \) such that \( s = \lambda_1 a_1 + \cdots + \lambda_n a_n \). As \( s \notin M^* + M^* \), we
can conclude that $s = a_i$, for some $i \in \{1, \ldots, n\}$, that is, $s \in A$. □

Now, we will see that $M^* \setminus (M^* + M^*)$ is finite. For this, we will need the concept of Apéry set of a numerical semigroup.

Let $S$ be a numerical semigroup and let $n \in S^*$. The Apéry set of $n$ in $S$ is
\[ \text{Ap}(S, n) = \{ s \in S | s - n \notin S \}. \]

**Lemma 2.0.8.** Let $S$ be a numerical semigroup and let $n \in S^*$. Then
\[ \text{Ap}(S, n) = \{ 0 = w(0), w(1), \ldots, w(n-1) \}, \]
where
\[ w(i) = \min \{ s \in S | s \equiv i \mod n \}, \]
for all $0 \leq i \leq n - 1$.

*Proof.* It follows directly from the fact that for every $i \in \{1, \ldots, n - 1\}$, there exists $k \in \mathbb{N}$ such that $i + kn \in S$. □

**Example 2.0.9.** Let $S$ be the numerical semigroup generated by $\{5, 7, 9\}$. Then
\[ S = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\} \]
(the symbol $\rightarrow$ means that every integer greater than 14 belongs to $S$). Hence $\text{Ap}(S, 5) = \{0, 7, 9, 16, 18\}$.

In particular, it follows from Lemma 2.0.8 that $\# \text{Ap}(S, n) = n$. □

**Lemma 2.0.10.** Let $S$ be a numerical semigroup and let $n \in S^*$. Then for all $s \in S$, there exists a unique pair $(k, w) \in \mathbb{N} \times \text{Ap}(S, n)$ such that $s = kn + w$.

*Proof.* This follows directly from the characterization of $\text{Ap}(S, n)$ given in Lemma 2.0.8. □

We say that a system of generators of a numerical semigroup is a **minimal** system of generators if none of its proper subsets generates the numerical semigroup.

**Theorem 2.0.11.** Every numerical semigroup admits a unique minimal system of generators. This minimal system of generators is finite.

*Proof.* By Lemma 2.0.7, $S^* \setminus (S^* + S^*)$ is the minimal system of generators of $S$. By Lemma 2.0.10 for any $n \in S^*$, we have that $S = \langle \text{Ap}(S, n) \cup \{n\} \rangle$. As $\#(\text{Ap}(S, n) \cup \{n\}) = n + 1$, we conclude that $S^* \setminus (S^* + S^*)$ is finite. □

Since every submonoid of $\mathbb{N}$ is isomorphic to a numerical semigroup, this property is also true for submonoids of $\mathbb{N}$.

**Corollary 2.0.12.** Let $M$ be a submonoid of $\mathbb{N}$. Then $M$ has a unique minimal system of generators, which in addition is finite.
Let $S$ be a numerical semigroup and let $\{m_1 < m_2 < \cdots < m_p\}$ be its minimal system of generators. Then $m_1$ is known as the multiplicity of $S$, denoted by $m(S)$. The cardinality $p$ of the minimal system of generators is called the embedding dimension of $S$ and will be denoted by $e(S)$. Notice that $e(S) = 1$ if and only if $S = \mathbb{N}$.

**Proposition 2.0.13.** Let $S$ be a numerical semigroup. Then

i) $m(S) = \min(S^*)$,

ii) $e(S) \leq m(S)$.

**Proof.** It follows directly from its definition that the multiplicity is the least positive integer in $S$. The item ii) follows from the fact that $\{m(S)\} \cup \text{Ap}(S, m(S)) \setminus \{0\}$ is a system of generators of $S$ with cardinality $m(S)$. □

**Example 2.0.14.** If $m$ is a positive integer, then $S = \{0, m, -\} \in \mathbb{N}$ is a numerical semigroup with multiplicity $m$. A minimal system of generators for $S$ is $\{m, m + 1, \cdots, 2m - 1\}$. Thus, $e(S) = m = m(S)$ (the upper bound of item ii) of Proposition 2.0.13 is reached).

Now, we will present other important invariants of a numerical semigroup: its Frobenius number and its genus. Given a numerical semigroup $S$, the Frobenius number of $S$ is the greatest integer not in $S$, which is denoted by $F(S)$. It is sometimes replaced by the conductor of $S$, which is the least integer $x$ such that $x + n \in S$, for all $n \in \mathbb{N}$, that is, $x = F(S) + 1$. We will denote the conductor of $S$ by $c(S)$. We denote by $G(S) = \mathbb{N} \setminus S$, which is known as the set of gaps of $S$. Its cardinality is called the genus of $S$ and is denoted by $g(S)$.

**Example 2.0.15.** Let $S = (5, 7, 9)$. We know that $S = \{0, 5, 7, 9, 10, 12, 14, -\} \in \mathbb{N}$ and thus $F(S) = 13$, $G(S) = \{1, 2, 3, 4, 6, 8, 11, 13\}$ and $g(S) = 8$.

