Optimal design problems for a degenerate operator in Orlicz-Sobolev spaces

Jefferson Abrantes dos Santos
Universidade Federal de Campina Grande

IMECC-UNICAMP
Setembro 2022

Joint work with **Sergio H. Monari Soares (USP/São Carlos)**

Research partially supported by FAPESP 16/16745-3 and CNPq/Brazil

The Problem

In this talk we are concerned with the minimization problem

The Problem

In this talk we are concerned with the minimization problem for the functional

$$J(u) = \int_{\Omega} G(|\nabla u|)\, dx$$

The Problem

In this talk we are concerned with the minimization problem for the functional

$$J(u) = \int_\Omega G(|\nabla u|)\, dx$$

in the class

$$\mathcal{K} = \Big\{ u \in L^1(\Omega)\ :$$

The Problem

In this talk we are concerned with the minimization problem for the functional

$$J(u) = \int_\Omega G(|\nabla u|)\, dx$$

in the class

$$\mathcal{K} = \left\{ u \in L^1(\Omega) \; : \; u = f \text{ on } \partial\Omega \right.$$

The Problem

In this talk we are concerned with the minimization problem for the functional

$$J(u) = \int_\Omega G(|\nabla u|)\, dx$$

in the class

$$\mathcal{K} = \left\{ u \in L^1(\Omega) \,:\, u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) = \alpha \right\}$$

The Problem

In this talk we are concerned with the minimization problem for the functional

$$J(u) = \int_\Omega G(|\nabla u|)\, dx$$

in the class

$$\mathcal{K} = \left\{ u \in L^1(\Omega) \,:\, u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) = \alpha \right\}$$

for a prescribed function

- $f \in L^\infty(\Omega)$

The Problem

In this talk we are concerned with the minimization problem for the functional

$$J(u) = \int_\Omega G(|\nabla u|)\, dx$$

in the class

$$\mathcal{K} = \left\{ u \in L^1(\Omega) \, : \, u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) = \alpha \right\}$$

for a prescribed function

- $f \in L^\infty(\Omega)$ with
- $f \geq 0$;

The Problem

In this talk we are concerned with the minimization problem for the functional

$$J(u) = \int_\Omega G(|\nabla u|)\, dx$$

in the class

$$\mathcal{K} = \left\{ u \in L^1(\Omega) \,:\, u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) = \alpha \right\}$$

for a prescribed function

- $f \in L^\infty(\Omega)$ with
- $f \geq 0$;
- Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$;

The Problem

In this talk we are concerned with the minimization problem for the functional

$$J(u) = \int_\Omega G(|\nabla u|)\, dx$$

in the class

$$\mathcal{K} = \left\{ u \in L^1(\Omega) \;:\; u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) = \alpha \right\}$$

for a prescribed function

- $f \in L^\infty(\Omega)$ with
- $f \geq 0$;
- Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$;
- $\mathcal{L}^N(E)$ denotes the N dimensional Lebesgue measure of the set E in $\mathbb{R}^N$.

The starting point of the study of this problem is the seminal work by **Aguilera, Alt and Caffarelli**[1] for the case $G(t) = t^2$.

[1]N. Aguilera, H. Alt, L. Caffarelli, SIAM J.Control Optim., 1986.

[2]J. Fernández Bonder, S. Martínez, N. Wolanski, J. Differential Equations, 2006.

[3]K. Oliveira and E.V. Teixeira, Differential Integral Equations, 2006.

[4]S. Martínez, J. Math. Anal. Appl., 2008.

The starting point of the study of this problem is the seminal work by **Aguilera, Alt and Caffarelli**[1] for the case $G(t) = t^2$.

The case $G(t) = t^p$, with $1 < p < \infty$, was considered by **Bonder, Martínez and Wolanski.**[2] and by **Oliveira and Teixeira**[3].

[1] N. Aguilera, H. Alt, L. Caffarelli, SIAM J.Control Optim., 1986.
[2] J. Fernández Bonder, S. Martínez, N. Wolanski, J. Differential Equations, 2006.
[3] K. Oliveira and E.V. Teixeira, Differential Integral Equations, 2006.
[4] S. Martínez, J. Math. Anal. Appl., 2008.

The starting point of the study of this problem is the seminal work by **Aguilera, Alt and Caffarelli**[1] for the case $G(t) = t^2$.

The case $G(t) = t^p$, with $1 < p < \infty$, was considered by **Bonder, Martínez and Wolanski.**[2] and by **Oliveira and Teixeira**[3].

Martínez[4] studied this problem under the following natural condition that allows a different behavior of $G(t)$ when t is close to **zero** or **infinity**:

$$(1) \qquad 0 < \delta_0 \leq \frac{tg'(t)}{g(t)} \leq g_0, \quad \forall\, t > 0,$$

where $g(t) = G'(t)$.

[1] N. Aguilera, H. Alt, L. Caffarelli, SIAM J.Control Optim., 1986.
[2] J. Fernández Bonder, S. Martínez, N. Wolanski, J. Differential Equations, 2006.
[3] K. Oliveira and E.V. Teixeira, Differential Integral Equations, 2006.
[4] S. Martínez, J. Math. Anal. Appl., 2008.

Condition (1) was used by **Liberman**[5]

[5] G.M. Lieberman, Comm. Partial Differential Equations 1991.

Condition (1) was used by **Liberman**[5] and its meaning is that the Euler-Lagrange equation associated with

$$\mathcal{J}(u) = \int_\Omega G(|\nabla u|)\, dx$$

is uniformly elliptic

[5] G.M. Lieberman, Comm. Partial Differential Equations 1991.

Condition (1) was used by **Liberman**[5] and its meaning is that the Euler-Lagrange equation associated with

$$\mathcal{J}(u) = \int_{\Omega} G(|\nabla u|)\, dx$$

is uniformly elliptic if and only if (1) (Lieberman's condition) holds.

[5] G.M. Lieberman, Comm. Partial Differential Equations 1991.

Rossi and Teixeira[6] considered the free boundary optimization problem

(2)

$$\min\left\{\int_\Omega |\nabla u|^p dx : u \in W^{1,p}(\Omega), u = f \text{ on } \partial\Omega\right.$$

[6] J. D. Rossi and E. V. Teixeira, Trans. Amer. Math. Soc. 2012

Rossi and Teixeira[6] considered the free boundary optimization problem

(2)

$$\min\left\{\int_\Omega |\nabla u|^p dx : u \in W^{1,p}(\Omega), u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) \leq \alpha\right\}$$

[6] J. D. Rossi and E. V. Teixeira, Trans. Amer. Math. Soc. 2012

Rossi and Teixeira[6] considered the free boundary optimization problem

(2)

$$\min\left\{\int_\Omega |\nabla u|^p dx : u \in W^{1,p}(\Omega), u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) \leq \alpha\right\}$$

and studied the asymptotic behavior as $p \to \infty$

[6] J. D. Rossi and E. V. Teixeira, Trans. Amer. Math. Soc. 2012

Rossi and Teixeira[6] considered the free boundary optimization problem

(2)

$$\min\left\{\int_\Omega |\nabla u|^p dx : u \in W^{1,p}(\Omega), u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) \leq \alpha\right\}$$

and studied the asymptotic behavior as $p \to \infty$ to find a limiting free boundary problem

[6] J. D. Rossi and E. V. Teixeira, Trans. Amer. Math. Soc. 2012

Rossi and Teixeira[6] considered the free boundary optimization problem

(2)

$$\min\left\{\int_\Omega |\nabla u|^p dx : u \in W^{1,p}(\Omega), u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) \le \alpha\right\}$$

and studied the asymptotic behavior as $p \to \infty$ to find a limiting free boundary problem given by the infinity Laplacian operator

$$\Delta_\infty u = DuD^2uDu$$

[6] J. D. Rossi and E. V. Teixeira, Trans. Amer. Math. Soc. 2012

Our problem

In the present work we are interested in a degenerate case.

