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Elliptic systems involving Schrödinger operators with vanishing potentials.

Join work with J. Arratia and D. Pereira

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Abstract

- This talk is concerned with existence of a bounded positive solution of the following elliptic system involving Schrödinger operators

$$\begin{cases} -\Delta u + V_1(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^p & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V_2(x)v = \mu\rho_2(x)(u+1)^q(v+1)^s & \text{in} \quad \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty. \end{cases}$$

where $p, q, r, s \geq 0$, V_i is a nonnegative vanishing potential, and ρ_i has the property (H) introduced by Brezis and Kamin [1].

Abstract

- This talk is concerned with existence of a bounded positive solution of the following elliptic system involving Schrödinger operators

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where $p, q, r, s \geq 0$, V_i is a nonnegative vanishing potential, and ρ_i has the property (H) introduced by Brezis and Kamin [1].

- Furthermore, by imposing some restrictions on the powers p, q, r, s without additional hypotheses of integrability on the weights ρ_i, we obtain a second solution using variational methods. In this context we consider two particular cases: a gradient system and a Hamiltonian system.

Introduction

- More precisely, we will study the following elliptic system involving Schrödinger operators

$$\begin{cases} -\Delta u + V_1(x)u = \lambda \rho_1(x)(u+1)^r(v+1)^p & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V_2(x)v = \mu \rho_2(x)(u+1)^q(v+1)^s & \text{in} \quad \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty. \end{cases} \qquad (\mathbf{S}_{\lambda,\mu})$$

where $\lambda, \mu > 0$, $p, q, r, s \geq 0$, $N \geq 3$.

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where $\lambda, \mu > 0$, $p, q, r, s \geq 0$, $N \geq 3$.

- V_i is a nonnegative vanish potential satisfying

$$\frac{a_i}{1+|x|^\alpha} \leq V_i(x) \leq \frac{A_i}{1+|x|^\alpha} \quad \text{for all} \quad x \in \mathbb{R}^N \qquad (H_V^\alpha)$$

for some constants $\alpha, A_i > 0$ and $a_i \geq 0$, $i = 1, 2$.

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  where $\lambda, \mu > 0$, $p, q, r, s \geq 0$, $N \geq 3$.
- V_i is a nonnegative vanish potential satisfying

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  for some constants $\alpha, A_i > 0$ and $a_i \geq 0$, $i = 1, 2$.
- The weight $\rho_i \in L^\infty(\mathbb{R}^N)$ satisfies

$$0 < \rho_i(x) \leq \frac{k_i}{1+|x|^\beta} \quad \text{in} \quad \mathbb{R}^N, \qquad (H_\rho)$$

  with $\alpha + \beta > 4$ and $k_i > 0$, $i = 1, 2$.

Introduction

- Before to deal the main results about System $(\mathbf{S}_{\lambda,\mu})$, we will give some know facts about the Poisson's equation

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The property (H) introduced by Brezis and Kamin

Let $\rho \in L^\infty_{loc}(\mathbb{R}^N)$, $\rho(x) \geq 0$ and ρ not identically zero. We said that ρ has the property property (H) if there exist a bounded solution of Poisson's equation (1)

- In the celebrated paper [1], Brezis and Kamin proved that the sublinear problem

$$\begin{cases} -\Delta u = \rho(x)u^\alpha & \text{in} \quad \mathbb{R}^N \\ u(x) \to 0 & \text{as} \quad |x| \to \infty, \end{cases} \tag{2}$$

where $N \geq 3$ and $0 < \alpha < 1$, has a bounded positive solution if and only if ρ has the property (H).

Introduction

- An important fact is that the authors prove that Problem (2) has a bounded solution if and only if

$$U(x) := \frac{1}{N(N-2)w_N} \int_{\mathbb{R}^N_+} \frac{\rho(y)}{|x-y|^{N-2}} dy \in L^\infty(\mathbb{R}^N). \tag{3}$$

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- Thus, if we consider potentials like

$$\rho(x) = \frac{1}{1+|x|^\beta} \quad \text{for any } \beta > 2,$$

(3) is satisfied.

Introduction

- Recently Cardoso, Cerda, Pereira and Ubilla [2] they have studied the existence of bounded solution for the *linear Schrödinger equation*

$$-\Delta u + V(x)u = \rho(x) \ \text{ in } \ \mathbb{R}^N, \qquad \textbf{(LS)}$$

giving the next condition of "compatibility" condition between ρ and V.

Introduction

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giving the next condition of "compatibility" condition between ρ and V.

Definition

Suppose that ρ has the property (H) and let U be the bounded solution of $-\Delta U = \rho(x)$ in $\mathbb{R}^N$. We say that V and ρ are compatible if

$$\frac{1}{|x|^{N-2}} * (VU) \in L^\infty(\mathbb{R}^N).$$

Introduction

Lemma

Assume that ρ *satisfies* **(H_ρ)** *and* V *satisfies* **(H_V^α)** *with* $\alpha \in (0,2)$. *Then* V *and* ρ *are compatible*

Theorem

If V *and* ρ *are compatible, then the linear Schrödinger equation* ***(LS)*** *has a bounded positive solution.*

Introduction

- Let us state our first result.

Theorem 1

Assume that $p, q, r, s \geq 0$ and in addition suppose hypotheses (H_ρ) and (H_V^α) hold with $\alpha \in (0, 2]$ and $\alpha + \beta > 4$. Then, there exists $\Lambda > 0$ such that System $(\mathbf{S}_{\lambda,\mu})$

$$\begin{cases} -\Delta u + V_1(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^p & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V_2(x)v = \mu\rho_2(x)(u+1)^q(v+1)^s & \text{in} \quad \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty, \end{cases}$$

has at least one bounded positive solution for every $0 < \lambda, \mu < \Lambda$.