**Lemma 2.0.16.** Let $S$ be a numerical semigroup. Then

$$g(S) \geq \frac{F(S) + 1}{2}.$$

**Proof.** It follows from the following remark: if $s \in S$, then $F(S) - s$ cannot be in $S$ (otherwise, we would have $F(S) = (F(S) - s) + s \in S$). □

**Example 2.0.17.** The sequence of number of numerical semigroups with Frobenius number equal to $n$ is

\[
1, 1, 2, 2, 5, 4, 11, 10, 21, 22, 51, 40, 106, 103, 200, 205, 465, 405, 961, 900, 1828, 1913, 4096, 3578, 8273, 8175, 16132, 16267, 34903, 31822, 70854, 68681, 137391, 140661, 292081, 270258, 591443, 582453, 1156012, \ldots \text{(sequence } \text{http://oeis.org/A124506).}
\]

**Example 2.0.18.** The sequence of number of numerical semigroups of genus $g$ is

\[
1, 1, 2, 4, 7, 12, 23, 39, 67, 118, 204, 343, 592, 1001, 1693, 2857, 4806, 8045, 13467, 22464, 37396, 62194, 103246, 170963, 282828, 467224, 770832, 1270267, 2091030, 3437839, 5646773, 9266788, 15195070, 24896206, 49761087, 66687201, 109032500, 178158289, \ldots \text{(sequence } \text{http://oeis.org/A007323).}
\]

Bras-Amarós in [4] has computed the number $n_g$ for $g \geq 50$ and detected a Fibonacci-like behaviour on the quotients $n_g/n_{g-1}$.
Now, we will finish our results about submonoids of $\mathbb{N}$.

**Lemma 2.0.19.** If $S$ and $T$ are isomorphic numerical semigroups, then there is only one isomorphism $\varphi : S \to T$, that is an increasing function.

**Proof.** Let $\varphi : S \to T$ be an isomorphism. Then $\varphi(m(S)) = m(T)$. By Proposition 2.0.13, we know that $m(S) = \min(S^*)$ (resp. $m(T) = \min(T^*)$). If we assume that $\varphi(m(S)) > m(T)$, then there is $s_1 \in S$ with $s_1 > m(S)$ such that $\varphi(s_1) = m(T)$. Then

$$s_1 \varphi(m(S)) = \varphi(s_1 m(S)) = m(S) \varphi(s_1) = m(S) m(T),$$

and since $s_1 > m(S)$ and $\varphi(m(S)) > m(T)$, also $s_1 \varphi(m(S)) > m(S) m(T)$, a contradiction. Therefore, $\varphi(m(S)) = m(T)$.

If we write $S = \{0 < s_0 = m(S) < s_1 < s_2 < \cdots \}$ and $T = \{0 < t_0 = m(T) < t_1 < t_2 < \cdots \}$, then $\varphi(s_0) = t_0$. Now, $\varphi|_{S \setminus \{s_0\}} : S \setminus \{s_0\} \to T \setminus \{t_0\}$ is an isomorphism of numerical semigroups, thus, by a similar argument as above, $\varphi(s_1) = t_1$. And by induction on $n$ we obtain that $\varphi(s_n) = t_n$ for all $n \in \mathbb{N}$. This shows that $\varphi$ is unique and is an increasing function. $\square$

**Lemma 2.0.20.** Suppose that $\varphi : S \to T$ is an isomorphism between numerical semigroups. If $U \subset S$ is a numerical semigroup, then $\varphi(U) \subset T$ is also a numerical semigroup.

**Proof.** Since $U$ is a numerical semigroup, $\mathbb{N} \setminus U$ is finite, and therefore $S \setminus U$ is finite. Then, $\varphi(S \setminus U) = T \setminus \varphi(U)$ is finite and $\mathbb{N} \setminus \varphi(U) = (\mathbb{N} \setminus T) \cup (T \setminus \varphi(U))$ is finite. Thus, $\varphi(U)$ is a numerical semigroup. $\square$

**Theorem 2.0.21.** If $S$ and $T$ are isomorphic numerical semigroups, then $S = T$.

**Proof.** Let $S$ and $T$ be numerical semigroups and let $\varphi : S \to T$ be the unique isomorphism, that by Lemma 2.0.19 is increasing.

First we prove that $\varphi(c(S)) = c(T)$. Let $s_0 \in S$ such that $\varphi(s_0) = c(T)$. If $t \in T$ is such that $t \geq c(T)$, then $t = c(T) + r$ for some $r \geq 0$ and there exists $s \in S$ such that $\varphi(s) = t$. Since $\varphi$ is increasing, we have that $s = s_0 + s(r)$ for some $s(r) \geq 0$. Then we have

$$s_0 t = s_0 \varphi(s) = \varphi(s_0 s) = s \varphi(s_0) = (s_0 + s(r)) c(T).$$

By replacing $t = c(T) + r$ we get $s_0 (c(T) + r) = (s_0 + s(r)) c(T)$, that yields to $r s_0 = s(r) c(T)$ and therefore

$$s(r) = \frac{s_0}{c(T)} r$$

for all $r \geq 0$. Let $c = s_0 / c(T)$. Since $cr = s(r)$ is a natural number for all $r \geq 0$, $c$ is a natural number. Hence $\varphi(s_0 + cr) = c(T) + r$ for all $r \geq 0$, and so $\varphi^{-1}(c(T) + r) = s_0 + cr$, which means that

$$\varphi^{-1}((c(T), c(T) + 1, c(T) + 2, \ldots)) = \{s_0 + cr | r \geq 0\}.$$ 

Now we apply Lemma 2.0.20 to the isomorphism $\varphi^{-1} : T \to S$ and the numerical semigroup $U = \{0, c(T), c(T) + 1, c(T) + 2, \ldots \} \subset T$ to obtain that $\varphi^{-1}(U) = \{0 \cup \{s_0 + cr | r \geq 0\}$ is a numerical semigroup. Then, for some $r$ big enough, $s_0 + rc$ and $s_0 + c(r + 1)$ must be consecutive natural numbers, that is, $s_0 + c(r + 1) = s_0 + cr + 1$, that yields to $c = 1$. 

Therefore \( \varphi^{-1}(\{c(T), c(T) + 1, c(T) + 2, \ldots \}) = \{s_0 + r | r \geq 0\} \). In particular, \( \{s_0 + r | r \geq 0\} \subset S \) and \( \deg c(S) = s_0 \). Then, \( \varphi(c(S)) \leq \varphi(s_0) = c(T) \).

