Mixed Tensorial Product of Modules

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Abstract

In this paper we define and construct a wider form of tensorial product of modules with different rings of scalars (which we suppose to be commutative with unity). For this, it is necessary to generalize the concepts of homomorphism and bilinear application for these cases.

Definition 1 *:*

a *Let M be an A-module, let N be a B-module and let $f : M \longrightarrow N$ be a function. If $\sigma : A \longrightarrow B$ is a ring homomorphism, we say that f is a σ-homomorphism of modules if*

$$f(x+y) = f(x) + f(y) \text{ and } f(ax) = \sigma(a)f(x).$$

b *Let M be an A-module, let N be a B-module, let P be an R-module and let $f : M \times N \longrightarrow P$ be an application. If $\sigma : A \longrightarrow R$ and $\mu : B \longrightarrow R$ are ring homomorphisms, we say that f is a (σ, μ)-bilinear application if*

$$f(ax+by, z) = \sigma(a)f(x,z) + \sigma(b)f(y,z) \text{ and}$$
$$f(x, ay+bz) = \mu(a)f(x,y) + \mu(b)f(x,z).$$

Definition 2 *:*
Let M be an A-module, let N be a B-module and let R be a ring together with the ring homomorphisms $\sigma : A \longrightarrow R$ and $\mu : B \longrightarrow R$. Thus, an R-tensorial product of M and N is an R-module T, which we denote by $T = M \otimes_R N$, together with a (σ, μ)-bilinear application $f : M \times N \longrightarrow T$ such that it satisfies the following universal property:

> *For all S-modules X, all ring homomorphisms $\eta : R \longrightarrow S$ and all $(\eta \circ \sigma, \eta \circ \mu)$-bilinear applications $g : M \times N \longrightarrow X$, there exists a unique η-homomorphism $h : T \longrightarrow X$ such that $h \circ f = g$.*

We prove that there exists a unique, up to isomorphism, R-tensorial product of M and N. And if $\mathcal{H}_\sigma(M,N)$ and $\mathcal{B}_{(\sigma,\mu)}(M \times N, P)$ denote the sets of σ-homomorphisms and (σ, μ)-bilinear applications, respectively, where M, N and P are as in definition 1b, then,

$$\mathcal{B}_{(\sigma,\mu)}(M \times N, P) \cong_R \mathcal{H}_\sigma\ (M, \mathcal{H}_\mu\ (N,P)) \cong_R \mathcal{H}\mathrm{om}_R(M \otimes_R N, P).$$