Our problem

In the present work we are interested in a degenerate case. We consider the minimization problem

$$\min\{J(u) : |\nabla u| \in K_{\Phi}(\Omega), u = f \text{ on } \partial\Omega,$$

Our problem

In the present work we are interested in a degenerate case. We consider the minimization problem

$$\min\{J(u) : |\nabla u| \in K_{\Phi}(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

Our problem

In the present work we are interested in a degenerate case. We consider the minimization problem

$$\min\{J(u) : |\nabla u| \in K_{\Phi}(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

for a prescribed

- $f \in C(\overline{\Omega})$;

Our problem

In the present work we are interested in a degenerate case. We consider the minimization problem

$$\min\{J(u) : |\nabla u| \in K_{\Phi}(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

for a prescribed

- $f \in C(\overline{\Omega})$;
- $|\nabla f| \in K_{\Phi}(\Omega) \Leftrightarrow \int_{\Omega} \Phi(|\nabla f|) < +\infty$;

Our problem

In the present work we are interested in a degenerate case. We consider the minimization problem

$$\min\{J(u) : |\nabla u| \in K_\Phi(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

for a prescribed

- $f \in C(\overline{\Omega})$;
- $|\nabla f| \in K_\Phi(\Omega) \Leftrightarrow \int_\Omega \Phi(|\nabla f|) < +\infty$;
- $f \geq 0$

Our problem

In the present work we are interested in a degenerate case. We consider the minimization problem

$$\min\{J(u) : |\nabla u| \in K_{\Phi}(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

for a prescribed

- $f \in C(\overline{\Omega})$;
- $|\nabla f| \in K_{\Phi}(\Omega) \Leftrightarrow \int_{\Omega} \Phi(|\nabla f|) < +\infty$;
- $f \geq 0$

and the functional J is given by

$$J(u) = \int_{\Omega} \Phi(|\nabla u|)\, dx, \ |\nabla u| \in K_{\Phi}(\Omega),$$

Our problem

In the present work we are interested in a degenerate case. We consider the minimization problem

$$\min\{J(u) : |\nabla u| \in K_\Phi(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

for a prescribed

- $f \in C(\overline{\Omega})$;
- $|\nabla f| \in K_\Phi(\Omega) \Leftrightarrow \int_\Omega \Phi(|\nabla f|) < +\infty$;
- $f \geq 0$

and the functional J is given by

$$J(u) = \int_\Omega \Phi(|\nabla u|)\, dx, \ |\nabla u| \in K_\Phi(\Omega),$$

where

- $\Phi(t) = \exp(t^2) - 1$

Our problem

In the present work we are interested in a degenerate case. We consider the minimization problem

$$\min\{J(u) : |\nabla u| \in K_\Phi(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

for a prescribed

- $f \in C(\overline{\Omega})$;
- $|\nabla f| \in K_\Phi(\Omega) \Leftrightarrow \int_\Omega \Phi(|\nabla f|) < +\infty$;
- $f \geq 0$

and the functional J is given by

$$J(u) = \int_\Omega \Phi(|\nabla u|)\, dx, \ |\nabla u| \in K_\Phi(\Omega),$$

where

- $\Phi(t) = \exp(t^2) - 1$
- $K_\Phi(\Omega)$ is the Orlicz class.

The function

$$\Phi(t) = \exp(t^2) - 1$$

The function

$$\Phi(t) = \exp(t^2) - 1$$

satisfies

$$1 \leq \frac{\phi'(t)t}{\phi(t)}$$

The function

$$\Phi(t) = \exp(t^2) - 1$$

satisfies

$$1 \leq \frac{\phi'(t)t}{\phi(t)} \quad \text{and} \quad \lim_{t \to +\infty} \frac{\phi'(t)t}{\phi(t)} = \infty,$$

The function

$$\Phi(t) = \exp(t^2) - 1$$

satisfies

$$1 \leq \frac{\phi'(t)t}{\phi(t)} \quad \text{and} \quad \lim_{t \to +\infty} \frac{\phi'(t)t}{\phi(t)} = \infty,$$

where $\phi(t) = \Phi'(t)$,

The function

$$\Phi(t) = \exp(t^2) - 1$$

satisfies

$$1 \leq \frac{\phi'(t)t}{\phi(t)} \quad \text{and} \quad \lim_{t \to +\infty} \frac{\phi'(t)t}{\phi(t)} = \infty,$$

where $\phi(t) = \Phi'(t)$, that is $\delta_0 = 1$

The function

$$\Phi(t) = \exp(t^2) - 1$$

satisfies

$$1 \leq \frac{\phi'(t)t}{\phi(t)} \quad \text{and} \quad \lim_{t\to+\infty} \frac{\phi'(t)t}{\phi(t)} = \infty,$$

where $\phi(t) = \Phi'(t)$, that is $\delta_0 = 1$ and $g_0 = \infty$ in condition (1).

The function

$$\Phi(t) = \exp(t^2) - 1$$

satisfies

$$1 \leq \frac{\phi'(t)t}{\phi(t)} \quad \text{and} \quad \lim_{t \to +\infty} \frac{\phi'(t)t}{\phi(t)} = \infty,$$

where $\phi(t) = \Phi'(t)$, that is $\delta_0 = 1$ and $g_0 = \infty$ in condition (1). So, in this case observe that the function Φ does not assume the Liberman condition.

Our problem

Since

$$\Phi \text{ does not satisfy } \Delta_2\text{-condition}^7,$$

[7] A satisfies the Δ_2-condition if there is a constant $k > 0$ such that $A(2t) \leq kA(t)$ for all $t \geq 0$.

Our problem

Since

$$\Phi \text{ does not satisfy } \Delta_2\text{-condition}^7,$$

the Banach space

$$W^{1,\Phi}(\Omega) \text{ is neither separable nor reflexive,}$$

[7] A satisfies the Δ_2-condition if there is a constant $k > 0$ such that $A(2t) \leq kA(t)$ for all $t \geq 0$.

Our problem

Since

$$\Phi \text{ does not satisfy } \Delta_2\text{-condition}^7,$$

the Banach space

$$W^{1,\Phi}(\Omega) \text{ is neither separable nor reflexive,}$$

as a result, the use of minimizing sequences to find solutions to the minimization problem breaks down.

---

[7] A satisfies the Δ_2-condition if there is a constant $k > 0$ such that $A(2t) \leq kA(t)$ for all $t \geq 0$.