By a symmetric argument applied to the isomorphism \( \varphi^{-1} : T \rightarrow S \) we obtain that \( \varphi^{-1}(c(T)) \leq c(S) \), so that \( c(T) \leq \varphi(c(S)) \). Thus, we have proved that \( \varphi(c(S)) = c(T) \).

Now we will show that \( c(S) = c(T) \). In fact, the restriction of \( \varphi \) to \( V = \{0, c(S), c(S) + 1, c(S) + 2, \ldots \} \) gives an isomorphism between \( V \) and \( U = \{0, c(T), c(T) + 1, c(T) + 2, \ldots \} \), so these two numerical semigroups have the same embedding dimension, but \( U \) has embedding dimension \( c(T) \) and \( V \) has embedding dimension \( c(S) \) (see Example 2.0.14), so that \( c(S) = c(T) \).

By Lemma 2.0.19 there is only one isomorphism from \( V \) to \( U = V \), that is indeed the identity map. The restriction \( \varphi|_V : V \rightarrow V \) is thus the identity of \( V \), so for all \( s \geq c(S) \), \( \varphi(s) = s \).

Now we can show that \( S = T \). If \( s \in S^* \), then for some \( m \in \mathbb{N}^* \), \( ms > c(S) \). Then, \( m \varphi(s) = \varphi(ms) = ms \), and since \( m > 0 \) it results that \( s = \varphi(s) \in T \). This shows that \( S \subset T \). The analog argument applied to \( \varphi^{-1} : T \rightarrow s \) shows the inclusion \( T \subset S \). Thus, \( S = T \).

Notice that, by Lemma 2.0.19 if \( S \) is a numerical semigroup, then the identity map \( \text{id}_S : S \rightarrow S \) is the only isomorphism that exists.

**Corollary 2.0.22.** Let \( M \neq \emptyset \) be a proper submonoid of \( \mathbb{N} \). Then \( M \) is isomorphic to a unique numerical semigroup.

Now, we will introduce a family of invariants we will need to study irreducible numerical semigroups: the pseudo-Frobenius numbers and the type of a numerical semigroup. Let \( S \) be a numerical semigroup. We say that an integer \( x \) is a pseudo-Frobenius number if \( x \notin S \) and \( x + s \in S \), for all \( s \in S^* \). We will denote by \( \text{PF}(S) \) the set of pseudo-Frobenius numbers of \( S \). The cardinality of \( \text{PF}(S) \) is called the type of \( S \) and is denoted by \( t(S) \).

We can define the following relation over the integers: say that \( a \leq_S b \) if \( b - a \in S \). It follows from the properties of numerical semigroup of \( S \) that \( \leq_S \) is a (partial) order relation, that is, \( \leq_S \) is reflexive, antisymmetric and transitive. From the definition of pseudo-Frobenius numbers, we can conclude that they are the maximal elements with respect to \( \leq_S \) of \( \mathbb{Z} \setminus S \).

**Proposition 2.0.23.** Let \( S \) be a numerical semigroup. Then

i) \( \text{PF}(S) = \text{Maximals}_{\leq_S}(\mathbb{Z} \setminus S) \),

ii) \( x \in \mathbb{Z} \setminus S \) if and only if \( f - x \in S \) for some \( f \in \text{PF}(S) \).

Therefore, we have a duality between minimal generators and pseudo-Frobenius numbers of a numerical semigroup, since \( \text{Minimals}_{\leq_S}(\mathbb{Z} \setminus \{0\}) \) is the minimal system of generators of \( S \).

**Example 2.0.24.** Let \( S = \langle 5, 7, 9 \rangle = \{0, 5, 7, 9, 10, 12, 14, \rightarrow \} \). Then \( \text{PF}(S) = \{11, 13\} \) and \( t(S) = 2 \).
3. Almost-symmetric, symmetric, pseudo-symmetric and irreducible numerical semigroups

Before presenting the irreducible numerical semigroups, let’s talk about some classes of numerical semigroups related to them: the almost-symmetric, symmetric and pseudo-symmetric numerical semigroups. In this part we are following, mainly, the references [3], [6] and the Chapter 3 of [12].

The following Proposition generalizes the inequality of Lemma 2.0.16 (for a demonstration, see the reference [8]):

Proposition 3.0.1. Let $S$ be a numerical semigroup. Then

$$g(S) \geq \frac{F(S) + t(S)}{2}.$$  

The numerical semigroups $S$ such that the equality occurs in Proposition 3.0.1 are called almost symmetric. The almost symmetric numerical semigroups of type 1 are called symmetric and those of type 2 are called pseudo-symmetric.

Example 3.0.2. The numerical semigroup $\langle 4, 6, 7 \rangle = \{0, 4, 6, 7, 8, 10, 11, \rightarrow \}$ is symmetric, $\langle 3, 4, 5 \rangle = \{0, 3, \rightarrow \}$ is pseudo-symmetric and $\langle 5, 8, 9, 12 \rangle = \{0, 5, 8, 9, 10, 12, \rightarrow \}$ is almost symmetric of type 3 (and therefore is neither symmetric nor pseudo-symmetric).

The next Proposition gives one of the many characterizations of symmetric and pseudo-symmetric numerical semigroups. Sometimes these are chosen as the definitions (we have chosen the definition in terms of equality in Proposition 3.0.1 to make a connection with the concept of almost symmetric numerical semigroups).

Proposition 3.0.3. Let $S$ be a numerical semigroup.

i) $S$ is symmetric if and only if $F(S)$ is odd and $x \in \mathbb{Z} \setminus S$ implies $F(S) - x \in S$.

ii) $S$ is pseudo-symmetric if and only if $F(S)$ is even and $x \in \mathbb{Z} \setminus S$ implies that either $F(S) - x \in S$ or $x = F(S)/2$.

