Tube structures with forms defined on closed manifolds

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Summary

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Let c be a smooth closed 1-form defined on a closed manifold M. We consider the operator $\mathbb{L} : C^\infty(M \times \mathbb{S}^1) \to C^\infty(M \times \mathbb{S}^1, \Lambda^{(1,0)})$:

$$\mathbb{L}u = d_t u + c(t) \wedge \partial_x u \,.$$

Assuming that $(t_1, \ldots, t_n)$ are local coordinates on M and C a local primitive of c, we have the vector fields

$$L_j = \frac{\partial}{\partial t_j} + \frac{\partial C}{\partial t_j}(t) \frac{\partial}{\partial x} \,, \quad j = 1, \ldots, n \,.$$

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They are local generators of $\mathcal{V} \doteq (T')^{\perp} \subset \mathbb{C} \otimes T(M \times \mathbb{S}^1)$ where T' is the line sub-bundle of $\mathbb{C} \otimes T^*(M \times \mathbb{S}^1)$ generated by the 1-form $dx - c$. Any involutive structure defines in a natural way a complex of differential operators - which in the case of $\mathcal{V}$ is given by $\mathbb{L}$ when acting on distributions:

$$\mathscr{D}'(M \times \mathbb{S}^1) \xrightarrow{\mathbb{L}} \mathfrak{U}^1(M \times \mathbb{S}^1) \xrightarrow{\mathbb{L}^1}$$

$$\xrightarrow{\mathbb{L}^1} \mathfrak{U}^2(M \times \mathbb{S}^1) \xrightarrow{\mathbb{L}^2} \cdots \xrightarrow{\mathbb{L}^{n-1}} \mathfrak{U}^n(M \times \mathbb{S}^1) \xrightarrow{\mathbb{L}^n} 0.$$

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We study the smooth global solvability of $\mathbb{L}$, i.e., the possibility of finding a globally defined solution $u \in \mathscr{D}'(M \times \mathbb{S}^1)$ to

$$\mathbb{L}u = d_t u + c(t) \wedge \partial_x u = f,$$

when f is smooth.

If f is in the range of $\mathbb{L}$ it must satisfy:

(i) $\mathbb{L}f = 0$ (a consequence of the complex property $\mathbb{L} \circ \mathbb{L} = 0$);

(ii) f must be orthogonal to the kernel of the adjoint operator $\mathbb{L}^*$.

While (i) is of local nature, the homology of M plays a role in (ii).

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They are usually referred to as the compatibility conditions for f (we write $f \in \mathbb{E}$) and are formulated in several equivalent ways.

We say that the operator $\mathbb{L}$ is globally hypoelliptic if

$$\mathbb{L}u \in C^\infty(M \times \mathbb{S}^1, \Lambda^{(1,0)}) \implies u \in C^\infty(M \times \mathbb{S}^1).$$

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Statement when c is exact

When c is smooth and exact, we can define a primitive of b on M by $B(t) = \int_{t_0}^{t} b$. In [Cardoso; Hounie, 1977] the authors characterized the global solvability as follows:

Theorem

If b is exact the following statements are equivalent:
(I) $\mathbb{L}$ is globally solvable.
(II) The semilevel sets $\{t \in M : B(t) < r\}$ and $\{t \in M : B(t) > r\}$ are connected for every $r \in \mathbb{R}$.

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Minimal covering space

- We are given a manifold M where a real smooth closed 1-form b is defined.
- We construct a special covering space $\widetilde{M}$ on which a primitive $\widetilde{B}$ of b is defined.
- Call D the group of deck transformations of $\widetilde{M}$.
- The primitive $\widetilde{B}$ is such that

$$\widetilde{B}(\sigma(t)) - \widetilde{B}(t) = b_\sigma \, ,$$

for $\sigma \in \mathrm{D}$, and $b_\sigma = 0 \Leftrightarrow \sigma = 1$.

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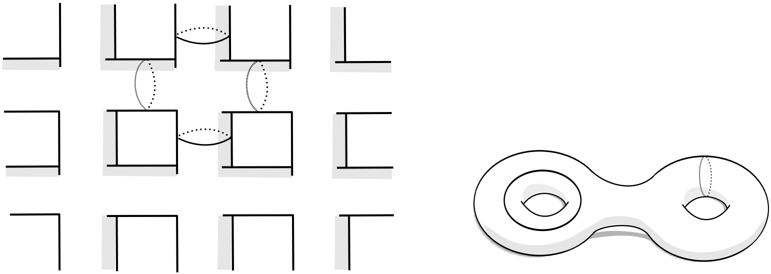
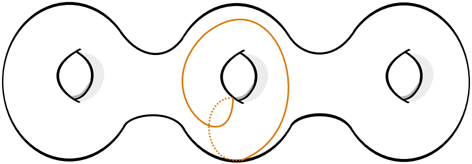
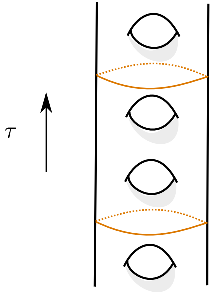