Introduction

- We also establish a converse of **Theorem 1**

Theorem 2

Suppose that $V \in L^\infty(\mathbb{R}^N)$ is a nonnegative potential and the weights ρ_i belong to $L^\infty(\mathbb{R}^N)$ with $\rho_i > 0$, for $i = 1, 2$. Suppose also that $\lambda, \mu > 0$, the powers satisfy $0 < r, s < 1$, $pq < (r-1)(s-1)$ and there exist positive constants b_1, b_2 such that $b_1\rho_1(x) \le \rho_2(x) \le b_2\rho_1(x)$ for every $x \in \mathbb{R}^N$. If System $(\mathbf{S}_{\lambda,\mu})$ admits a bounded positive solution, then, the linear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = \rho_i(x) & \text{in} \quad \mathbb{R}^N \\ u(x) \to 0 & \text{as} \quad |x| \to \infty \end{cases}$$

has a bounded positive solution, for $i = 1, 2$.

Introduction

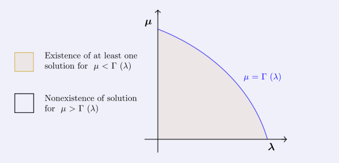
- Note that when $r, s > 1$ we can construct a function that is the border between the region of existence and nonexistence.

Theorem 3

Suppose hypotheses (H_ρ) and (H_V^α) hold with $\alpha \in (0, 2]$ and $\alpha + \beta > 4$. Assume also that $r, s > 1$ and $p, q \geq 0$. Then, there is a positive constant λ^* and a continuous function $\Gamma : (0, \lambda^*) \to [0, \infty)$ such that if $\lambda \in (0, \lambda^*)$ then System $(\mathbf{S}_{\lambda,\mu})$:

i) has at least one bounded positive solution if $0 < \mu < \Gamma(\lambda)$;

ii) has no bounded positive solution if $\mu > \Gamma(\lambda)$.



Introduction

- The second solution will be obtained employing variational methods. The first one case is the following gradient system:

$$\begin{cases} -\Delta u + V(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^{s+1} & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V(x)v = \lambda\rho_2(x)(u+1)^{r+1}(v+1)^s & \text{in} \quad \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty \end{cases} \qquad \mathbf{(GS_\lambda)}$$

with $\rho_1(x) = (r+1)\rho(x)$ and $\rho_2(x) = (s+1)\rho(x)$.

Introduction

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  with $\rho_1(x) = (r+1)\rho(x)$ and $\rho_2(x) = (s+1)\rho(x)$.
- The main result in this context is the following:

Theorem 4

Suppose hypotheses (H_ρ) and (H_V^α) hold with $\alpha \in (0,2]$ and $\alpha + \beta > 4$,

i) If $r, s \geq 0$, then there exists $\lambda^* > 0$ such that the gradient System $(\mathbf{GS}_\lambda)$ possesses at least one bounded positive solution $(u_{1,\lambda}, v_{1,\lambda})$ for all $0 < \lambda < \lambda^*$ while for $r, s > 1$ and $\lambda > \lambda^*$ there are no bounded positive solutions.

ii) If $r, s > 1$ and $r + s < 2^* - 2$, then there exists $0 < \lambda^{**} \leq \lambda^*$ such that the gradient System $(\mathbf{GS}_\lambda)$ possesses a second positive solution of the form $(u_{1,\lambda} + u, v_{1,\lambda} + v)$ for all $0 < \lambda < \lambda^{**}$, where $u, v \in H^1(\mathbb{R}^N)$.

Introduction

- The second particular situation involves the following Hamiltonian system

$$\begin{cases} -\Delta u + V(x)u = \lambda\rho(x)(v+1)^p & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V(x)v = \lambda\rho(x)(u+1)^q & \text{in} \quad \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty \end{cases} \qquad \textbf{(HS}_\lambda\textbf{)}$$

for some conditions in the powers $p, q > 0$.

Introduction

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  for some conditions in the powers $p, q > 0$.
- The main result involving the Hamiltonian system is the following:

Theorem 5

Suppose hypotheses (H_ρ) and (H_V^α) hold with $\alpha \in (0, 2]$. Also, suppose also that $\alpha + \beta > 4$ and $p, q \geq 0$, then

i) There exists $\lambda^* > 0$ such that Hamiltonian System $(\mathbf{HS}_\lambda)$ possesses at least one bounded positive solution $(u_{1,\lambda}, v_{1,\lambda})$ for all $0 < \lambda < \lambda^*$ while for $p, q > 1$ and $\lambda > \lambda^*$ there are no bounded positive solutions.

ii) If $pq < 1$, then Hamiltonian System $(\mathbf{HS}_\lambda)$ possesses at least one bounded positive solution $(u_{1,\lambda}, v_{1,\lambda})$ for all $\lambda > 0$.

iii) If $1 < pq$ and $p, q < 2^* - 1$, then there exists $0 < \lambda^{**} \leq \lambda^*$ such that Hamiltonian System $(\mathbf{HS}_\lambda)$ possesses a second positive solution of the form $(u_{1,\lambda} + u, v_{1,\lambda} + v)$ for all $0 < \lambda < \lambda^{**}$, where $u, v \in H^1(\mathbb{R}^N)$.