Proof. i) If $S$ is symmetric, then $g(S) = \frac{F(S) + t(S)}{2} = \frac{F(S) + 1}{2}$. Since $g(S)$ is an integer, it follows that $F(S)$ is odd. As $S$ is symmetric, we have $t(S) = 1$ and $PF(S) = \{F(S)\}$. By item ii) of Proposition 2.0.23 $x \in \mathbb{Z} \setminus S$ if and only if $f - x \in S$ for some $f \in PF(S)$, that is, $f = F(S)$. Therefore, $F(S) - x \in S$.

Now consider $G(S) \subset \mathbb{Z} \setminus S$. Then we have $F(S) - x \in S$, for all $x \in G(S)$. Therefore, each gap of $S$ is sent to a element of $S$ less than $F(S)$ (and $G(S)$ is sent to 0). Since $F(S)$ is odd, this is well defined, that is, there is no element $x$ of $G(S)$ such that $F(S) - x = x$. Thus, $g(S) = \frac{F(S) + 1}{2}$ and $S$ is symmetric.

ii) Since $S$ is pseudo-symmetric, then $g(S) = \frac{F(S) + t(S)}{2} = \frac{F(S) + 2}{2}$. As $g(S)$ is an integer, it follows that $F(S)$ is even. Let $y \neq F(S)$ the other element of $PF(S)$. In fact, we have $g(S) = \frac{F(S) + 1}{2}$. If $s \in \{1, \ldots, F(S)\}$ is such that $s \in S$, then $F(S) - s \notin S$ (otherwise, if $F(S) - s \in S$, we would have $F(S) = s + (F(S) - s) \in S$, which is a contradiction). If $s < F(S)/2$ (resp. $s < F(S)/2$), then $F(S) - s > F(S)/2$ (resp. $F(S) - s < F(S)/2$). Moreover, $F(S)/2$ is a gap, since $2F(S)/2 = F(S) \notin S$. Thus, $x \in \mathbb{Z} \setminus S$, implies that $F(S) - x \in S$ or $x = F(S)/2$. 

The proof of this part is analogue to corresponding on item i). Consider \( G(S) \subset \mathbb{Z} \setminus S \). Then we have \( F(S) - x \in S \), for all \( x \in G(S), x \neq F(S)/2 \). Thus, \( g(S) = \frac{F(S)}{2} + 1 = \frac{F(S)+2}{2} \) and \( S \) is pseudo-symmetric.

\[ \square \]

Example 3.0.4. The number of symmetric numerical semigroups with Frobenius number \( 2n - 1 \) is:

\[ 1, 1, 2, 3, 3, 6, 8, 7, 15, 20, 18, 36, 44, 45, 83, 109, 101, 174, 246, 227, \ldots \]

(sequence \texttt{https://oeis.org/A158278}).

Example 3.0.5. The number of pseudo-symmetric numerical semigroups with Frobenius number \( 2n \) is:

\[ 1, 1, 1, 2, 3, 2, 6, 7, 7, 11, 20, 14, 35, 37, 36, 70, 106, 77, 182, \ldots \]

(sequence \texttt{https://oeis.org/A158279}).

Symmetric numerical semigroups are exactly those numerical semigroups with type one:

Proposition 3.0.6. Let \( S \) be a numerical semigroup. The following are equivalent.

1. \( S \) is symmetric.
2. \( PF(S) = \{F(S)\} \).
3. \( t(S) = 1 \).

Proof. The implication i)\( \Rightarrow \) iii) follows from definition of symmetric numerical semigroup and iii)\( \Rightarrow \) ii) follows from definition of pseudo-Frobenius numbers. By item i) of Proposition 3.0.3, we can conclude ii)\( \Rightarrow \) i) if we prove that \( F(S) \) is odd and that \( x \in \mathbb{Z} \setminus S \) implies \( F(S) - x \in S \). In fact, the second assertion follows from item ii) of Proposition 2.0.23. If \( F(S) \) is even, then \( F(S)/2 \notin S \) (otherwise, we would have \( F(S) = 2F(S)/2 \in S \)). Then, again by item ii) of Proposition 2.0.23 \( f - F(S)/2 \in S \), for some \( f \in PF(S) \). Since \( PF(S) = \{F(S)\} \), we conclude that \( F(S)/2 = F(S) - F(S)/2 \in S \), which is a contradiction. Thus, \( F(S) \) is odd and then, \( S \) is symmetric.

\[ \square \]

The analogue to Proposition 3.0.6 for pseudo-symmetric semigroups is stated as follows.

Proposition 3.0.7. Let \( S \) be a numerical semigroup. The following conditions are equivalent.

1. \( S \) is pseudo-symmetric.
2. \( PF(S) = \{F(S), F(S)/2\} \).

Proof. i)\( \Rightarrow \) ii) If \( S \) is pseudo-symmetric, then \( t(S) = 2 \). Therefore, \( PF(S) = \{F(S), y\} \), where \( y \neq F(S) \). By item ii) of Proposition 3.0.3, since \( y \in \mathbb{Z} \setminus S \), if \( y \neq F(S)/2 \), then we have \( F(S) - y \in S^* \). By definition of a pseudo-Frobenius number applied to \( y \), we can deduce that \( F(S) = y + (F(S) - y) \in S \), which is a contradiction. Thus, \( y = F(S)/2 \).

ii)\( \Rightarrow \) i) If \( x \in \mathbb{Z} \setminus S \), then by item ii) of Proposition 2.0.23 \( f - x \in S \) for some \( f \in PF(S) \), that is, \( F(S) - x \in S \) or \( \frac{F(S)}{2} - x \in S \). Suppose that \( \frac{F(S)}{2} - x \in S \) and \( x \neq \frac{F(S)}{2} \), that is, \( \frac{F(S)}{2} - x \in S^* \). Then, by definition of a pseudo-Frobenius number applied to \( F(S)/2 \) we have that \( F(S) - x = \left( \frac{F(S)}{2} \right) + \left( \frac{F(S)}{2} - x \right) \in S \). Thus, by item ii) of Proposition 3.0.3, \( S \) is pseudo-symmetric.

\[ \square \]

We saw in Proposition 3.0.6 that \( t(S) = 1 \) implies that \( S \) is symmetric. But, \( t(S) = 2 \) cannot ensure that \( S \) is pseudo-symmetric.
Example 3.0.8. Let \( S = \langle 5, 7, 8 \rangle = \{0, 5, 7, 8, 10, 12, \rightarrow \} \). The set of pseudo-Frobenius numbers of \( S \) is \( \text{PF}(S) = \{9, 11\} \). This semigroup has type two, but it is not pseudo-symmetric. In fact, \( g(S) = 7 \) and \( \frac{F(S)+t(S)}{2} = \frac{11+2}{2} = \frac{13}{2} \).

