ENDOMORPHISM RINGS OF JACOBIANS

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1. Introduction

The object of this talk is to show some results about the decomposition of abelian varieties with group action. Applying this decomposition we will give some examples of Jacobians varieties with complex multiplication.

2. Group Actions on Abelian Varieties

Let G be a finite group acting on an abelian variety $\mathcal{A}$. This action induces a natural algebra homomorphism

$$\Phi \ : \mathbb{Q}[G] \ \rightarrow \ \mathrm{End}_{\mathbb{Q}}(\mathcal{A})$$

The group algebra $\mathbb{Q}[G]$ admits a direct product decomposition

$$\mathbb{Q}[G] \cong \mathcal{Q}_0 \times \mathcal{Q}_1 \times \mathcal{Q}_2 \times .... \times \mathcal{Q}_r$$

where for each $i = 0, 1, ..., r$, $\mathcal{Q}_i$ is a simple G-invariant $\mathbb{Q}$-algebra.

Also, in a natural form we may write the decomposition of $1 \in \mathbb{Q}[G]$ as follows

$$1 = e_0 + e_1 + e_2 + ... + e_r$$

Put $\mathcal{A}_i = \mathrm{Im}\,\Phi(e_i)$. We have that $\mathcal{A}_i$ is a G-invariant subvariety of $\mathcal{A}$. In this way, it is obtained an isogeny decomposition, called the *isotypical decomposition* of $\mathcal{A}$:

$$\mathcal{A}_0 \times \mathcal{A}_1 \times \mathcal{A}_2 \times .... \times \mathcal{A}_r \ \rightarrow \ \mathcal{A}$$

As the simple $\mathbb{Q}$-algebra $\mathcal{Q}_i$ decomposes into a direct sum of minimal left ideals, we may write:

$$e_i = f_{i_1} + f_{i_2} + ... + f_{i_{k_i}}$$

which gives rise to an isogeny decomposition:

$$\mathcal{B}_{i_1} \times \mathcal{B}_{i_2} \times .... \times \mathcal{B}_{i_k} \ \rightarrow \ \mathcal{A}_i$$

where $\mathcal{B}_{i_j} = \mathrm{Im}\,\Phi(f_{i_j})$. An essential point here is that we may describe the f_{i_j} explicitly in terms of the irreducible rational representations of G, and it can be shown that the varieties $\mathcal{B}_{i_1}$, $\mathcal{B}_{i_2}$, $....\mathcal{B}_{i_k}$ are isogenous to each other, (Carocca-Rodríguez have given an algorithm to find explicit primitive rational idempotents f_{i_j} in [C-R])

In [L-R] Lange and Recillas have recently given the general decomposition for any abelian variety admitting a group action

Theorem 2.1. *Let G be a finite group acting on an abelian variety $\mathcal{A}$. Let $\mathcal{W}_1, \dots, \mathcal{W}_r$ denote the irreducible $\mathbb{Q}$-representations of G. Then there are abelian subvarieties $\mathcal{B}_1, \dots, \mathcal{B}_r$ such that each $\mathcal{B}_j^{n_j}$ is $G-$stable and associated to the representation $\mathcal{W}_j$, and also an isogeny*

$$\mathcal{A} \sim \mathcal{B}_1^{n_1} \times \dots \times \mathcal{B}_r^{n_r} \ . \tag{2.1}$$

Here the integers n_j satisfy $n_j = \frac{\dim V_j}{m_j}$, where V_j is a complex irreducible representation associated to $\mathcal{W}_j$ and $m_j = m_{\mathbb{Q}}(V_j)$ is the Schur index of V_j.