Outline

Elliptic system. General case

The gradient system

The Hamiltonian system

Some Nonhomogeneous Elliptic System

Elliptic system. General case

- The proof of existence of the first solution of System $(\mathbf{S}_{\lambda,\mu})$ follows the line of Brezis-Kamin [1], Cardoso-Cerda-Pereira-Ubilla [2] and Montenegro [3], that is to say, we will apply some monotonicity methods.
- First, we will use the lower and upper solution technique developed by Montenegro [3], to obtain a solution of

$$\begin{cases} -\Delta u + V_1(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^p & \text{in} \quad B_R \\ -\Delta v + V_2(x)v = \mu\rho_2(x)(u+1)^q(v+1)^s & \text{in} \quad B_R \\ u = 0 = v & \text{on} \quad \partial B_R \end{cases} \qquad (\mathbf{S}_{\mathbf{R},\lambda,\mu})$$

- More precisely:

Lemma 1.1

Assume that $p, q, r, s \geq 0$. Let U_{V_i} be a bounded positive solution of

$$\begin{cases} -\Delta u + V_i(x)u = \rho_i(x) & \text{in} \quad \mathbb{R}^N \\ u(x) \to 0 & \text{as} \quad |x| \to \infty. \end{cases} \qquad (4)$$

Then there is $\Lambda > 0$, which does not depend on R, such that if $0 < \lambda, \mu < \Lambda$, the System $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$ has a minimal positive solution (u_R, v_R), which is increasing with R and satisfies $u_R \leq U_{V_1}$ and $v_R \leq U_{V_2}$.

Elliptic system. General case

Proof

Existence of bounded solution

- $(\underline{u}, \underline{v}) = (0, 0)$ is a lower solution of $\mathbf{(S_{R,\lambda,\mu})}$ for any $\lambda, \mu \in (0, \infty)$.

Elliptic system. General case

Proof

Existence of bounded solution

- $(\underline{u}, \underline{v}) = (0,0)$ is a lower solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$ for any $\lambda, \mu \in (0,\infty)$.
- Since $U_{V_1}, U_{V_2} \in L^\infty(\mathbb{R}^N)$ there exists $\Lambda > 0$ such that for $0 < \lambda, \mu \leq \Lambda$, the pair $(\overline{u}, \overline{v}) = (U_{V_1}, U_{V_2})$ is an upper solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$, for any $R > 0$. Therefore there is a solution $(\overline{u}_R, \overline{v}_R)$ of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$.

Elliptic system. General case

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Existence of minimal solution

Elliptic system. General case

Proof

Existence of bounded solution

- $(\underline{u}, \underline{v}) = (0, 0)$ is a lower solution of $(\mathbf{S_{R,\lambda,\mu}})$ for any $\lambda, \mu \in (0, \infty)$.
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Existence of minimal solution

(u_R, v_R) **is increasing with** R**.**

Elliptic system. General case

Proof

Existence of bounded solution

- $(\underline{u}, \underline{v}) = (0,0)$ is a lower solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$ for any $\lambda, \mu \in (0,\infty)$.
- Since $U_{V_1}, U_{V_2} \in L^{\infty}(\mathbb{R}^N)$ there exists $\Lambda > 0$ such that for $0 < \lambda, \mu \leq \Lambda$, the pair $(\overline{u}, \overline{v}) = (U_{V_1}, U_{V_2})$ is an upper solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$, for any $R > 0$. Therefore there is a solution $(\overline{u}_R, \overline{v}_R)$ of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$.

Existence of minimal solution

(u_R, v_R) is increasing with R.

- Since (u_R, v_R) is the minimal solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$, it follows that if $R' > R$, then

$$u_R \leq u_{R'} \quad \text{and} \quad v_R \leq v_{R'} \quad \text{in} \quad B_R$$

and

$$u_R \leq U_{V_1} \quad \text{and} \quad v_R \leq U_{V_2}.$$

□

Elliptic system. General case

Proof of Theorem 1

Elliptic system. General case

Proof of Theorem 1

- Let $0 < \lambda, \mu < \Lambda$, $R > 0$ and (u_R, v_R) be the increasing sequence of solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$ given by **Lemma 1.1**. Thus, there exist the limits

$$\lim_{R\to\infty} u_R(x) := u(x) \quad \text{and} \quad \lim_{R\to\infty} v_R(x) := v(x) \quad \text{for every} \quad x \in \mathbb{R}^N.$$

Elliptic system. General case

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- Using Green's representation in the ball B_R, convergence theorems and property (H) it is possible to show that (u, v) is a bounded positive solution of $(\mathbf{S}_{\lambda,\mu})$.

□

Elliptic system. General case

Now, we prove the converse of **Theorem 1**.

Theorem 2

Suppose that $V \in L^\infty(\mathbb{R}^N)$ is a nonnegative potential and the weights ρ_i belong to $L^\infty(\mathbb{R}^N)$ with $\rho_i > 0$, for $i = 1, 2$. Suppose also that $\lambda, \mu > 0$, the powers satisfy $0 < r, s < 1$, $pq < (r-1)(s-1)$ and there exist positive constants b_1, b_2 such that $b_1\rho_1(x) \leq \rho_2(x) \leq b_2\rho_1(x)$ for every $x \in \mathbb{R}^N$. If System $(\mathbf{S}_{\lambda,\mu})$ admits a bounded positive solution, then, the linear Schrödinger equation

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has a bounded positive solution, for $i = 1, 2$.

Elliptic system. General case

Proof of Theorem 2

Elliptic system. General case

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- Let (u, v) be a bounded positive solution of system $(\mathbf{S}_{\lambda,\mu})$.