If \( S \) is not symmetric (\( t(S) \geq 2 \)), by Proposition 3.0.6, define \( h(S) = \max(\text{PF}(S) \setminus \{F(S)\}) \), that is, the second largest pseudo-Frobenius number of \( S \). If \( S \) is pseudo-symmetric, \( h(S) = F(S)/2 \). We have the following Lemma which will be useful for us in the sequel.

Lemma 3.0.9. Let \( S \) be a numerical semigroup such that \( S \) is not symmetric. Then \( 2h(S) \geq F(S) \).

Proof. In fact, by definition of pseudo-Frobenius number applied to \( h(S) \), we have that \( h(S) + s \in S \), for all \( s \in S^* \). Therefore, since \( F(S) \in \Z \setminus S \), we conclude that \( F(S) - h(S) \in \Z \setminus S \). Now, by item ii) of Proposition 2.0.23, \( f - (F(S) - h(S)) \in S \), for some \( f \in \text{PF}(S) \). As \( F(S) - (F(S) - h(S)) = h(S) \notin S \), we must have \( f \leq h(S) \). Then, \( 2h(S) - F(S) \geq f - (F(S) - h(S)) \in S \), that is, \( 2h(S) - F(S) \geq 0 \).

Let \( n \) be a fixed integer. Define \( \mathcal{S}(n) = \{S|S \text{ is a numerical semigroup with } F(S) = n\} \). The elements of \( \mathcal{S}(n) \) are partially ordered with respect to inclusion. The following characterization of symmetric and pseudo-symmetric numerical semigroups is what relates them to the irreducible numeric semigroups:

Proposition 3.0.10. Let \( S \) be a numerical semigroup with Frobenius number \( F(S) = n \).

i) If \( n \) is odd we have that \( S \) is maximal in \( \mathcal{S}(n) \) if and only if \( S \) is symmetric.

ii) If \( n \) is even we have that \( S \) is maximal in \( \mathcal{S}(n) \) if and only if \( S \) is pseudo-symmetric.

Proof. i) If \( S \) is symmetric and \( x \in \Z \setminus S \), then it follows from item ii) of Proposition 2.0.23 and from Proposition 3.0.6 that \( F(S) - x = s \), for some \( s \in S \). Thus, \( F(S) = s + x \in \langle S, x \rangle \) so \( F(\langle S, x \rangle) < F(S) \) and hence \( S \) is maximal in \( \mathcal{S}(n) \).

If \( S \) is not symmetric, we have \( F(S) \notin \langle S, h(S) \rangle \). In fact, if \( F(S) = s + m \cdot h(S) \) for some \( s \in S \) and some \( m \in \N \), we must have \( n = 1 \) since (by Lemma 3.0.9) \( 2h(S) \geq F(S) \) and thus this would give \( h(S) + s = F(S) \), a contradiction. Thus \( S \) is not maximal in \( \mathcal{S}(n) \).

ii) If \( S \) is pseudo-symmetric and \( x \in \Z \setminus S \), then by item ii) of Proposition 3.0.3, we have \( F(S) - x \in S \) or \( x = F(S)/2 \). In both cases we get \( F(S) \in \langle S, x \rangle \), thus \( F(\langle S, x \rangle) < F(S) \) and \( S \) is maximal in \( \mathcal{S}(n) \).

If \( S \) is not pseudo-symmetric, then by item ii) of Proposition 3.0.7, \( \text{PF}(S) \neq \{F(S), F(S)/2\} \). Then \( h(S) > F(S)/2 \) and we can conclude as in i) that \( F(\langle S, h(S) \rangle) = F(S) \) and that \( S \) is not maximal in \( \mathcal{S}(n) \).

We say that a numerical semigroup is irreducible if it can not be expressed as an intersection of two numerical semigroups containing it properly. We are going to show that irreducible numerical semigroups are maximal in the set of numerical semigroups with fixed Frobenius number. First we prove the following Lemma:

Lemma 3.0.11. Let \( S \) be a numerical semigroup other than \( \N \). Then \( S \cup \{F(S)\} \) is again a numerical semigroup.
Proof. We have $0 \in S \subset S \cup \{F(S)\}$. Now, take $a, b \in S \cup \{F(S)\}$. If both $a$ and $b$ are in $S$, then $a + b \in S \subset S \cup \{F(S)\}$. If any of them is $F(S)$, then $a + b \geq F(S)$ and thus $a + b \in S \cup \{F(S)\}$. Lastly, $\mathbb{N} \setminus (S \cup \{F(S)\}) \subset \mathbb{N} \setminus S$. Therefore, $\mathbb{N} \setminus (S \cup \{F(S)\})$ is finite, since $\mathbb{N} \setminus S$ is finite. This proves that $S \cup \{F(S)\}$ is a numerical semigroup.

\[ \Box \]

Theorem 3.0.12. Let $S$ be a numerical semigroup. The following conditions are equivalent.

i) $S$ is irreducible.

ii) $S$ is maximal in $S(F(S))$.

iii) $S$ is maximal in the set of all numerical semigroups that do not contain $F(S)$.