Our problem

Since

$$\Phi \text{ does not satisfy } \Delta_2\text{-condition}^{7},$$

the Banach space

$$W^{1,\Phi}(\Omega) \text{ is neither separable nor reflexive},$$

as a result, the use of minimizing sequences to find solutions to the minimization problem breaks down.
To overcome this difficulty and **"others"**, for each $k \in \mathbb{N}$, we consider the truncated function G_k defined for $t \in \mathbb{R}$ by

$$G_k(t) = \sum_{n=1}^{k} \frac{|t|^{2n}}{n!}, \quad t \in \mathbb{R}.$$

[7] A satisfies the Δ_2-condition if there is a constant $k > 0$ such that $A(2t) \leq kA(t)$ for all $t \geq 0$.

Let $g_k(t) = G'_k(t)$, $t \in \mathbb{R}$.

Let $g_k(t) = G_k'(t)$, $t \in \mathbb{R}$. The function g_k satisfies

(3) $$\delta_0 \le \frac{t g_k'(t)}{g_k(t)} \le g_0, \ \ t \ge 0,$$

for $\delta_0 = 1$

Let $g_k(t) = G_k'(t)$, $t \in \mathbb{R}$. The function g_k satisfies

(3) $$\delta_0 \leq \frac{t g_k'(t)}{g_k(t)} \leq g_0, \ \ t \geq 0,$$

for $\delta_0 = 1$and $g_0 = 2k - 1$.

Let $g_k(t) = G_k'(t)$, $t \in \mathbb{R}$. The function g_k satisfies

(3) $$\delta_0 \leq \frac{t g_k'(t)}{g_k(t)} \leq g_0, \;\; t \geq 0,$$

for $\delta_0 = 1$and $g_0 = 2k - 1$.

Since

$$f \in W^{1,G_k}(\Omega) \cap C(\overline{\Omega}) \text{ for sufficiently large } k,$$ [8]

[8] because $f \in W^{1,\Phi}(\Omega)$, with $|\nabla f| \in K_\Phi(\Omega)$, and the immersion of $W^{1,\Phi}(\Omega)$ in $W^{1,G_k}(\Omega)$ is continuous (for every k) and in $C(\overline{\Omega})$ (for k sufficiently large)

Since

$$f \in W^{1,G_k}(\Omega) \cap C(\overline{\Omega}) \text{ for sufficiently large } k,^{8}$$

the result of Martínez implies the existence of a minimizer u_k

[8]because $f \in W^{1,\Phi}(\Omega)$, with $|\nabla f| \in K_\Phi(\Omega)$, and the immersion of $W^{1,\Phi}(\Omega)$ in $W^{1,G_k}(\Omega)$ is continuous (for every k) and in $C(\overline{\Omega})$ (for k sufficiently large)

Since

$$f \in W^{1,G_k}(\Omega) \cap C(\overline{\Omega}) \text{ for sufficiently large } k,$$ [8]

the result of Martínez implies the existence of a minimizer u_k to the minimization problem

$$\text{(4)} \qquad \min\left\{ \int_\Omega G_k(|\nabla u|)\, dx : u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha \right\}$$

[8] because $f \in W^{1,\Phi}(\Omega)$, with $|\nabla f| \in K_\Phi(\Omega)$, and the immersion of $W^{1,\Phi}(\Omega)$ in $W^{1,G_k}(\Omega)$ is continuous (for every k) and in $C(\overline{\Omega})$ (for k sufficiently large)

Existence

Theorem (1)

Let

$$u_k \in W^{1,G_k}(\Omega) \text{ be a minimizer of (4).}$$

Existence

Theorem (1)

Let

$$u_k \in W^{1,G_k}(\Omega) \text{ be a minimizer of (4).}$$

Then, there is a subsequence (not renamed) such that

$$u_k \to u_\Phi, \text{ as } k \to \infty, \text{ uniformly on } \overline{\Omega},$$

Existence

Theorem (1)

Let

$$u_k \in W^{1,G_k}(\Omega) \text{ be a minimizer of (4).}$$

Then, there is a subsequence (not renamed) such that

$$u_k \to u_\Phi, \text{ as } k \to \infty, \text{ uniformly on } \overline{\Omega},$$

where

$$u_\Phi \in W^{1,\Phi}(\Omega) \text{ is a solution}$$

Existence

Theorem (1)

Let

$$u_k \in W^{1,G_k}(\Omega) \text{ be a minimizer of (4).}$$

Then, there is a subsequence (not renamed) such that

$$u_k \to u_\Phi, \text{ as } k \to \infty, \text{ uniformly on } \overline{\Omega},$$

where

$$u_\Phi \in W^{1,\Phi}(\Omega) \text{ is a solution}$$

to problem

$$\min\left\{J(u) \,:\, u = f \text{ on } \partial\Omega, |\nabla u| \in K_\Phi(\Omega), \mathcal{L}^N(\{u > 0\}) = \alpha\right\}, \tag{5}$$

Existence

Theorem (1)

Let

$$u_k \in W^{1,G_k}(\Omega) \text{ be a minimizer of (4).}$$

Then, there is a subsequence (not renamed) such that

$$u_k \to u_\Phi, \text{ as } k \to \infty, \text{ uniformly on } \overline{\Omega},$$

where

$$u_\Phi \in W^{1,\Phi}(\Omega) \text{ is a solution}$$

to problem

$$(5) \qquad \min \left\{ J(u) \,:\, u = f \text{ on } \partial\Omega, |\nabla u| \in K_\Phi(\Omega), \mathcal{L}^N(\{u > 0\}) = \alpha \right\},$$

where

$$J(u) = \int_\Omega \Phi(|\nabla u|)\, dx.$$

Theorem (1)

Moreover, the function

$$u_\Phi \textit{ is a viscosity sense,}$$

Theorem (1)

Moreover, the function

$$u_\Phi \text{ is a viscosity sense,}$$

to the equation

$$\Delta u_\Phi + 2\Delta_\infty u_\Phi = 0 \textit{ in } \{u_\Phi > 0\}.$$

Theorem (1)

Moreover, the function

$$u_\Phi \text{ is a viscosity sense,}$$

to the equation

$$\Delta u_\Phi + 2\Delta_\infty u_\Phi = 0 \textit{ in } \{u_\Phi > 0\}.$$

Some fine properties of u_Φ

Theorem (2)

Let

$$u_\Phi \in W^{1,\Phi}(\Omega) \text{ be the solution to (5)}$$

Some fine properties of u_Φ

Theorem (2)

Let

$$u_\Phi \in W^{1,\Phi}(\Omega) \text{ be the solution to (5)}$$

given by Theorem 1,

$$D \subset\subset \Omega \text{ be any set}$$

Some fine properties of u_Φ

Theorem (2)

Let

$$u_\Phi \in W^{1,\Phi}(\Omega) \text{ be the solution to (5)}$$

given by Theorem 1,

$$D \subset\subset \Omega \text{ be any set}$$

and

$B_r(x) \subset D \cap \{u_\Phi > 0\}$ *be a ball touching the free boundary* $\partial\{u_\Phi > 0\}$

for $r > 0$ *is sufficiently small.*

Theorem (2)

Then,

(i) ***Non-degeneracy.*** *There are positive constants c and C depending only on N and f*

Theorem (2)

Then,

(i) ***Non-degeneracy.*** *There are positive constants c and C depending only on N and f such that*

$$cr \leq u_{\Phi}(x) \leq Cr.$$

Theorem (2)