Elliptic system. General case

Proof of Theorem 2

- Let (u, v) be a bounded positive solution of system $(\mathbf{S}_{\lambda,\mu})$.
- Consider the auxiliary function $w = (u+1)^a(v+1)^b$, with $a = 1 - r$ and $b = 1 - s$ and define $z = \frac{1}{1-\eta} w^{1-\eta}$, where

$$\frac{1}{\eta} = \frac{1}{\frac{b+p}{b}} + \frac{1}{\frac{a+q}{a}}.$$

Elliptic system. General case

Proof of Theorem 2

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$$\frac{1}{\eta} = \frac{1}{\frac{b+p}{b}} + \frac{1}{\frac{a+q}{a}}.$$

- Using that $b_1 \rho(x) \le \rho_2(x)$, $0 < (1-\eta)(a+b) < 1$ and V be a nonnegative potential, we obtain

$$\begin{cases} -\Delta z + V(x) z \ge c_1 \rho_1(x) & \text{in} \quad \mathbb{R}^N \\ z(x) \to 0 & \text{as} \quad |x| \to \infty \end{cases}$$

Elliptic system. General case

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$$\frac{1}{\eta} = \frac{1}{\frac{b+p}{b}} + \frac{1}{\frac{a+q}{a}}.$$

- Using that $b_1 \rho(x) \le \rho_2(x)$, $0 < (1-\eta)(a+b) < 1$ and V be a nonnegative potential, we obtain

$$\begin{cases} -\Delta z + V(x) z \ge c_1 \rho_1(x) & \text{in} \quad \mathbb{R}^N \\ z(x) \to 0 & \text{as} \quad |x| \to \infty \end{cases}$$

- This allows us to demonstrate the existence of a bounded positive solution of the linear Schrödinger equation **(LS)**, when $\rho = \rho_1$.

□

A Sobolev embedding

- Now, we obtain a second solution of System $(\mathbf{S}_{\lambda,\mu})$ using variational methods.

A Sobolev embedding

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- For this purpose, we denote by $H_V^1(\mathbb{R}^N)$ the Sobolev subspace of $H^1(\mathbb{R}^N)$ endowed with the scalar product

$$\langle u, v\rangle_{H_V^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\,dx,$$

and the corresponding norm

$$\|u\|_{H_V^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx\right)^{\frac{1}{2}}.$$

A Sobolev embedding

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- For this purpose, we denote by $H^1_V(\mathbb{R}^N)$ the Sobolev subspace of $H^1(\mathbb{R}^N)$ endowed with the scalar product

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and the corresponding norm

$$\|u\|_{H^1_V(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx\right)^{\frac{1}{2}}.$$

- For $q > 1$, let us denote by $L^q_\rho(\mathbb{R}^N)$ the weighted Lebesgue space

$$L^q_\rho(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and } \; ||u||_{L^q_\rho(\mathbb{R}^N)} < +\infty\right\},$$

where

$$||u||_{L^q_\rho(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \rho(x)|u|^q dx\right)^{\frac{1}{q}}$$

A Sobolev embedding

The following embedding result due to A. Ambrosetti, V. Felli and A. Malchiodi [1].

Lemma 1.2

Suppose hypotheses (H_ρ) and (H_V^α) hold with $\alpha \in (0,2]$. Then the embedding

$$H_V^1(\mathbb{R}^N) \hookrightarrow L_\rho^q(\mathbb{R}^N)$$

is continuous for $2 \leq q \leq 2^*$ and is compact if $2 \leq q < 2^*$.

- The Hilbert space in which we will work is $E = H_V^1(\mathbb{R}^N) \times H_V^1(\mathbb{R}^N)$ endowed with the inner product given by

$$\langle (u,v),(\varphi,\psi)\rangle = \int_{\mathbb{R}^N} \Big(\nabla u \nabla \varphi + \nabla v \nabla \psi + V(x)u\varphi + V(x)v\psi \Big) dx$$

and corresponding norm

$$\|(u,v)\| = \left(\int_{\mathbb{R}^N} \Big(|\nabla u|^2 + V(x)u^2 + |\nabla v|^2 + V(x)v^2 \Big) dx \right)^{1/2}.$$

[1] A. Ambrosetti, V. Felli and A. Malchiodi. *Ground states of Nonlinear Schrödinger Equations with Potentials Vanishing at Infinity.* J. Eur. Math. Soc. 7, 2005, 117-144.

The gradient system

- This section is devoted to the proof of **Theorem 4**, which involves the gradient system $(\mathbf{GS}_\lambda)$.

The gradient system

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- Observe that the *most natural energy functional* $\mathfrak{J}_\lambda : E \to \mathbb{R}$, associated to the gradient system **$(\mathbf{GS}_\lambda)$** is given by

$$\mathfrak{J}_\lambda(u,v) = \frac{1}{2}\|(u,v)\|^2 - \lambda \int_{\mathbb{R}^N} \rho(x) F(u,v)dx,$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$F(u,v) = (u+1)^{r+1}(v+1)^{s+1},$$

where we have assumed that $r, s > 1$ and $r + s < 2^* - 2$.

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  where we have assumed that $r, s > 1$ and $r + s < 2^* - 2$.
- However it is not well defined because the Sobolev embeddings do not work.
- This is mainly due to the behaviour near zero of the nonlinearities and the fact that the $\rho(x)$ coefficient does not necessarily satisfy any integrability hypothesis.

The gradient system

For this reason, in order to show the existence of a second solution for System $(\mathbf{GS}_\lambda)$, we will consider the following auxiliary system

$$\begin{cases} -\Delta u + V(x)u = \lambda\rho(x)f(x,u,v) & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V(x)v = \lambda\rho(x)g(x,u,v) & \text{in} \quad \mathbb{R}^N \end{cases} \qquad (\mathbf{GS}^\lambda_\mathbf{A})$$

where the functions f, g are defined by

$$f(x,u,v) = f_1\big(u_{1,\lambda} + u^+, v_{1,\lambda} + v^+\big) - f_1\big(u_{1,\lambda}, v_{1,\lambda}\big)$$

and

$$g(x,u,v) = f_2\big(u_{1,\lambda} + u^+, v_{1,\lambda} + v^+\big) - f_2\big(u_{1,\lambda}, v_{1,\lambda}\big),$$

where for simplicity we have denoted $u_{1,\lambda}, v_{1,\lambda}$ instead of $u_{1,\lambda}(x), v_{1,\lambda}(x)$, and where

$$f_1(u,v) = \frac{\partial F}{\partial u} \quad \text{and} \quad f_2(u,v) = \frac{\partial F}{\partial v}.$$

