

Chain Codes and Tits Buildings

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Let \mathbb{F}_q^n be an n -dimensional vector space on the finite field \mathbb{F}_q . For a linear code $C \subset \mathbb{F}_q^n$ of dimension k we define its r -th *minimum weight* by

$$d_r(C) = \min \{ \|D\| : D \subset C, \dim(D) = r \}$$

where $\|D\| = \# \bigcup_{v \in D} \text{Supp}(v)$ with $\text{Supp}(v) = \{i : v_i \neq 0\}$, where $v = (v_1, \dots, v_n)$. We say that a code $C \subset \mathbb{F}_q^n$ is of *chain type* if there is a chain $D_1 \subset D_2 \subset \dots \subset D_k = C$ of sub-codes of C such that $\|D_i\| = d_i(C)$ for all $i \in \{1, \dots, k\}$. For a chain type code the sequence of sub-codes that achieve the minimum weights is not unique. The numeric sequence $(d_1(C), \dots, d_k(C))$ is called *weights hierarchy*.

We denote by $(D_i)_{i=1}^{n-1}$ a maximal chain $D_1 \subset \dots \subset D_{n-1}$ of subspaces of \mathbb{F}_q^n . The set $A_{n-1}(q)$ of all maximal chains has a structure of Tits building. In this case, the *apartments* are simplicial complex of the sequences $(\langle v_{\sigma(1)}, \dots, v_{\sigma(i)} \rangle)_{i=1}^{n-1}$ obtained by fixing a base $\{v_i\}_{i=1}^n$ of \mathbb{F}_q^n , where σ is an element of the Coxeter group \mathbf{S}_n . The subspaces sequences are called *chambers*, and two chambers in a building are said *adjacent* when there exists a transposition $\sigma \in \mathbf{S}_n$ that transforms a chambers in the other. A subset S of the Tits building is *connected* if any two chambers in S can be connected by a sequence of adjacent chambers. To this sequence the chambers and adjacent we give the name of *gallery*.

From now on we will work with the Tits building $\Delta = A_{n-1}(2)$. We denote by $\Delta_2(d_1, \dots, d_{n-1})$ the set all chambers in Δ that achieve a weight hierarchy (d_1, \dots, d_{n-1}) of their codes of codimension 1 (the possible weights are $(1, 2, \dots, n-1)$, $(1, \widehat{2}, 3, 4, \dots, n)$, $(1, 2, \widehat{3}, 4, 5, \dots, n)$, \dots , $(1, 2, 3, \dots, n-2, \widehat{n-1}, n)$ e $(2, 3, \dots, n)$, where $\widehat{(\cdot)}$ means to omit), and find that

$$\bigcup_{(d_1, \dots, d_{n-1})} \Delta_2(d_1, \dots, d_{n-1})$$

have $(2^n - n)$ connected components. The component $\Delta_2(1, 2, 3, \dots, n-1)$ is clearly an apartment of Δ . We also give an algorithm for the construction of some galleries in the component $\Delta_2(2, 3, \dots, n)$. Denoting by $\Delta_2 I$ the set of all chambers the type $(1, 2, \dots, \widehat{m+1}, \dots, n)$ with codes of codimension 1 given by kernel of the functional

$$\varphi(v_1, \dots, v_n) = v_1 + \dots + \widehat{v_{i_1}} + \dots + \widehat{v_{i_m}} + \dots + v_n.$$

The other sets

$$\Delta_2(1, 2, \dots, \widehat{m+1}, \dots, n),$$

$m \in \{1, 2, \dots, n-2\}$, are unions disconnected of the sets of the J -connected chambers of the shape $\Delta_2 I$, with $I = \{i_1, \dots, i_m\}$ and $J = \{1, 2, \dots, \widehat{m}, \dots, n-2\}$.