Then,

(i) ***Non-degeneracy.*** *There are positive constants*

$$c \text{ and } C \text{ depending only on } N \text{ and } f$$

such that

$$cr \leq u_{\Phi}(x) \leq Cr.$$

(ii) ***Harnack inequality in a touching ball.*** *There is a positive constant*

$$C \text{ depending only on } r \text{ and } M := \sup_{\overline{\Omega}} f$$

Theorem (2)

Then,

(*i*) ***Non-degeneracy.*** *There are positive constants*

c *and* C *depending only on* N *and* f

such that

$$cr \leq u_\Phi(x) \leq Cr.$$

(*ii*) ***Harnack inequality in a touching ball.*** *There is a positive constant*

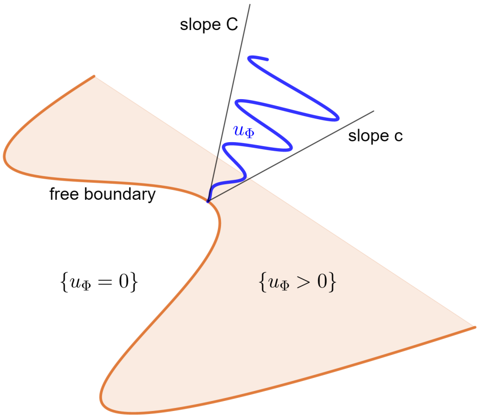
C *depending only on* r *and* $M := \sup_{\overline{\Omega}} f$

such that

$$\sup_{B_{\sigma r}(x)} u_\Phi \leq C \inf_{B_{\sigma r}(x)} u_\Phi,$$

for any $\sigma \in (0,1)$.

Figure: A pictorial figure of the non-degeneracy



In order to prove Theorem 2,

[9] G.M Lieberman, Comm. Partial Differential Equations, 1991.

[10] J. Abrantes Santos and S.H. Monari Soares, Rev. Mat. Iberoam., 2020.

In order to prove Theorem 2, we use refined Local maximum principle

[9] G.M Lieberman, Comm. Partial Differential Equations, 1991.

[10] J. Abrantes Santos and S.H. Monari Soares, Rev. Mat. Iberoam., 2020.

In order to prove Theorem 2, we use refined Local maximum principle and Harnack's inequality for G_k-subharmonic and harmonic functions, respectively,

[9] G.M Lieberman, Comm. Partial Differential Equations, 1991.

[10] J. Abrantes Santos and S.H. Monari Soares, Rev. Mat. Iberoam., 2020.

In order to prove Theorem 2, we use refined Local maximum principle and Harnack's inequality for G_k-subharmonic and harmonic functions, respectively, developed by Lieberman [9],

[9] G.M Lieberman, Comm. Partial Differential Equations, 1991.

[10] J. Abrantes Santos and S.H. Monari Soares, Rev. Mat. Iberoam., 2020.

In order to prove Theorem 2, we use refined Local maximum principle and Harnack's inequality for G_k-subharmonic and harmonic functions, respectively, developed by Lieberman [9], where G_k is given by

$$G_k(t) = \sum_{n=1}^{k} \frac{|t|^{2n}}{n!}, \quad t \in \mathbb{R}.$$

[9] G.M Lieberman, Comm. Partial Differential Equations, 1991.

[10] J. Abrantes Santos and S.H. Monari Soares, Rev. Mat. Iberoam., 2020.

In order to prove Theorem 2, we use refined Local maximum principle and Harnack's inequality for G_k-subharmonic and harmonic functions, respectively, developed by Lieberman [9], where G_k is given by

$$G_k(t) = \sum_{n=1}^{k} \frac{|t|^{2n}}{n!}, \quad t \in \mathbb{R}.$$

Such refinement was proved in a joint work with Monari Soares [10]

[9] G.M Lieberman, Comm. Partial Differential Equations, 1991.

[10] J. Abrantes Santos and S.H. Monari Soares, Rev. Mat. Iberoam., 2020.

Theorem (Local maximum principle)

Set $k \geq 2$ *and* $R \in (0, 1]$.

Theorem (Local maximum principle)

Set $k \geq 2$ *and* $R \in (0,1]$*. Suppose that* $u \in L^\infty(B_R) \cap W^{1,G_k}(B_R)$

Theorem (Local maximum principle)

Set $k \geq 2$ and $R \in (0, 1]$. Suppose that $u \in L^\infty(B_R) \cap W^{1,G_k}(B_R)$ satisfies

$$-\mathrm{div}\left(g_k(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) \leq 0 \text{ in } B_R,$$

Theorem (Local maximum principle)

Set $k \geq 2$ *and* $R \in (0,1]$. *Suppose that* $u \in L^\infty(B_R) \cap W^{1,G_k}(B_R)$ *satisfies*

$$-\mathrm{div}\left(g_k(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) \leq 0 \text{ in } B_R,$$

with $0 \leq u \leq M$ *in* B_R,

Theorem (Local maximum principle)

Set $k \geq 2$ and $R \in (0, 1]$. Suppose that $u \in L^\infty(B_R) \cap W^{1,G_k}(B_R)$ satisfies

$$-\mathrm{div}\left(g_k(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) \leq 0 \text{ in } B_R,$$

with $0 \leq u \leq M$ in B_R, then for any $s \geq (g_0 + 1)^2 N =: \theta N$ and $\sigma \in (0, 1)$,

Theorem (Local maximum principle)

Set $k \geq 2$ and $R \in (0,1]$. Suppose that $u \in L^\infty(B_R) \cap W^{1,G_k}(B_R)$ satisfies

$$-\mathrm{div}\left(g_k(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) \leq 0 \text{ in } B_R,$$

with $0 \leq u \leq M$ in B_R, then for any $s \geq (g_0+1)^2 N =: \theta N$ and $\sigma \in (0,1)$, there is a constant $c = c(N, g_0, k, R, M, \sigma)$ such that

$$\sup_{B_{\sigma R}} u^2 \leq c\left(\int_{B_R} u^s \, dx\right)^{\frac{1}{s}}.$$

Theorem (Local maximum principle)

Set $k \geq 2$ and $R \in (0,1]$. Suppose that $u \in L^{\infty}(B_R) \cap W^{1,G_k}(B_R)$ satisfies

$$-\mathrm{div}\left(g_k(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) \leq 0 \text{ in } B_R,$$

with $0 \leq u \leq M$ in B_R, then for any $s \geq (g_0+1)^2 N =: \theta N$ and $\sigma \in (0,1)$, there is a constant $c = c(N, g_0, k, R, M, \sigma)$ such that

$$\sup_{B_{\sigma R}} u^2 \leq c\left(\int_{B_R} u^s \, dx\right)^{\frac{1}{s}}.$$

Moreover, there is $k_0 \in \mathbb{N}$ depending only on N, σ and R

Theorem (Local maximum principle)

Set $k \geq 2$ and $R \in (0,1]$. Suppose that $u \in L^\infty(B_R) \cap W^{1,G_k}(B_R)$ satisfies

$$-\operatorname{div}\left(g_k(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) \leq 0 \text{ in } B_R,$$

with $0 \leq u \leq M$ in B_R, then for any $s \geq (g_0+1)^2 N =: \theta N$ and $\sigma \in (0,1)$, there is a constant $c = c(N, g_0, k, R, M, \sigma)$ such that

$$\sup_{B_{\sigma R}} u^2 \leq c\left(\int_{B_R} u^s \, dx\right)^{\frac{1}{s}}.$$

Moreover, there is $k_0 \in \mathbb{N}$ depending only on N, σ and R such that

$$c < \frac{2R^2}{M}\left(\exp(M/R)^2 - 1\right)$$

provided that $k \geq k_0$.