The gradient system

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$$\begin{cases} -\Delta u + V(x)u = \lambda \rho(x) f(x,u,v) & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V(x)v = \lambda \rho(x) g(x,u,v) & \text{in} \quad \mathbb{R}^N \end{cases} \qquad (\mathbf{GS}^\lambda_\mathbf{A})$$

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$$f_1(u,v) = \frac{\partial F}{\partial u} \quad \text{and} \quad f_2(u,v) = \frac{\partial F}{\partial v}.$$

- Clearly, if (u,v) is a solution for the auxiliary system $(\mathbf{GS}^\lambda_\mathbf{A})$, then $(u_{1,\lambda} + u, v_{1,\lambda} + v)$ is a solution of System $(\mathbf{GS}_\lambda)$.

The gradient system

- Now, we define $G : \mathbb{R}^{N+2} \to \mathbb{R}$ by

$$G = F\left(u_{1,\lambda}+u^{+}, v_{1,\lambda}+v^{+}\right) - F\left(u_{1,\lambda}, v_{1,\lambda}\right) - \Big(f_1\left(u_{1,\lambda}, v_{1,\lambda}\right)u^{+} + f_2\left(u_{1,\lambda}, v_{1,\lambda}\right)v^{+}\Big).$$

The gradient system

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  $G = F(u_{1,\lambda}+u^+, v_{1,\lambda}+v^+) - F(u_{1,\lambda}, v_{1,\lambda}) - \Big(f_1(u_{1,\lambda}, v_{1,\lambda})u^+ + f_2(u_{1,\lambda}, v_{1,\lambda})v^+\Big).$

- Then

$$\frac{\partial G}{\partial u} = f(x,u,v) \quad \text{and} \quad \frac{\partial G}{\partial v} = g(x,u,v).$$

The gradient system

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- This shows that the auxiliary problem $\mathbf{(GS_A^\lambda)}$ is also a gradient system.

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- Then
$$\frac{\partial G}{\partial u} = f(x,u,v) \quad \text{and} \quad \frac{\partial G}{\partial v} = g(x,u,v).$$
- This shows that the auxiliary problem $(\mathbf{GS_A^\lambda})$ is also a gradient system.
- The energy functional associated to the auxiliary system $(\mathbf{GS_A^\lambda})$ is given by
$$J_\lambda(u,v) = \frac{1}{2}\|(u,v)\|^2 - \lambda \int_{\mathbb{R}^N} \rho(x)G(x,u,v)dx.$$

The gradient system

Lemma 2.1

The functional J_λ associated to $(GS^\lambda_{\mathbf{A}})$ is well defined in E.

Proof.
Using the inequality (I_{tl}) given by:

$$(a+b)^t(c+d)^l - a^t c^l \leq \begin{cases} t(a+b)^{t-1}(c+d)^l b + l(a+b)^t(c+d)^{l-1}d & \text{if } t,l \geq 1 \\ ta^{t-1}(c+d)^l b + l(a+b)^t(c+d)^{l-1}d & \text{if } 0 \leq t < 1, l \geq 1 \\ t(a+b)^{t-1}(c+d)^l b + l(a+b)^t c^{l-1}d & \text{if } t \geq 1, 0 \leq l < 1 \\ ta^{t-1}(c+d)^l b + l(a+b)^t c^{l-1}d & \text{if } 0 < t, l < 1, \end{cases}$$

is possible to show that there exists $C > 0$ such that

$$G(x,u,v) \leq C\left(u^2 + v^2 + (u+v)^{r+s+2}\right) \quad \text{for all } x \in \mathbb{R}^N \text{ and } u,v \geq 0. \tag{5}$$

This fact allows us to easily prove the **Lemma 2.1**

□

The gradient system

The next lemma says that J_λ has the mountain pass geometry.

Lemma 2.2

i) There exist $\lambda_1^* > 0$ and $r_0,\ a > 0$ such that

$$J_\lambda(u, v) \geq a \text{ if } \|(u, v)\| = r_0 \text{ for every } \ \lambda \in (0, \lambda_1^*).$$

ii) There exists $(u, v) \in E$ with

$$\|(u, v)\| > r_0 \text{ and } J_\lambda(u, v) < 0.$$

The gradient system

The nonlinearity G satisfies the following property which is more general than the classical Ambrosetti-Rabinowitz condition:

Lemma 2.3

There exist $\theta \in \left(2, 2^*\right)$ and $C > 0$ such that

$$uf(x,u,v) + vg(x,u,v) - \theta G(x,u,v) \geq -C\left(u^2 + v^2\right)$$

for all $x \in \mathbb{R}^N$ and $u, v > 0$.

Lemma 2.4

There exists $\lambda_2^* > 0$ enough small such that the functional J_λ satisfies the Palais-Smale condition for every $\lambda \in (0, \lambda_2^*)$.

- Finally, from **Lemma 2.1, 2.3** and **Lemma 2.4** there exists $0 < \lambda^{**} \leq \lambda^*$ such that the functional J_λ is well defined and satisfies the conditions of the Mountain Pass Theorem for every $\lambda \in (0, \lambda^{**})$, which allows us to conclude the proof of **Theorem 4** part ii).

The Hamiltonian system

- This section is devoted to the proof of **Theorem 5**, which involves the Hamiltonian system **(HS$_\lambda$)**.

The Hamiltonian system

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- If $pq < 1$, by choosing $\gamma > q$ such that $p\gamma < 1$ is possible to find $M > 1$ large enough such that
$$\begin{cases} M \geq \lambda(M^{\gamma}\|U_{V_2}\|_\infty + 1)^p \\ M^{\gamma} \geq \mu(M\|U_{V_1}\|_\infty + 1)^q, \end{cases}$$
where U_{V_1}, U_{V_2} is a bounded positive solution of (4).