For the case of a $G-$action on the Jacobian variety JY of a curve Y, we can be even more explicit. The following is proved in [C-R] (see also [L-R])

Theorem 2.2. *Given a Galois cover $Y \rightarrow Y_G = Y/G$, consider the associated isotypical decomposition of JY given as*

$$JY \sim JY_G \times \mathcal{B}_2^{\frac{\dim V_2}{m_2}} \times \dots \times \mathcal{B}_r^{\frac{\dim V_r}{m_r}}$$

Let $H \leq G$ and $Y_H = Y/H$ be the quotient curve given by H.

1. *Then the corresponding isotypical decomposition of JY_H is given as follows.*

   $$JY_H \sim JY_G \times \mathcal{B}_2^{\frac{\dim V_2^H}{m_2}} \times \dots \times \mathcal{B}_r^{\frac{\dim V_r^H}{m_r}} \ ,$$

2. *and for any subgroups $H \subseteq N \subseteq G$ the corresponding decomposition of the Prym variety $P(Y_H/Y_N)$ is given as follows.*

   $$P(Y_H/Y_N) \sim \mathcal{B}_2^{s_2} \times \dots \times \mathcal{B}_r^{s_r}$$

   where

   $$s_j = \frac{\dim V_j^H}{m_j} - \frac{\dim V_j^N}{m_j} \ .$$

   and V_j^H is the subspace of V_j fixed under H.

Returning to the decomposition

$$\mathbb{Q}[G] \cong \mathcal{Q}_0 \times \mathcal{Q}_1 \times \mathcal{Q}_2 \times .... \times \mathcal{Q}_r$$

each simple factor $\mathcal{Q}_i$ is isomorphic to $M_{n_i}(\Delta_i)$, where Δ_i is a central simple algebra whose center K_i is an abelian extension of $\mathbb{Q}$ and $n_i = \dfrac{\dim(V_i)}{m(V_i)}$.

Let e_{jj} be the matrix in $M_{n_i}(\Delta_i)$ which has a one in the (j, j) position and zeroes elsewhere. Then $\mathcal{W}_i := M_{n_i}(\Delta_i)e_{11}$ is an irreducible representation of $\mathbb{Q}[G]$, and in fact every irreducible representation of $\mathbb{Q}[G]$ arises as $\mathcal{W}_i$ for exactly one i. We can think of $\mathcal{W}_i$ as a vector space over Δ_i^{op}, in which case

$$\mathcal{Q}_i = \operatorname{End}_{\Delta_i^{op}}(\mathcal{W}_i)$$

Let $H \leq G$ and

$$P_H = \frac{1}{|H|}\sum_{h\in H} h$$

The Hecke algebra $\mathbb{Q}[H\backslash G/H]$ is defined as the subalgebra of $\mathbb{Q}[G]$ generated by $P_H g P_H$ for all $g \in G$.

Since P_H commutes with the action of Δ_i^{op} on $\mathcal{W}_i$, it follows that

$$\mathcal{W}_i^H = P_H \mathcal{W}_i$$

as a sub-Δ_i^{op}-vector space of $\mathcal{W}_i$ and that

$$P_H \mathcal{Q}_i P_H = P_H \operatorname{End}_{\Delta_i^{op}}(\mathcal{W}_i) P_H = \operatorname{End}_{\Delta_i^{op}}(\mathcal{W}_i^H) \cong M_{d_i}(\Delta_i)$$

where $d_i = \dfrac{\dim(V_i^H)}{m(V_i)}$. In particular, if $\dim(U_i^H) = m(U_i)$, then $M_{d_i}(\Delta_i^{op}) = \Delta_i$.

In general, we have a decomposition

$$\mathbb{Q}[H\backslash G/H] \cong \bigoplus_i M_{d_i}(\Delta_i^{op})$$

Now consider JY, the Jacobian variety of a curve Y with G-action and $H \leq G$.

