Branched Crystals and Tensor Products in Category O

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The theory of crystal bases was introduced by Kashiwara in to study the category of integrable representations of the Drinfeld-Jimbo quantization of symmetrizable Kac–Moody Lie algebra. This theory has proved to be a very important development in the combinatorial approach to representation theory. In particular Kashiwara defined the tensor product of crystal bases and showed that it corresponded to the tensor product of representations. Later he defined the abstract notion of a crystal, the tensor product of crystals and showed that the tensor product was commutative and associative.

We approach the problem of describing the tensor product decomposition of representations in the Bernstein–Gelfand–Gelfand category $\mathcal{O}$, where we have plenty of non-integrable representations, e.g., Verma Modules. For the moment we concentrate on developing the sl(2) "brick" of the theory. We define the notion of branched crystals which is adapted to study the indecomposable objects in $\mathcal{O}$. We then define a tensor product rule for these crystals and prove, in a purely combinatorial manner, that the tensor product decomposes in the same way as the corresponding representations. As it should be expected, our definition of the tensor product rule is somewhat more complicated than that of Kashiwara. In order to guarantee the compatibility of the tensor product rule we introduce an extra ingredient, a map $\chi : B_1 \otimes B_2 \to \mathbb{Z}_2$. Roughly speaking, this map measures when the roles of the factors in the tensor product can be interchanged or not.

Our notion of crystals for $\mathcal{O}$, coincides with the usual definition of crystals in the case of finite dimensional representations and the new tensor product rule in this case is also just the usual one in a an appropriate order.