Cutting where the periods are zero

$$\int_{\gamma_k} b = c_k, \int_{\delta_k} b = d_k, \text{ and } pc_k + qd_k = 0$$





$$B(\tau(t)) = B(t) + d'$$

$$B(t) = P(t) + d' t_3$$

Summary

- Denote by $\mathscr{A}$ the set of the connected components $\mathcal{O}$ of regular semilevel sets of $\widetilde{B}$ such that $\widetilde{B}$ is bounded on $\mathcal{O}$. Then consider the inclusion $j : \mathcal{O} \hookrightarrow \widetilde{M}$.
- We will associate to $\mathcal{O} \in \mathscr{A}$ the vector $I(\mathcal{O}) = (\int_{\alpha_1} a, \ldots, \int_{\alpha_\mu} a)$, where $\{\alpha_1, \ldots \alpha_\mu\}$ is a basis of the free part of $j^* H_1(\mathcal{O}, \mathbb{Z})$.

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Theorem [Hounie; Zugliani, 2021]

Assume that M is a closed surface and that the 1-form $c = a + ib$ is smooth and closed. The following statements are equivalent:

(I) $\mathbb{L}$ is globally solvable.

(II) One of the conditions below is satisfied:

- $\mathscr{A} = \emptyset$ or, if $\mathcal{O} \in \mathscr{A}$, $I(\mathcal{O})$ is neither a rational nor a Liouville vector.
- b is exact, the semilevel sets of $\widetilde{B}$ are connected; a is rational, and, if $q \in \mathbb{Z}$ is such that $qI(\mathcal{O}) \in (2\pi\mathbb{Z})^{\mu}$ for $\mathcal{O} \in \mathscr{A}$, then qa is integral.

Compatibility conditions

Definition

We say that a 1-form $f \in C^\infty(M \times \mathbb{S}^1, \Lambda^{1,0})$ belongs to $\mathbb{E}$ if:

- for each $\xi \in \mathbb{Z}$ and each smooth curve γ connecting t to $\sigma(t)$ in $\mathscr{U}$ with $i\xi c_\sigma \in 2\pi\mathbb{Z}$,

$$\int_\gamma e^{i\xi C(s)} \hat{f}(s,\xi) = 0\,.$$

- $d_t(e^{i\xi C(t)} \hat{f}(t,\xi)) = 0$ for every $\xi \in \mathbb{Z}$.

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Summary

- One can compute the Fourier coefficients of a candidate to a problem's solution on $\widetilde{M}$ by solving a differential equation for each $\xi \in \mathbb{Z}$, which yields

$$\widehat{u}(t,\xi) = \int_{t_0}^{t} \upsilon + K_\xi e^{\xi C(t)},$$

where $\upsilon = e^{i\xi[C(s)-C(t)]}\hat{f}(s,\xi)$.

- Imposing the periodicity in order to define a solution on the manifold, we determine K_ξ and the coefficients, namely

$$\widehat{u}(t,\xi) = \frac{1}{e^{\xi(ia_\sigma - b_\sigma)} - 1} \int_{t}^{\sigma(t)} \upsilon,$$

where $C(\sigma(t)) - C(t) = c_\sigma = a_\sigma + ib_\sigma$.

- We wish to prove that $\{\widehat{u}(t,\xi)\}$ decays rapidly.

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Regarding the proof of (II) $\implies$ (I)

If $a \equiv 0$, we will have the desired control for $\xi > 0$ if along the curve

$$B(s) \geqslant B(t) + \frac{1}{1 + |\xi|},$$

holds true, since

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Lemma [Maire, Comm. Partial Differential Equations, 1980]

Let O be an open set in $\mathbb{R}^m$ and $\Phi \in C^\omega(O)$. For $s \in O$ with $\nabla\Phi(s) \neq 0$, the solution $\gamma_s : [0, \delta(s)) \to O$ of

$$\begin{cases} y' = \dfrac{\nabla\Phi(y)}{\|\nabla\Phi(y)\|} \\ y(0) = s\,. \end{cases}$$

satisfies

$$\Phi(\gamma_s(\tau)) \geqslant \Phi(s) + C_0 \tau^{\frac{1}{1-\theta}},$$

for $\tau \in [0, \delta(s))$.