Proof. i)$\Rightarrow$ ii) Let $\bar{S}$ be a numerical semigroup such that $S \subset \bar{S}$ and $F(\bar{S}) = F(S)$. Then $S = (S \cup \{F(S)\}) \cap \bar{S}$. Since $S$ is irreducible, we conclude that $S = \bar{S}$.

ii)$\Rightarrow$ iii) Let $\bar{S}$ be a numerical semigroup such that $S \subset \bar{S}$ and $F(S) \notin \bar{S}$. Then $S \cup \{F(S) + 1, \rightarrow\}$ is a numerical semigroup that contains $S$ with Frobenius number $F(S)$. Hence, $S = \bar{S} \cup \{F(S) + 1, \rightarrow\}$ and so $S = \bar{S}$.

iii)$\Rightarrow$ i) Let $S_1$ and $S_2$ be two numerical semigroups that contain $S$ properly. Then, by hypothesis, $F(S) \in S_1$ and $F(S) \in S_2$. Thus $S \neq S_1 \cap S_2$ and then $S$ is irreducible.

\[ \Box \]

From Proposition 3.0.10 and Theorem 3.0.12 we can deduce the next result.

Corollary 3.0.13. Let $S$ be a numerical semigroup.

i) If $F(S)$ is odd, then $S$ is irreducible if and only if $S$ is symmetric.

ii) If $F(S)$ is even, then $S$ is irreducible if and only if $S$ is pseudo-symmetric.

Therefore, the set of irreducible numerical semigroups covers exactly the symmetric and pseudo-symmetric numerical semigroups.

Example 3.0.14. The number of irreducible numerical semigroups with Frobenius number $n$ is: 1, 1, 1, 2, 1, 3, 2, 3, 6, 2, 8, 6, 7, 15, 7, 20, 11, 18, 20, 36, 14, 44, 35, 45, 37, 83, 36, 70, 101, 106, 174, 77, 246, 182, 227, ... (sequence https://oeis.org/A158206). By Corollary 3.0.13, this sequence can be obtained by intercalating the sequences of Examples 3.0.4 and 3.0.5.
4. DECOMPOSITION OF A NUMERICAL SEMIGROUP AS AN INTERSECTION OF IRREDUCIBLE NUMERICAL SEMIGROUPLS

Before study the decompositions of numerical semigroups as intersection of irreducible numerical semigroups, we will need the concept of special gap of a numerical semigroup. Its definition is motivated by the problem of building unitary extensions of a numerical semigroup. In this part we are following the Chapter 3 of [12] and the papers [10] and [13].

Given a numerical semigroup $S$, we denote by
\[
SG(S) = \{x \in PF(S)| 2x \in S\}.
\]
Its elements are called the \textit{special gaps} of $S$. They are precisely those gaps $x$ of $S$ such that $S \cup \{x\}$ is also a numerical semigroup.

**Proposition 4.0.1.** Let $S$ be a numerical semigroup and let $x \in G(S)$. The following properties are equivalent:

i) $x \in SG(S)$,

ii) $S \cup \{x\}$ is a numerical semigroup.

*Proof.* i)⇒ ii) We have $0 \in S \subset S \cup \{x\}$. Let $a, b \in S \cup \{x\}$. If $a, b \in S$, then $a + b \in S \subset S \cup \{x\}$. If only one of them belongs to $S$, say $a$ and $b = x$, then $x + a \in S \subset S \cup \{x\}$ is a numerical semigroup, by definition of a pseudo-Frobenius number. If $a = b = x$, then $a + b = 2x \in S$, by definition of a special gap. Lastly, $\mathbb{N} \setminus (S \cup \{x\}) \subset \mathbb{N} \setminus S$, then $\mathbb{N} \setminus (S \cup \{x\})$ is finite. Thus, $S \cup \{x\}$ is a numerical semigroup.

ii)⇒ i) Suppose that $S \cup \{x\}$ is a numerical semigroup. Then, we have $x \notin S$ and $x + s \in S$, for all $s \in S^*$, that is, $x \in PF(S)$. Moreover, since $x \neq 0$, $2x \neq x$ and then, $2x \in S$. Hence, $x \in SG(S)$.

*Example 4.0.2.* Let $S = \{0, 7, \rightarrow\}$. Then $S$ is a numerical semigroup with $PF(S) = \{1, 2, 3, 4, 5, 6\}$ and $SG(S) = \{4, 5, 6\}$. Therefore, $\{0, 4, 7, \rightarrow\}$, $\{0, 5, 7, \rightarrow\}$ and $\{0, 6, \rightarrow\}$ are also numerical semigroups.

**Lemma 4.0.3.** Let $S$ and $T$ be two numerical semigroups such that $S \subset T$. Then $S \cup \{\max(T \setminus S)\}$ is a numerical semigroup.

*Proof.* By Proposition [4.0.1], it suffices to prove that $x = \max(T \setminus S) \in SG(S)$. In fact, $x + s \in T$ and $x + s > x$ for all $s \in S^*$. Thus $x + s \in S$ and then $x \in PF(S)$. Moreover, $2x \in T$ and $2x > x$, which implies that $2x \in S$ and $x \in SG(S)$. Hence, $S \cup \{\max(T \setminus S)\}$ is a numerical semigroup.

Given a numerical semigroup $S$, we denote by $\mathcal{O}(S)$ the set of all numerical semigroups that contain $S$. This is called the set of \textit{overseminens} of $S$. Since $\mathbb{N} \setminus S$ is finite, $\mathcal{O}(S)$ is finite.

Given two numerical semigroups $S$ and $T$ with $S \subset T$, we define recursively

i) $S_0 = S$,

ii) $S_{n+1} = S_n \cup \{\max(T \setminus S_n)\}$ if $S_n \neq T$ and $S_n = S_{n+1}$, otherwise.