Let

$$\Phi\ : \mathbb{Q}[G]\ \rightarrow\ \operatorname{End}_{\mathbb{Q}}(JY)$$

be the induced homomorphism.
The action of Φ restricts to the natural action

$$\mathbb{Q}[H\backslash G/H] \rightarrow \operatorname{End}_{\mathbb{Q}}(\Phi(P_H)JY)$$

The natural map

$$JY_H \rightarrow JY$$

restricts to an isogeny

$$JY_H \to \Phi(P_H)(JY)$$

Therefore, we have an embedding

$$\mathbb{Q}[H\backslash G/H] \to \mathrm{End}_{\mathbb{Q}}(JY_H)$$

3. JACOBIANS WITH COMPLEX MULTIPLICATION

Let $\mathcal{A}$ be a principally polarized abelian variety. It has long been known which algebras arise as $\mathrm{End}_{\mathbb{Q}}(\mathcal{A})$. If we restrict our attention to Jacobian varieties of curves of a given genus, the question is less well understood. In [El] Ellenberg gives examples of families of curves whose Jacobians are acted on by certain real subfields of cyclotomic fields.

Let K be a totally complex quadratic extension of a totally real field K_0 with $[K_0 : \mathbb{Q}] = e_0$, and $\mathcal{A}$ an abelian variety. We say that $\mathcal{A}$ is of *CM Type* if there is an embedding $K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(\mathcal{A})$ and

$$\dim(\mathcal{A}) = e_0$$

In [Co] R. Coleman formulated the following conjecture:

Conjecture (Coleman)

Fix an integer $g \geq 4$. *Then there are only finitely many complex algebraic curves* Y *of genus* g *such that* JY *is of CM type*

In [DJ-N] De Jong and Noot give examples of curves that contradict this conjecture; these examples are for infinitely many curves of genus 4 and 6 whose Jacobians are of *CM Type.*

4. SOME EXAMPLES OF JACOBIANS WITH COMPLEX MULTIPLICATION

Let F be a skew field of degree d^2 over its center K which is a totally complex quadratic extension of a totally real field K_0 with $|K_0 : \mathbb{Q}| = e_0$ and $\mathcal{A}$ an abelian variety. We say that $\mathcal{A}$ admits complex multiplication by F if there is an embedding $F \hookrightarrow \mathrm{End}_{\mathbb{Q}}(\mathcal{A})$ and

$$\dim(\mathcal{A}) = d^2 e_0 m$$

for some integer $m \geq 1$.

Applying the isotypical decomposition of Jacobians with group actions, we will give some examples of Jacobians with complex multilication.

Let $p,\ q$ be odd prime numbers such that $p\ /\ q-1$. Consider the group

$$G =< x,\ y,\ z\ \ /\ \ x^q, y^p, z^2,\ y^{-1}xyx^{-r},\ zxzx^{-1},\ zyzy^{-1} >$$

where $1 < r < q$ and $r^p \equiv 1 \mod q$.

We have that G is a metacyclic group of order $|G| = 2qp$.
Let $a = x,\ b = y,\ c = y^{-1}x^{-1},\ d = z$ and $e = z$. We have that $G =< a, b, c, d, e >$, $abcde = 1$ and there exists an Galois Covering $Y \to \mathbb{P}^1$ with Galois group G, branched at 5 points with monodromy $a,\ b,\ c,\ d,\ e$.

Let $H_1 =< x >$, $H_2 =< y >$, $H_3 =< z >$ $H_4 =< xz >$ and $H_5 =< yz >$.
For the intermediate covers $Y_i = Y/H_i$ we have that the genera are given by:

$$\begin{aligned}
g(Y) &= (2q-1)(p-1)\\
g(Y_1) &= (p-1)\\
g(Y_2) &= \frac{2(p-1)(q-1)}{p}\\
g(Y_3) &= \frac{(q-1)(p-2)}{2}\\
g(Y_4) &= 0\\
g(Y_5) &= \frac{(q-1)(p-2)}{2p}
\end{aligned}$$