The Hamiltonian system

- This section is devoted to the proof of **Theorem 5**, which involves the Hamiltonian system **(HS$_\lambda$)**.
- If $pq < 1$, by choosing $\gamma > q$ such that $p\gamma < 1$ is possible to find $M > 1$ large enough such that
$$\begin{cases} M \geq \lambda(M^\gamma \|U_{V_2}\|_\infty + 1)^p \\ M^\gamma \geq \mu(M\|U_{V_1}\|_\infty + 1)^q, \end{cases}$$
where U_{V_1}, U_{V_2} is a bounded positive solution of (4).
- Thus, the couple $(MU_{V_1}, M^\gamma U_{V_2})$ is an upper solution of **(S$_{\mathbf{R},\lambda,\mu}$)** for every $R, \lambda, \mu > 0$, and since $(\underline{u}, \underline{v}) = (0, 0)$ is a lower solution of **(S$_{\mathbf{R},\lambda,\mu}$)**, following the argument in **Theorem 1**, we obtain existence of at least one bounded positive solution of Hamiltonian System **(HS$_\lambda$)** for all $\lambda > 0$, which proves **Theorem 5** part ii).

The Hamiltonian system

- Now, we assume that $pq > 1$ and let $(u_{1,\lambda}, v_{1,\lambda})$ be a bounded positive solution of **(HS$_\lambda$)**, given by **Theorem 5** i).

The Hamiltonian system

- Now, we assume that $pq > 1$ and let $(u_{1,\lambda}, v_{1,\lambda})$ be a bounded positive solution of **(HS$_\lambda$)**, given by **Theorem 5** i).
- In a similar way as in a gradient system, to show the existence of a second solution for the System **(HS$_\lambda$)** we will show the existence of at least one solution for the following auxiliary Hamiltonian system

$$\begin{cases} -\Delta u + V(x)u = \lambda\rho(x)f(x,v) & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V(x)v = \lambda\rho(x)g(x,u) & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{$\mathbf{HS_A^\lambda}$}$$

with

$$f(x,v) := h_1(v_{1,\lambda} + v^+) - h_1(v_{1,\lambda}), \quad g(x,u) := h_2(u_{1,\lambda} + u^+) - h_2(u_{1,\lambda})$$

and

$$h_1(v) = \frac{\partial \mathcal{H}}{\partial v}, \quad h_2(u) = \frac{\partial \mathcal{H}}{\partial u},$$

where $\mathcal{H} : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\mathcal{H}(u,v) = \frac{(u+1)^{q+1}}{q+1} + \frac{(v+1)^{p+1}}{p+1}.$$

The Hamiltonian system

- Define $H : \mathbb{R}^{N+2} \to \mathbb{R}$ by

  $$H(x,u,v) = \mathcal{H}(u_{1,\lambda}+u^+, v_{1,\lambda}+v^+) - \mathcal{H}(u_{1,\lambda}, v_{1,\lambda}) - \big(h_1(v_{1,\lambda})v^+ + h_2(u_{1,\lambda})u^+\big).$$

- Then
  $$\frac{\partial H}{\partial v} = f(x,v) \quad \text{and} \quad \frac{\partial H}{\partial u} = g(x,u).$$

- This shows that the auxiliary problem **($\mathbf{HS}_{\mathbf{A}}^{\lambda}$)** is also a Hamiltonian system.
- The energy functional associated to the auxiliary system **($\mathbf{HS}_{\mathbf{A}}^{\lambda}$)** is given by

  $$I_\lambda(u,v) = \int_{\mathbb{R}^N} \Big(\nabla u \nabla v + V(x)uv\Big)dx - \lambda \int_{\mathbb{R}^N} \rho(x)H(x,u,v)dx$$

Lemma 3.1

The functional I_λ associated to **($\mathbf{HS}_{\mathbf{A}}^{\lambda}$)** is well defined in E.

The Hamiltonian system

- To show the existence of a nontrivial solution of the auxiliary problem **(HS_A^λ)**, we will use the technique developed in [2], in which the authors show the existence of at least one positive solution for a Hamiltonian system of the form:

$$\begin{cases} -\Delta u + V(x)u = \rho_1(x)f(v) & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V(x)v = \rho_2(x)g(u) & \text{in} \quad \mathbb{R}^N, \end{cases}$$

[2] E. Toon and P. Ubilla. *Hamiltonian systems of Schrödinger equations with vanishing potentials.* Commun. Contemp. Math, 2020, 2050074.

The Hamiltonian system

- To show the existence of a nontrivial solution of the auxiliary problem $(\mathbf{HS}_{\mathbf{A}}^{\lambda})$, we will use the technique developed in [2], in which the authors show the existence of at least one positive solution for a Hamiltonian system of the form:

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- Since the nonlinearities of our system $(\mathbf{HS}_{\mathbf{A}}^{\lambda})$ are not of separate variables, we cannot directly use their argument. However by taking λ small enough, we can adapt their argument for our case.

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The Hamiltonian system

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- Since the nonlinearities of our system $(\mathbf{HS}^{\lambda}_{\mathbf{A}})$ are not of separate variables, we cannot directly use their argument. However by taking λ small enough, we can adapt their argument for our case.
- Let E be a Hilbert space and $\Phi \in C^1(E, \mathbb{R})$. Recall that $(u_n) \subset E$ is a *Cerami sequence* at the level c ($(C)_c$-sequence for short) if

$$\Phi(u_n) \underset{n\to\infty}{\longrightarrow} c \quad \text{and} \quad (1 + \|u_n\|)\Phi'(u_n) \underset{n\to\infty}{\longrightarrow} 0.$$

[2] E. Toon and P. Ubilla. *Hamiltonian systems of Schrödinger equations with vanishing potentials.* Commun. Contemp. Math, 2020, 2050074.