Theorem (Harnack's inequality)

Let $R \in (0, c_N^{-1/N})$, where c_N is the constant given by the John and Nirenberg lemma. For any

- $\sigma \in (0, 1)$,

Theorem (Harnack's inequality)

Let $R \in (0, c_N^{-1/N})$, where c_N is the constant given by the John and Nirenberg lemma. For any

- $\sigma \in (0, 1)$,
- $M > 0$

Theorem (Harnack's inequality)

Let $R \in (0, c_N^{-1/N})$, where c_N is the constant given by the John and Nirenberg lemma. For any

- $\sigma \in (0, 1)$,
- $M > 0$ *and*
- $k \geq k_0$ *with* $k_0 = k_0(N, \sigma, R, \delta_0)$

Theorem (Harnack's inequality)

Let $R \in (0, c_N^{-1/N})$, where c_N is the constant given by the John and Nirenberg lemma. For any

- $\sigma \in (0,1)$,
- $M > 0$ *and*
- $k \geq k_0$ *with* $k_0 = k_0(N, \sigma, R, \delta_0)$

given by Theorem [Local maximum principle], supposing that $u \in W^{1,G_k}(B_R)$ satisfies

$$-div\left(g_k(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = 0, \quad in\ B_R$$

Theorem (Harnack's inequality)

Let $R \in (0, c_N^{-1/N})$, where c_N is the constant given by the John and Nirenberg lemma. For any

- $\sigma \in (0,1)$,
- $M > 0$ *and*
- $k \geq k_0$ *with* $k_0 = k_0(N, \sigma, R, \delta_0)$

given by Theorem [Local maximum principle], supposing that $u \in W^{1,G_k}(B_R)$ *satisfies*

$$-div\left(g_k(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = 0, \quad in\ B_R$$

with $0 \leq u \leq M$ *in* B_R.

Theorem (Harnack's inequality)

Let $R \in (0, c_N^{-1/N})$, where c_N is the constant given by the John and Nirenberg lemma. For any

- $\sigma \in (0,1)$,
- $M > 0$ *and*
- $k \geq k_0$ *with* $k_0 = k_0(N, \sigma, R, \delta_0)$

given by Theorem [Local maximum principle], supposing that $u \in W^{1,G_k}(B_R)$ satisfies

$$-div\left(g_k(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = 0, \quad in\ B_R$$

with $0 \leq u \leq M$ in B_R. Then there is a positive constant C depending only on R and M such that

$$\sup_{B_{\sigma R}} u \leq C \inf_{B_{\sigma R}} u.$$

Interesting Result

Theorem (Harnack's inequality for the degenerate case)

Let $R \in (0, c_N^{-1/N})$, where c_N is the constant given by the John and Nirenberg lemma. For any $\sigma \in (0,1)$ given by Theorem [Local maximum principle], supposing that $u_k \in W^{1,G_k}(\Omega)$ is a minimizer of the

$$\min \left\{ \int_\Omega G_k\left(|\nabla u|\right)\, dx : u = f \text{ on } \partial\Omega \right\},$$

such that

$$u_k \to v_\Phi, \quad \text{uniformly on } \overline{\Omega}$$

and $v_\Phi \in W^{1,\Phi}(\Omega)$ is a minimizer of the

$$\min \left\{ \int_\Omega \Phi\left(|\nabla u|\right)\, dx : u = f \text{ on } \partial\Omega \right\},$$

Then there is a positive constant C depending only on R and M such that

$$\sup_{B_{\sigma R}} v_\Phi \leq C \inf_{B_{\sigma R}} v_\Phi, \ \sigma \in (0,1).$$

Recently

Theorem (Beck and Mingione CPAM 2020)

If u is a local minimizer of the functional

$$J(v) = \int_\Omega \Phi(|\nabla v|)\, dx.$$

Then

$$|\nabla u|_{L^\infty(B_{R/2})} \leq \Phi^{-1}\left(\frac{1}{|B_R|}\int_{B_R} \Phi(|\nabla u|)\, dx\right) + 1.$$

Regarding Regularity

As a consequence of Theorem 2, we are able to prove that the solution to (5), given by Theorem 1, is Lipschitz continuous along the free boundary $\partial\{u_\Phi > 0\}$.

Regarding Regularity

As a consequence of Theorem 2, we are able to prove that the solution to (5), given by Theorem 1, is Lipschitz continuous along the free boundary $\partial\{u_\Phi > 0\}$.

Corollary

Let

$$u_\Phi \text{ be the solution to problem (5)}$$

Regarding Regularity

As a consequence of Theorem 2, we are able to prove that the solution to (5), given by Theorem 1, is Lipschitz continuous along the free boundary $\partial\{u_\Phi > 0\}$.

Corollary

Let

$$u_\Phi \text{ be the solution to problem (5)}$$

and

$$x_0 \in \partial\{u_\Phi > 0\} \cap \Omega.$$

Regarding Regularity

As a consequence of Theorem 2, we are able to prove that the solution to (5), given by Theorem 1, is Lipschitz continuous along the free boundary $\partial\{u_\Phi > 0\}$.

Corollary

Let

$$u_\Phi \text{ be the solution to problem (5)}$$

and

$$x_0 \in \partial\{u_\Phi > 0\} \cap \Omega.$$

Then there exists a positive constant $c > 0$ depending only N, such that

$$u_\Phi(x) \leq c|x - x_0|$$

for $x \in \{u_\Phi > 0\}$ near x_0.

Since the quasilinear operator

$$\Delta u + 2\Delta_\infty u \text{ is elliptic,}$$

[11] L. C. Evans and C. K. Smart, Calc. Var. Partial Differential Equations 2011.

Since the quasilinear operator

$$\Delta u + 2\Delta_\infty u \text{ is elliptic,}$$

a theorem of **Evans and C. K. Smart**[11] says that

u_Φ is **locally Lipschitz continuous** within its set of positivity, $\{u_\Phi > 0\}$.

[11] L. C. Evans and C. K. Smart, Calc. Var. Partial Differential Equations 2011.

Since the quasilinear operator

$$\Delta u + 2\Delta_\infty u \text{ is elliptic,}$$

a theorem of **Evans and C. K. Smart**[11] says that

u_Φ is **locally Lipschitz continuous** within its set of positivity, $\{u_\Phi > 0\}$.

If we combine this with the corollary, we see that any minimizer of (5) belongs to $C^{0,1}_{\text{loc}}(\Omega)$.

[11] L. C. Evans and C. K. Smart, Calc. Var. Partial Differential Equations 2011.