If the cardinality of $T \setminus S$ is $k$, then

$$S = S_0 \subset S_1 \subset \cdots \subset S_k = T.$$
By using this idea we can construct the set $\mathcal{O}(S)$.

The next example illustrates this construction.

**Example 4.0.4.** Let $S = \langle 5, 7, 9, 11 \rangle$. We have $SG(S) = \{13\}$ and then $S \cup \{13\} = \langle 5, 7, 9, 11, 13 \rangle$ is a semigroup containing $S$. As $SG(S \cup \{13\}) = \{6, 8\}$, from $S \cup \{13\}$ we obtain two new semigroups: $S \cup \{6, 13\}$ and $S \cup \{8, 13\}$. By repeating this process we obtain $\mathcal{O}(S)$, shown in the following figure:

![Figure 1. The oversemigroups of $S = \langle 5, 7, 9, 11 \rangle$.](image)

**Proposition 4.0.5.** Let $S$ be a numerical semigroup and let $\{g_1, \ldots, g_t\} \subset G(S)$. The following conditions are equivalent.

i) $S$ is maximal (with respect to set inclusion) in the set of all numerical semigroups $T$ such that $T \cap \{g_1, \ldots, g_t\} = \emptyset$.

ii) $SG(S) \subset \{g_1, \ldots, g_t\}$.

**Proof.** i)⇒ ii) Let $x \in SG(S)$. By Proposition 4.0.1, $S \cup \{x\}$ is a numerical semigroup containing $S$ properly. Thus if Condition i) holds, then $(S \cup \{x\}) \cap \{g_1, \ldots, g_t\} \neq \emptyset$. Hence $x \in \{g_1, \ldots, g_t\}$.

ii)⇒ i) Suppose that $SG(S) \subset \{g_1, \ldots, g_t\}$ and that $S$ is not maximal in the set of all numerical semigroups $T$ such that $T \cap \{g_1, \ldots, g_t\} = \emptyset$. Then, there exists a numerical semigroup $\bar{S}$ such that $S \subset \bar{S}$ and $\bar{S} \cap \{g_1, \ldots, g_t\} = \emptyset$. By Lemma 4.0.3, we can conclude that $\max(\bar{S} \setminus S) \in SG(S) \subset \{g_1, \ldots, g_t\}$, which contradicts $\bar{S} \cap \{g_1, \ldots, g_t\} = \emptyset$.

As a corollary we have the following characterization of irreducible numerical semigroups.
Corollary 4.0.6. Let $S$ be a numerical semigroup. Then $S$ is irreducible if and only if $SG(S)$ has at most one element.

Proof. In fact, by Theorem 3.0.12, $S$ is irreducible if and only if it is maximal in the set of numerical semigroups $T$ such that $T \cap \{F(S)\} = \emptyset$. By Proposition 4.0.5, this is equivalent to $SG(S) \subseteq \{F(S)\}$, that is, $SG(S)$ has at most one element.

□

Proposition 4.0.7. Every numerical semigroup can be expressed as the intersection of finitely many irreducible numerical semigroups.

Proof. Let $S$ be a numerical semigroup. If $S$ is not irreducible, then there exists $S_1$ and $S_2$ properly containing it and such that $S_1 \cap S_2 = S$. If $S_1$ or $S_2$ are not irreducible, we can write them as an intersection of two other numerical semigroups. We can repeat several times this process, but only a finite number of times, since every numerical semigroup appearing in this procedure belongs to $\mathcal{O}(S)$, which is finite.

□

Now, we will present a procedure to compute a decomposition of a given numerical semigroup into irreducible numerical semigroups. Denote by

\[ \mathcal{I}(S) = \{T \in \mathcal{O}(S)|T \text{ is irreducible}\}. \]

Proposition 4.0.8. Let $S$ be a numerical semigroup and let

\[ \{S_1, \ldots, S_n\} = \text{Minimals}_\subset(\mathcal{I}(S)). \]

Then

\[ S = S_1 \cap \cdots \cap S_n. \]

Proof. Recall that we have a procedure to construct $\mathcal{O}(S)$, based on the computation of the set $SG(S)$. While performing this procedure we can choose those oversemigroups with at most on special gap, which in view of Corollary 4.0.6 are exactly those irreducible oversemigroups of $S$, that is, we can write $S = \bigcap_{T \in \mathcal{I}(S)} T$. Now, we can remove from this intersection those elements that are not minimal with respect to set inclusion, and the resulting semigroup remains unchanged, that is, $S = S_1 \cap \cdots \cap S_n$.

□

The decomposition of Proposition 4.0.8 does not have to be minimal (in the sense of minimal number of irreducibles involved) as the following example shows.

Example 4.0.9. Let $S = \langle 5, 6, 8 \rangle = \{0, 5, 6, 8, 10, \rightarrow\}$. We will compute the set $\text{Minimals}_\subset(\mathcal{I}(S))$. Since $SG(S) = \{7, 9\}$, by Proposition 4.0.1, $S \cup \{7\}$ and $S \cup \{9\}$ are numerical semigroups. As $SG(S \cup \{7\}) = \{9\}$, $S \cup \{7\}$ is irreducible by Corollary 4.0.6, which implies that it belongs to $\text{Minimals}_\subset(\mathcal{I}(S))$. About the semigroup $S \cup \{9\}$, we have $SG(S \cup \{9\}) = \{3, 4, 7\}$ and it is not irreducible. By Proposition 4.0.1, the sets $S \cup \{3, 9\}$, $S \cup \{4, 9\}$ and $S \cup \{7, 9\}$ are also numerical semigroups. Both $S \cup \{3, 9\}$ and $S \cup \{4, 9\}$ are irreducible numerical semigroups, and $S \cup \{7, 9\}$ contains the irreducible numerical semigroup $S \cup \{7\}$. Hence the set

\[ \text{Minimals}_\subset(\mathcal{I}(S)) = \{S \cup \{7\}, S \cup \{3, 9\}, S \cup \{4, 9\}\}. \]

Finally,

\[ S = (S \cup \{7\}) \cap (S \cup \{3, 9\}) \cap (S \cup \{4, 9\}) = (S \cup \{7\}) \cap (S \cup \{4, 9\}). \]
Although the decomposition of Proposition \[4.0.8\] being not minimal, when looking for the least \(n\) such that \(S = S_1 \cap \cdots \cap S_n\), with \(S_1, \ldots, S_n \in \mathcal{I}(S)\), it suffices to search among the decompositions with elements in \(\text{Minimals}_{\subseteq}(\mathcal{I}(S))\).