The Hamiltonian system

In this line, we will use the linking result due to Li and Szulkin [3]:

Lemma 3.2

Let $E = E^+ \oplus E^-$ be a separable Hilbert space with E^- orthogonal to E^+ and $\Phi \in C^1(E, \mathbb{R})$. Suppose

i) $\Phi(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \Psi(z)$, where $\Psi \in C^1(E, \mathbb{R})$ is bounded from below, weakly sequentially lower semicontinuous and Ψ' is weakly sequentially continuous.

ii) There exist $z_0 \in E^+ \setminus \{0\}$, $\alpha > 0$ and $R > r > 0$ such that $\Phi|_{N_r} \geq \alpha$ and $\Phi|_{\partial M_{R,z_0}} \leq 0$.

Then there exists a $(C)_c$-sequence for Φ, with $c \geq \alpha$ and where

$$c := \inf_{h \in \Gamma} \sup_{u \in M_{R,z_0}} \Phi(h(u,1)).$$

[3] G. Li and A. Szulkin. *An asymptotically periodic Schrödinger equation with indefinite linear part.* Commun. Contemp. Math., V. 4, n.4, 2002, 763-776.

The Hamiltonian system

- The following result is a key point in our argument to obtain a second solution to the Hamiltonian system.

Lemma 3.3

Let $(z_n) \subset E$ is a $(C)_c$-sequence of I_λ. Then (z_n) is bounded in E, for sufficiently small values of λ.

- Since $pq > 1$, without loss of generality we will assume that $p > 1$. Then, there exists $C > 0$ such that

$$f(x,v) \leq C(v + v^p) \quad \text{and} \quad g(x,u) \leq \begin{cases} u & \text{if} \quad 0 < q \leq 1 \\ C(u + u^q) & \text{if} \quad q > 1, \end{cases}$$

for all $x \in \mathbb{R}^N$ and every $u, v \geq 0$.

The Hamiltonian system

Proof of Lemma 3.3

- We may assume, by contradiction, that $\|z_n\| \to \infty$ and set

$$w_n = \frac{z_n}{\|z_n\|} = \left(\frac{u_n}{\|z_n\|}, \frac{v_n}{\|z_n\|} \right) := (w_n^1, w_n^2).$$

The Hamiltonian system

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- It follows by Cerami condition that

$$\lim_{n\to\infty} \lambda \int_{\mathbb{R}^N} \rho(x) \left(\frac{f(x, v_n)u_n}{\|z_n\|^2} + \frac{g(x, u_n)v_n}{\|z_n\|^2} \right) dx = 1. \tag{6}$$

The Hamiltonian system

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- Let $0 \le a < b \le +\infty$ and define

$$A_n(a, b) = \{x \in \mathbb{R}^N \; ; \; a \le v_n(x) < b\}.$$

The Hamiltonian system

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- Let $0 \le a < b \le +\infty$ and define

$$A_n(a,b) = \{x \in \mathbb{R}^N \ ; \ a \le v_n(x) < b\}.$$

- There is $a > 0$ small enough such that $f(x,v) \le Cv$ for each $0 \le v \le a$, uniformly in $x \in \mathbb{R}^N$, then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} \int_{A_n(0,a)} \rho(x) \frac{f(x,v_n)u_n}{\|z_n\|^2} dx &\le C \int_{A_n(0,a)} \rho(x) \frac{v_n u_n}{\|z_n\|^2} dx \\ &= C \int_{A_n(0,a)} \rho(x) w_n^1 w_n^2 dx \\ &\le C \|w_n^1\|_{H^1_V(\mathbb{R}^N)} \|w_n^2\|_{H^1_V(\mathbb{R}^N)} \\ &\le C. \end{aligned}$$

The Hamiltonian system

- It follows by Cerami condition that, for n sufficiently large,

$$\int_{A_n(b,+\infty)} \rho(x)dx \to 0, \text{ as } b \to +\infty.$$

The Hamiltonian system

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- Let $t_1 \in \left(\frac{N}{2}, N\right)$ and $s_1 = \dfrac{1}{\frac{1}{2} + \frac{1}{N} - \frac{1}{t_1}}$. For n sufficiently large, we obtain

$$\int_{A_n(b,+\infty)} \rho(x)|w_n^1|^{s_1}dx \le C\left(\int_{A_n(b,+\infty)} \rho(x)dx\right)^{\frac{2^*-s_1}{2^*}}.$$

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$$\int_{A_n(b,+\infty)} \rho(x)|w_n^1|^{s_1} dx \le C \left(\int_{A_n(b,+\infty)} \rho(x)dx \right)^{\frac{2^* - s_1}{2^*}}.$$

- Thus, for n sufficiently large, using generalized Hölder's inequality we have

$$\begin{aligned}
\int_{A_n(b,+\infty)} \rho(x) \frac{f(x,v_n)u_n}{\|z_n\|^2} dx &= \int_{A_n(b,+\infty)} \rho^{\frac{1}{t_1}}(x) \rho^{\frac{1}{s_1}}(x) \rho^{\frac{1}{2^*}}(x) \frac{f(x,v_n)}{v_n} \frac{v_n}{\|z_n\|} \frac{u_n}{\|z_n\|} dx \\
&\le C \left(\int_{A_n(b,+\infty)} \rho(x) \left(\frac{|f(x,v_n)|}{|v_n|} \right)^{t_1} dx \right)^{\frac{1}{t_1}} \\
&\quad \cdot \left(\int_{A_n(b,+\infty)} \rho(x)|w_n^1|^{s_1} dx \right)^{\frac{1}{s_1}} \\
&\le C \left(\int_{A_n(b,+\infty)} \rho(x)|w_n^1|^{s_1} dx \right)^{\frac{1}{s_1}} \to 0, \text{ as } b \to +\infty.
\end{aligned}$$