Consequences of the non-degeneracy

- **Uniform positive density.** Given any domain

$$D \subset\subset \Omega,$$

Consequences of the non-degeneracy

- **Uniform positive density.** Given any domain

$$D \subset\subset \Omega,$$

there exists a constant $\theta > 0$ such that if

$$x_0 \in \Omega \cap \partial\{u_\Phi > 0\}$$

Consequences of the non-degeneracy

- **Uniform positive density.** Given any domain
$$D \subset\subset \Omega,$$
there exists a constant $\theta > 0$ such that if
$$x_0 \in \Omega \cap \partial\{u_\Phi > 0\}$$
then
$$\frac{\mathcal{L}^N(B_r(x_0) \cap \{u_\Phi > 0\})}{\mathcal{L}^N(B_r(x_0))} \geq \theta,$$
for every $r \in (0, \mathrm{dist}(\partial D, \partial\Omega))$,

Consequences of the non-degeneracy

- **Uniform positive density.** Given any domain
$$D \subset\subset \Omega,$$
there exists a constant $\theta > 0$ such that if
$$x_0 \in \Omega \cap \partial\{u_\Phi > 0\}$$
then
$$\frac{\mathcal{L}^N (B_r(x_0) \cap \{u_\Phi > 0\})}{\mathcal{L}^N (B_r(x_0))} \geq \theta,$$
for every $r \in (0, \mathrm{dist}(\partial D, \partial\Omega))$,where $\mathcal{L}^N(E)$ denotes the N-dimensional Lebesgue measure of the set E.

Therefore the region $\{u_\Phi > 0\}$ can not present cusps along the free boundary in the direction of the phase $\{u_\Phi = 0\}$.

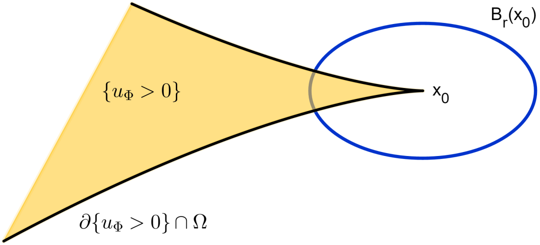


Figure: The free boundary has no cusp points.

Consequences of the non-degeneracy

- **Hausdorff dimension.** The set $\partial\{u_\Phi > 0\} \cap \Omega$ has Hausdorff dimension strictly less than N.

Consequences of the non-degeneracy

- **Hausdorff dimension.** The set

  $\partial\{u_\Phi > 0\} \cap \Omega$ has Hausdorff dimension strictly less than N.

  As a consequence,

  $\partial\{u_\Phi > 0\}$ has Lebesgue measure zero ($\mathcal{L}^N(\partial\{u_\Phi > 0\}) = 0$).

An Asymptotic result

Repeating the argument used in the proof of Theorem 1,

An Asymptotic result

Repeating the argument used in the proof of Theorem 1, for each $\ell > 0$ there is a solution u_ℓ

An Asymptotic result

Repeating the argument used in the proof of Theorem 1, for each $\ell > 0$ there is a solution u_ℓ to the minimization problem

$$\text{(6)} \qquad \min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega,\ \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

An Asymptotic result

Repeating the argument used in the proof of Theorem 1, for each $\ell > 0$ there is a solution u_ℓ to the minimization problem

$$\text{(6)} \qquad \min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\, dx,$$

An Asymptotic result

Repeating the argument used in the proof of Theorem 1, for each $\ell > 0$ there is a solution u_ℓ to the minimization problem

$$(6) \qquad \min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega,\ \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\, dx,$$

and

$$\Phi_\ell(t) = \exp(t^2/\ell) - 1.$$

An Asymptotic result

Repeating the argument used in the proof of Theorem 1, for each $\ell > 0$ there is a solution u_ℓ to the minimization problem

(6) $$\min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\, dx,$$

and

$$\Phi_\ell(t) = \exp(t^2/\ell) - 1.$$

Our objective now is to obtain an asymptotic result for solutions u_ℓ for sufficiently small ℓ.

A central issue in Rossi and Teixeira is the study of optimal design problems ruled by degenerate quasilinear operators.

A central issue in Rossi and Teixeira is the study of optimal design problems ruled by degenerate quasilinear operators.

In that paper, Rossi and Teixeira introduce a geometric compatibility condition upon the Lipschitz constant of the boundary data f,

A central issue in Rossi and Teixeira is the study of optimal design problems ruled by degenerate quasilinear operators.

In that paper, Rossi and Teixeira introduce a geometric compatibility condition upon the Lipschitz constant of the boundary data f, Ω

A central issue in Rossi and Teixeira is the study of optimal design problems ruled by degenerate quasilinear operators.

In that paper, Rossi and Teixeira introduce a geometric compatibility condition upon the Lipschitz constant of the boundary data f, Ω and α

A central issue in Rossi and Teixeira is the study of optimal design problems ruled by degenerate quasilinear operators.

In that paper, Rossi and Teixeira introduce a geometric compatibility condition upon the Lipschitz constant of the boundary data f, Ω and α to prove that any sequence of solutions to problem

$$\min \left\{ \int_\Omega |\nabla u|^p \, dx \; : \; u \in W^{1,p}(\Omega), \; u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) \leq \alpha \right\}$$

converges uniformly to the unique minimizer u_∞

A central issue in Rossi and Teixeira is the study of optimal design problems ruled by degenerate quasilinear operators.

In that paper, Rossi and Teixeira introduce a geometric compatibility condition upon the Lipschitz constant of the boundary data f, Ω and α to prove that any sequence of solutions to problem

$$\min\left\{\int_\Omega |\nabla u|^p\, dx \;:\; u \in W^{1,p}(\Omega),\; u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) \le \alpha\right\}$$

converges uniformly to the unique minimizer u_∞ for problem

$$\min\{\mathrm{Lip}(u) : u \in W^{1,\infty}(\Omega), u = f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\}) \le \alpha\},$$

A central issue in Rossi and Teixeira is the study of optimal design problems ruled by degenerate quasilinear operators.

In that paper, Rossi and Teixeira introduce a geometric compatibility condition upon the Lipschitz constant of the boundary data f, Ω and α to prove that any sequence of solutions to problem

$$\min\left\{\int_\Omega |\nabla u|^p\,dx \,:\, u\in W^{1,p}(\Omega),\ u=f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u>0\})\le\alpha\right\}$$

converges uniformly to the unique minimizer u_∞ for problem

$$\min\{\mathrm{Lip}(u) : u\in W^{1,\infty}(\Omega), u=f \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u>0\})\le\alpha\},$$

where $\mathrm{Lip}(u)$ is the Lipschitz constant of u.

Furthermore, u_∞ satisfies a PDE

Furthermore, u_∞ satisfies a PDE governed by the infinity-Laplacian operator

Furthermore, u_∞ satisfies a PDE governed by the infinity-Laplacian operator and it is given by the formula

$$u_\infty(x) = \max_{y\in\partial\Omega}(f(y) - \lambda^*|x-y|)_+,$$

Furthermore, u_∞ satisfies a PDE governed by the infinity-Laplacian operator and it is given by the formula

$$u_\infty(x) = \max_{y \in \partial\Omega} (f(y) - \lambda^* |x - y|)_+,$$

for some positive real number λ^* uniquely determined.

Bocea and Mihăilescu[12] study the limit behavior of the solution u_ℓ

[12] M. F. Bocea, M. Mihăilescu, Israel J. Math., 2019.

Bocea and Mihăilescu[12] study the limit behavior of the solution u_ℓ of the problems

$$
(7) \qquad \begin{cases} \ell \Delta u + 2\Delta_\infty u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}
$$

[12] M. F. Bocea, M. Mihăilescu, Israel J. Math., 2019.