**Proposition 4.0.10.** Let \(S\) be a numerical semigroup. If \(S = S_1 \cap \cdots \cap S_n\) with \(S_1, \ldots, S_n \in \mathcal{I}(S)\), then there exists \(S'_1, \ldots, S'_n \in \text{Minimals}_{\subseteq}(\mathcal{I}(S))\) such that \(S = S'_1 \cap \cdots \cap S'_n\).

**Proof.** For every \(i \in \{1, \ldots, n\}\), if \(S_i\) does not belong to \(\text{Minimals}_{\subseteq}(\mathcal{I}(S))\), then take \(S'_i \in \text{Minimals}_{\subseteq}(\mathcal{I}(S))\) such that \(S'_i \subset S_i\).

The next proposition gives an indication on which numerical semigroups must appear in a minimal decomposition.

**Proposition 4.0.11.** Let \(S\) be a numerical semigroup and let \(S_1, \ldots, S_n \in \mathcal{O}(S)\). The following conditions are equivalent.

i) \(S = S_1 \cap \cdots \cap S_n\).

ii) For all \(h \in \text{SG}(S)\), there exists \(i \in \{1, \ldots, n\}\) such that \(h \notin S_i\).

**Proof.** i)\(\Rightarrow\) ii) If \(h \in \text{SG}(S)\), then \(h \notin S\) and thus \(h \notin S_i\) for some \(i \in \{1, \ldots, n\}\).

ii)\(\Rightarrow\) i) If \(S \subsetneq S_1 \cap \cdots \cap S_n\), then by Lemma \[4.0.3\] \(h = \max((S_1 \cap \cdots \cap S_n) \setminus S)\) is in \(\text{SG}(S)\) and in all the \(S_i\) in a contradiction with the hypothesis.

Now, we are able to compute \(\text{Minimals}_{\subseteq}(\mathcal{I}(S)) = \{S_1, \ldots, S_n\}\). For every \(i \in \{1, \ldots, n\}\), set \(C(S_i) = \{h \in \text{SG}(S) | h \notin S_i\}\).

By Proposition \[4.0.11\] we know that \(S = S_{i_1} \cap \cdots \cap S_{i_r}\) if and only if \(C(S_{i_1}) \cup \cdots \cup C(S_{i_r}) = \text{SG}(S)\).

From the above results we can obtain a method for computing a decomposition of \(S\) as an intersection of irreducible numerical semigroups with the least possible number of them.

**Algorithm 4.0.12.** Let \(S\) be a non-irreducible semigroup.

1) Compute the set \(\text{SG}(S)\).
2) Set \(I = \emptyset\) and \(C = \{S\}\).
3) For all \(S' \in C\), compute (using Proposition \[4.0.1\]) all the numerical semigroups \(\tilde{S}\) such that \(#(\tilde{S} \setminus S') = 1\). Remove \(S'\) from \(C\). Let \(B\) be the set formed by the numerical semigroups constructed in this way.
4) Remove from \(B\) the numerical semigroups \(S'\) fulfilling that \(\text{SG}(S) \subset S'\).
5) Remove from \(B\) the numerical semigroups \(S'\) such that there exists \(\tilde{S} \in I\) with \(\tilde{S} \subset S'\).
6) Set \(C = \{S' \in B | S'\text{ is not irreducible}\}\).
7) Set \(I = I \cup \{S' \in B | S'\text{ is irreducible}\}\).
8) If \(C = \emptyset\), go to Step 3).
9) For every \(S \in I\), compute \(C(S)\).
10) Choose \(\{S_1, \ldots, S_r\}\) such that \(r\) is minimum fulfilling that \(C(S_1) \cup \cdots \cup C(S_r) = \text{SG}(S)\).
11) Return $S_1, \ldots, S_r$.

The next example will illustrates this method.

**Example 4.0.13.** We take again the numerical semigroup $S = \langle 5, 6, 8 \rangle$ from Example 4.0.9. We know that $SG(S) = \{7, 9\}$. Performing the steps of the Algorithm 4.0.12 we get (in Steps 6 and 7) that $I = \{\langle 5, 6, 7, 8 \rangle, \langle 3, 5 \rangle, \langle 4, 5, 6 \rangle\}$ and $C = \{(5, 6, 8, 9)\}$. Since $C \neq \emptyset$, we go back to Step 3) obtaining that $I = \{\langle 5, 6, 7, 8 \rangle, \langle 3, 5 \rangle, \langle 4, 5, 6 \rangle\}$ and $C = \emptyset$. Step 8) yields

\[
C(\langle 5, 6, 7, 8 \rangle) = \{9\}, C(\langle 3, 5 \rangle) = \{7\} \text{ and } C(\langle 4, 5, 6 \rangle) = \{7\}.
\]

Thus, the minimal decompositions of $S$ are

\[
S = \langle 5, 6, 7, 8 \rangle \cap \langle 3, 5 \rangle \text{ and } S = \langle 5, 6, 7, 8 \rangle \cap \langle 4, 5, 6 \rangle.
\]
References