The group G have 6 rational irreducible representations whose dimensions are given by:

$$\begin{array}{lllll}
\dim(W_1) = 1 & ; & K_1 = \mathbb{Q} & ; & m(V_1) = 1\\
\dim(W_2) = 1 & ; & K_2 = \mathbb{Q} & ; & m(V_2) = 1\\
\dim(W_3) = p-1 & ; & |K_3 : \mathbb{Q}| = p-1 & ; & m(V_3) = 1\\
\dim(W_4) = p-1 & ; & |K_4 : \mathbb{Q}| = p-1 & ; & m(V_4) = 1\\
\dim(W_5) = q-1 & ; & |K_5 : \mathbb{Q}| = \dfrac{(q-1)}{p} & ; & m(V_5) = 1\\
\dim(W_6) = q-1 & ; & |K_6 : \mathbb{Q}| = \dfrac{(q-1)}{p} & ; & m(V_6) = 1
\end{array}$$

where $K_i = \mathbb{Q}(\chi_i(g) \ / \ g \in G)$, V_i is a complex irreducible representation associated to $\mathcal{W}_i$ and $m(V_i)$ is the Schur index of V_i.

Now, following formulae given by A. Rojas in [Ro] for the dimensions of the components B_i, we can compute that under our assumptions

$$\begin{aligned} \dim(B_1) &= 0 \\ \dim(B_2) &= 0 \\ \dim(B_3) &= 0 \\ \dim(B_4) &= p-1 \\ \dim(B_5) &= \frac{(q-1)(p-2)}{2p} \\ \dim(B_6) &= \frac{(q-1)(3p-2)}{2p} \end{aligned}$$

The decomposition of JY is as follows:

$$JY \sim B_4 \times (B_5)^p \times (B_6)^p$$

Also we have:

$$\begin{aligned} JY_1 &\sim B_4 \\ JY_5 &\sim B_5 \\ JY_2 &\sim B_5 \times B_6 \end{aligned}$$

and the image by Φ of the corresponding Hecke algebra to $H_5 =< yz >$ is

$$\Phi(\mathbb{Q}[H\backslash G/H]) \cong K(\chi_5) \hookrightarrow \mathrm{End}_{\mathbb{Q}}(JY_5)$$

Note that, if K_i is considering as a quadratic extension of a totally real field $K_{0\,i}$ with $|K_{0\,i} : \mathbb{Q}| = e_{0\,i} = \dfrac{|K_i : \mathbb{Q}|}{2}$, we have that

$$\begin{aligned} \dim(B_5) &= \frac{(q-1)(p-2)}{2p} = e_{05}(p-2) \\ \dim(B_6) &= \frac{(q-1)(3p-2)}{2p} = e_{06}(3p-2) \end{aligned}$$

so JY_5 and B_6 have complex multiplication.

An interesting particular case of above situation is when $p = 3$ and $q = 6k + 1$ for some $k \in \mathbb{N}$.

In this case we have that $\dim(B_5) = \dim(JY_5) = k = e_0$ and JY_5 is of *CM Type*.

References

C-R. A. Carocca and R. Rodríguez, *Jacobians with group actions and rational idempotents*, submitted, Archiv.Math. AG/0305328

Co. R. Coleman, *Torsion points on curves*, In Galois representations and arithmetic algebraic geometry (Y. Ihara ed.), Advanced studies in pure mathematics,**12**, (1987), 235 - 247.

DJ-N. A.J de Jong and R. Noot, *Jacobians with complex multiplication*, In "Arithmetic Algebraic Geometry", Prog. Math. Birkhäuser, Boston, 1991.

El. J. Ellenberg, *Endomorphism Algebras of Jacobians*, Advances in Math,**162**, 243 - 271, (2001).

L-R. H. Lange and S. Recillas, *Abelian varieties with group action*, to appear in J. reine. Ang. Math.

Ro. A. Rojas, *Acciones de grupos en Variedades Jacobianas*, Ph. D. thesis, P. Universidad Católica de Chile, 2002.