Bocea and Mihăilescu[12] study the limit behavior of the solution u_ℓ of the problems

(7) $$\begin{cases} \ell \Delta u + 2\Delta_\infty u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

as $\ell \to 0^+$ to explore the connections between these solutions and infinity harmonic maps.

[12] M. F. Bocea, M. Mihăilescu, Israel J. Math., 2019.

Bocea and Mihăilescu[12] study the limit behavior of the solution u_ℓ of the problems

$$(7) \qquad \begin{cases} \ell\Delta u + 2\Delta_\infty u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

as $\ell \to 0^+$ to explore the connections between these solutions and infinity harmonic maps. More precisely, the sequence

$$\{u_\ell\} \text{ is bounded in } W^{1,2m}(\Omega), \text{ for every integer } m > N.$$

[12] M. F. Bocea, M. Mihăilescu, Israel J. Math., 2019.

Bocea and Mihăilescu[12] study the limit behavior of the solution u_ℓ of the problems

$$(7) \qquad \begin{cases} \ell\Delta u + 2\Delta_\infty u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

as $\ell \to 0^+$ to explore the connections between these solutions and infinity harmonic maps. More precisely, the sequence

$$\{u_\ell\} \text{ is bounded in } W^{1,2m}(\Omega), \text{ for every integer } m > N.$$

As $\ell \to 0^+$, the sequence $\{u_\ell\}$ converges uniformly in $\overline{\Omega}$

[12] M. F. Bocea, M. Mihăilescu, Israel J. Math., 2019.

Bocea and Mihăilescu[12] study the limit behavior of the solution u_ℓ of the problems

$$(7)\qquad \begin{cases} \ell\Delta u + 2\Delta_\infty u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

as $\ell \to 0^+$ to explore the connections between these solutions and infinity harmonic maps. More precisely, the sequence

$$\{u_\ell\} \text{ is bounded in } W^{1,2m}(\Omega), \text{ for every integer } m > N.$$

As $\ell \to 0^+$, the sequence $\{u_\ell\}$ converges uniformly in $\overline{\Omega}$ to the unique viscosity solution to the problem

$$(8)\qquad \begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

[12] M. F. Bocea, M. Mihăilescu, Israel J. Math., 2019.

For fixed $\ell > 0$,

For fixed $\ell > 0$, consider the minimization problem

$$\min\{J_{\ell}(u) : |\nabla u| \in K_{\Phi_{\ell}}(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\}, \tag{9}$$

For fixed $\ell > 0$, consider the minimization problem

$$\text{(9)} \qquad \min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega,\ \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\, dx,$$

For fixed $\ell > 0$, consider the minimization problem

(9) $$\min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega,\ \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\, dx,$$

and

$$\Phi_\ell(t) = \exp(t^2/\ell) - 1$$

For fixed $\ell > 0$, consider the minimization problem

$$\min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega,\ \mathcal{L}^N(\{u > 0\}) = \alpha\}, \tag{9}$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\, dx,$$

and

$$\Phi_\ell(t) = \exp(t^2/\ell) - 1$$

and the function

$$f : \overline{\Omega} \to \mathbb{R} \text{ is positive}$$

For fixed $\ell > 0$, consider the minimization problem

$$\text{(9)} \qquad \min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega,\ \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\, dx,$$

and

$$\Phi_\ell(t) = \exp(t^2/\ell) - 1$$

and the function

$$f : \overline{\Omega} \to \mathbb{R} \text{ is positive and uniform Lipschitz continuous.}$$

Combining the arguments of Bocea- Mihăilescu and Rossi-Teixeira,

Combining the arguments of Bocea- Mihăilescu and Rossi-Teixeira, we establish the optimal shape for the limiting problem as $\ell \to 0^+$.

Combining the arguments of Bocea- Mihăilescu and Rossi-Teixeira, we establish the optimal shape for the limiting problem as $\ell \to 0^+$.

To do this, we assume the geometric compatibility condition:

$$(H) \qquad \mathcal{L}^N\left(\bigcup_{x\in\partial\Omega} B_{\frac{f(x)}{\mathrm{Lip}(f)}}(x)\cap\Omega\right) \geq \alpha.$$

Combining the arguments of Bocea- Mihăilescu and Rossi-Teixeira, we establish the optimal shape for the limiting problem as $\ell \to 0^+$.

To do this, we assume the geometric compatibility condition:

$$(H) \qquad \mathcal{L}^N\left(\bigcup_{x\in\partial\Omega} B_{\frac{f(x)}{\mathrm{Lip}(f)}}(x)\cap\Omega\right) \geq \alpha.$$

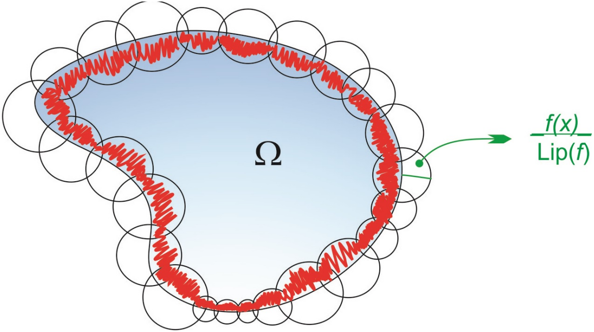


Figure: Compatibility condition.

Theorem

Suppose that (H) holds

Theorem

Suppose that (H) holds and let $\lambda^* \geq Lip(f)$

Theorem

Suppose that (H) holds and let $\lambda^* \geq Lip(f)$ *be the unique positive real number*

Theorem

Suppose that (H) holds and let $\lambda^* \geq Lip(f)$ *be the unique positive real number such that the measure of the set*

$$\Omega^* = \bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\lambda^*}}(x) \cap \Omega$$

Theorem

Suppose that (H) holds and let $\lambda^* \geq Lip(f)$ be the unique positive real number such that the measure of the set

$$\Omega^* = \bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\lambda^*}}(x) \cap \Omega$$

is equal to α.

Theorem

Suppose that (H) holds and let $\lambda^* \geq Lip(f)$ *be the unique positive real number such that the measure of the set*

$$\Omega^* = \bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\lambda^*}}(x) \cap \Omega$$

is equal to α*. If* u_ℓ *is a minimizer of (9),*

Theorem

Suppose that (H) holds and let $\lambda^* \geq Lip(f)$ *be the unique positive real number such that the measure of the set*

$$\Omega^* = \bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\lambda^*}}(x) \cap \Omega$$

is equal to α*. If* u_ℓ *is a minimizer of (9), then*

$$u_\ell \text{ converges to } u_0 \text{ uniformly in } \overline{\Omega} \text{ as } \ell \to 0^+.$$

Theorem

Suppose that (H) holds and let $\lambda^* \geq Lip(f)$ *be the unique positive real number such that the measure of the set*

$$\Omega^* = \bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\lambda^*}}(x) \cap \Omega$$

is equal to α. *If* u_ℓ *is a minimizer of (9), then*

$$u_\ell \text{ converges to } u_0 \text{ uniformly in } \overline{\Omega} \text{ as } \ell \to 0^+.$$

The function u_0 *is the unique solution*

Theorem

Suppose that (H) holds and let $\lambda^* \geq Lip(f)$ be the unique positive real number such that the measure of the set

$$\Omega^* = \bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\lambda^*}}(x) \cap \Omega$$

is equal to α. If u_ℓ is a minimizer of (9), then

u_ℓ converges to u_0 uniformly in $\overline{\Omega}$ as $\ell \to 0^+$.