The Hamiltonian system

- In a similar way it is possible to show that
$$\int_{A_n(a,b)} \rho(x)\,\frac{f(x,v_n)u_n}{\|z_n\|^2}\,dx \le 1 \text{ for } n \text{ sufficiently large.}$$

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- Using a similar argument for the nonlinearity g, we obtain:

$$\int_{\mathbb{R}^N} \rho(x) \left(\frac{f(x, v_n)u_n}{\|z_n\|^2} + \frac{g(x, u_n)v_n}{\|z_n\|^2} \right) dx \leq 2(1 + C) \quad \text{for } n \text{ sufficiently large.}$$

The Hamiltonian system

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$$\int_{\mathbb{R}^N} \rho(x) \left(\frac{f(x, v_n)u_n}{\|z_n\|^2} + \frac{g(x, u_n)v_n}{\|z_n\|^2} \right) dx \leq 2(1 + C) \quad \text{for } n \text{ sufficiently large.}$$

- If we consider $2\big(1 + C\big)\lambda < 1$, this fact contradicts (6). Therefore, (z_n) is bounded in E, for small values of λ, and the lemma is proved.
- Then, up to a subsequence, we may assume that $z_n \rightharpoonup z$ in E.

The Hamiltonian system

- In a similar way it is possible to show that

$$\int_{A_n(a,b)} \rho(x) \frac{f(x,v_n)u_n}{\|z_n\|^2} dx \leq 1 \text{ for } n \text{ sufficiently large.}$$

- Using a similar argument for the nonlinearity g, we obtain:

$$\int_{\mathbb{R}^N} \rho(x) \left(\frac{f(x,v_n)u_n}{\|z_n\|^2} + \frac{g(x,u_n)v_n}{\|z_n\|^2} \right) dx \leq 2(1+C) \quad \text{for } n \text{ sufficiently large.}$$

- If we consider $2\big(1+C\big)\lambda < 1$, this fact contradicts (6). Therefore, (z_n) is bounded in E, for small values of λ, and the lemma is proved.
- Then, up to a subsequence, we may assume that $z_n \rightharpoonup z$ in E.
- Using that z_n is a $(C)_c$-sequence it is possible to show that $z_n \to z$ in E.

The Hamiltonian system

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$$\int_{A_n(a,b)} \rho(x)\frac{f(x,v_n)u_n}{\|z_n\|^2}dx \leq 1 \text{ for } n \text{ sufficiently large.}$$
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- Then, up to a subsequence, we may assume that $z_n \rightharpoonup z$ in E.
- Using that z_n is a $(C)_c$-sequence it is possible to show that $z_n \to z$ in E.
- Therefore, $z = (u,v)$ is a nontrivial solution of problem $(\mathbf{HS_A^\lambda})$ with $I_\lambda(u,v) = c \geq a > 0$. Moreover by maximum principle $u > 0$ and $v > 0$.

The Hamiltonian system

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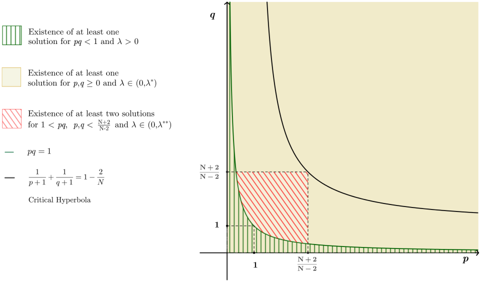
- If we consider $2\big(1+C\big)\lambda < 1$, this fact contradicts (6). Therefore, (z_n) is bounded in E, for small values of λ, and the lemma is proved.
- Then, up to a subsequence, we may assume that $z_n \rightharpoonup z$ in E.
- Using that z_n is a $(C)_c$-sequence it is possible to show that $z_n \to z$ in E.
- Therefore, $z = (u,v)$ is a nontrivial solution of problem $(\mathbf{HS}_{\mathbf{A}}^{\lambda})$ with $I_\lambda(u,v) = c \geq a > 0$. Moreover by maximum principle $u > 0$ and $v > 0$.
- Therefore $(u_{1,\lambda} + u, v_{1,\lambda} + v)$ is a positive solution of System $(\mathbf{HS}_\lambda)$. This concludes the proof of **Theorem 5** part iii).

The Hamiltonian system



This graph illustrates the results obtained for System ***(HS$_\lambda$)****, which may be compared to works about Hamiltonian systems involving the critical hyperbola.*

Some Nonhomogeneous Elliptic System

- Now, we give an important application of **Theorem 1**.

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Theorem 6

Let ρ_i satisfying (H_ρ) with $\beta > 2$ and consider the nonhomogeneous elliptic system

$$\begin{cases} -\Delta z = \rho_1(x) z^r w^p & \text{in} \quad \mathbb{R}^N \\ -\Delta w = \rho_2(x) z^q w^s & \text{in} \quad \mathbb{R}^N, \\ z(x) \to c_1, w(x) \to c_2 & \text{as} \quad |x| \to \infty. \end{cases} \tag{7}$$

Then System (7) has a bounded positive solution $(\bar{z}, \bar{w})$ in the following two cases:

i) $p, q > 0$ and $r, s > 1$, for some $c_1, c_2 \geq 0$ small enough.

ii) $r = s = 0$ and $0 < pq < 1$, for some c large enough, $c_1 = c$, $c_2 = c^\gamma$ and where $\gamma > q$.

- • We would like to mention the paper [3], where the class of type ii) problems was studied with $c = 0$ (see [3, Theorem 5.1]).

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