The function u_0 is the unique solution to

$$(10) \qquad \begin{cases} \Delta_\infty u = 0 & \text{in } \Omega^*, \end{cases}$$

Theorem

Suppose that (H) holds and let $\lambda^* \geq Lip(f)$ be the unique positive real number such that the measure of the set

$$\Omega^* = \bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\lambda^*}}(x) \cap \Omega$$

is equal to α. If u_ℓ is a minimizer of (9), then

$$u_\ell \text{ converges to } u_0 \text{ uniformly in } \overline{\Omega} \text{ as } \ell \to 0^+.$$

The function u_0 is the unique solution to

$$(10) \qquad \begin{cases} \Delta_\infty u = 0 & \text{in } \Omega^*, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

Theorem

Suppose that (H) holds and let $\lambda^* \geq Lip(f)$ be the unique positive real number such that the measure of the set

$$\Omega^* = \bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\lambda^*}}(x) \cap \Omega$$

is equal to α. If u_ℓ is a minimizer of (9), then

$$u_\ell \text{ converges to } u_0 \text{ uniformly in } \overline{\Omega} \text{ as } \ell \to 0^+.$$

The function u_0 is the unique solution to

$$(10) \qquad \begin{cases} \Delta_\infty u = 0 & \text{in } \Omega^*, \\ u = f & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial\Omega^* \cap \Omega. \end{cases}$$

Theorem

Suppose that (H) holds and let $\lambda^* \geq Lip(f)$ be the unique positive real number such that the measure of the set

$$\Omega^* = \bigcup_{x \in \partial\Omega} B_{\frac{f(x)}{\lambda^*}}(x) \cap \Omega$$

is equal to α. If u_ℓ is a minimizer of (9), then

$$u_\ell \text{ converges to } u_0 \text{ uniformly in } \overline{\Omega} \text{ as } \ell \to 0^+.$$

The function u_0 is the unique solution to

$$\text{(10)} \qquad \begin{cases} \Delta_\infty u = 0 & \text{in } \Omega^*, \\ u = f & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial\Omega^* \cap \Omega. \end{cases}$$

Furthermore, u_0 is given by the formula,

$$\text{(11)} \qquad u_0(x) = \max_{y \in \partial\Omega} (f(y) - \lambda^* |x - y|)_+, \quad x \in \overline{\Omega}.$$

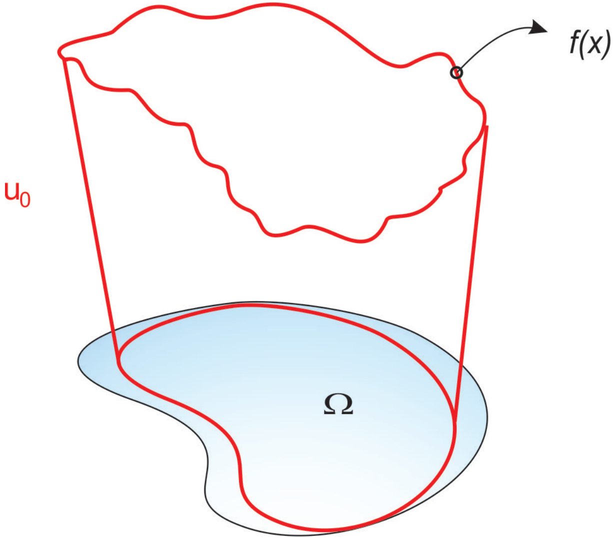


Figure: Unique solution u_0.

Corollary

If u_ℓ is a solution

Corollary

If u_ℓ is a solution to problem

$$\min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\}, \tag{12}$$

Corollary

If u_ℓ is a solution to problem

$$\min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\}, \tag{12}$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\, dx,$$

Corollary

If u_ℓ is a solution to problem

$$\text{(12)} \qquad \min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\, dx,$$

and

$$\Phi_\ell(t) = \exp(t^2/\ell) - 1,$$

Corollary

If u_ℓ is a solution to problem

$$\text{(12)} \qquad \min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega,\ \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\,dx,$$

and

$$\Phi_\ell(t) = \exp(t^2/\ell) - 1,$$

then, for sufficiently small ℓ,

Corollary

If u_ℓ is a solution to problem

$$(12) \qquad \min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega, \mathcal{L}^N(\{u > 0\}) = \alpha\},$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\,dx,$$

and

$$\Phi_\ell(t) = \exp(t^2/\ell) - 1,$$

then, for sufficiently small ℓ,

u_ℓ is uniformly Lipschitz

Corollary

If u_ℓ is a solution to problem

$$\min\{J_\ell(u) : |\nabla u| \in K_{\Phi_\ell}(\Omega), u = f \text{ on } \partial\Omega,\ \mathcal{L}^N(\{u > 0\}) = \alpha\}, \tag{12}$$

where

$$J_\ell(u) = \int_\Omega \Phi_\ell(|\nabla u|)\, dx,$$

and

$$\Phi_\ell(t) = \exp(t^2/\ell) - 1,$$

then, for sufficiently small ℓ,

$$u_\ell \text{ is uniformly Lipschitz}$$

and

$$u_\ell(x) = \max_{y \in \partial\Omega}(f(y) - \lambda^*|x - y|)_+ + o_\ell(1), \quad x \in \overline{\Omega}.$$

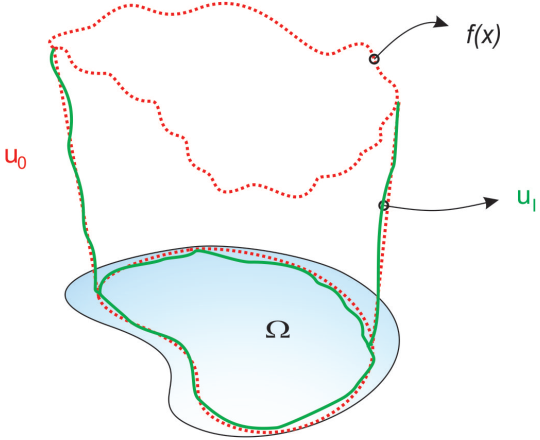


Figure: Asymptotic solution u_ℓ.

Essential references

1. Lieberman, G.M.: *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations 16 (1991), 311–361.
2. Martínez, S. and Wolanski, N.: *A minimum problem with free boundary in Orlicz spaces*, Adv. Math. 218 (2008), 1914–1971.
3. Rossi, J.D. and Teixeira, E.V.: *A limiting free boundary problem ruled by Aronsson's equation*, Trans. Amer. Math. Soc. 364 (2012), 703–719.
4. Abrantes Santos, J. and Monari Soares, S.H.: *A limiting free boundary problem for a degenerate operator in Orlicz-Sobolev spaces*, Rev. Mat. Iberoam. v. 36, p. 1687/6-1720, 2020.
5. Abrantes Santos, J. and Monari Soares, S.H. : *Optimal design problems for a degenerate operator in Orlicz-Sobolev spaces*, Calc. Var. Partial Differential Equations, v. 59, p. 1/183-23, 2020.

Thank you for your attention!