Proposition [Teissier, Acta Math., 1983]

Given a compact set $\mathscr{K} \subset U$, there exists $C_1 \doteq C_1(\mathscr{K}) > 0$ such that, for every $r \in B^\dagger(\mathscr{K})$, any pair of points in a component of $(B^\dagger)^{-1}(r) \cap \mathscr{K}$ can be joined by a real analytic path in $(B^\dagger)^{-1}(r) \cap \mathscr{K}$ with length less than C_1.

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Regarding the proof of (II) $\implies$ (I)

The approach will depend on b: if b is not exact, we consider a division of the pairs $(t, \xi) \in \widetilde{M} \times \mathbb{Z}^-$ in two classes.

The class (A) will consist of the pairs (t, ξ) for which there is $\sigma \in \mathrm{D}$ with $b_\sigma < 0$ such that

$$t \text{ and } \sigma(t) \text{ are in the same component of } \Omega_{\widetilde{B}(t) + \frac{1}{1+|\xi|}} .$$

As for the pairs in the class (B), for each $\sigma \in \mathrm{D}$ with $b_\sigma < 0$,

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Lemma

For each pair (t,ξ) in the class (B), there is a piecewise smooth closed curve $\gamma(t,\xi)$ in $\widetilde{M}$ based on t such that:

- $\gamma(t,\xi)$ is contained in $\Omega_{\widetilde{B}(t)+\frac{1}{1+|\xi|}}$;
- $|\gamma(t,\xi)| \leqslant C_0(1+|\xi|)$;
- $\left| e^{i\xi \int_{\gamma(t,\xi)} a} - 1 \right| \geqslant \dfrac{K}{|\xi|^s}$, for $K > 0$.

The hypothesis on the dimension is not required here as well.

A similar division and statement are true when b is exact.

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Summary

Example 1

Assume that M is a closed manifold and b has only isolate singular points. The following statements are equivalent:

(I) $\mathbb{L}$ is globally solvable.

(II) One of the two conditions below is satisfied:

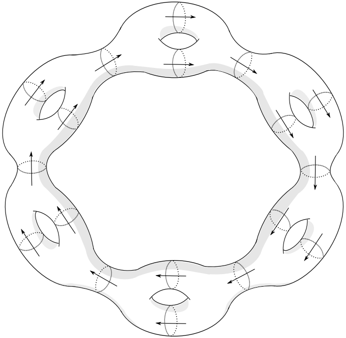
- The local primitives of b are open at any singular point.
- The form b is exact, the semilevel sets $\{t \in M : \widetilde{B}(t) > r\}$ and $\{t \in M : \widetilde{B}(t) < r\}$ are connected for every $r \in \mathbb{R}$, and a is integral.

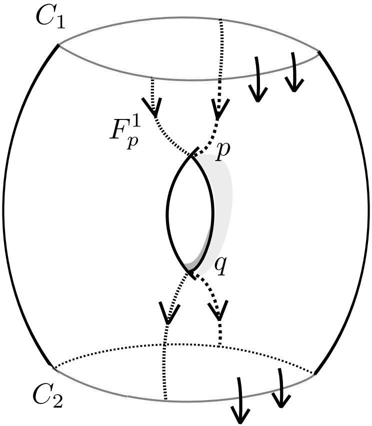
Example 2

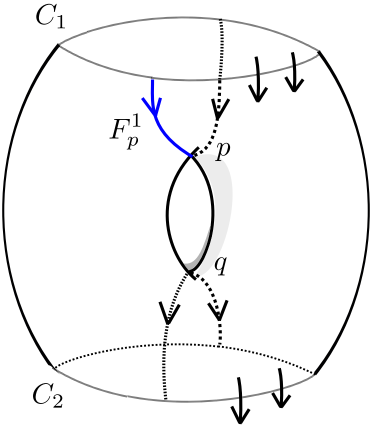
Assume that M is a closed manifold and $\text{rank}(b) = 1$. The following statements are equivalent:
(I) $\mathbb{L}$ is globally solvable.
(II) $\mathscr{A} = \emptyset$, or, for every $\mathcal{O} \in \mathscr{A}$, $I(\mathcal{O})$ is neither a rational nor a Liouville vector.

Levitt's Theorem

Every Morse foliation on a surface of genus $g > 1$ having only saddles as critical points (and not connected by leaves of the foliation), there are $3g - 3$ pairwise disjoint cycles, transversal to the foliation, decomposing the surface into *pants*. There is only one saddle on each *pant*.




C_1
F_p^1
p
q
C_2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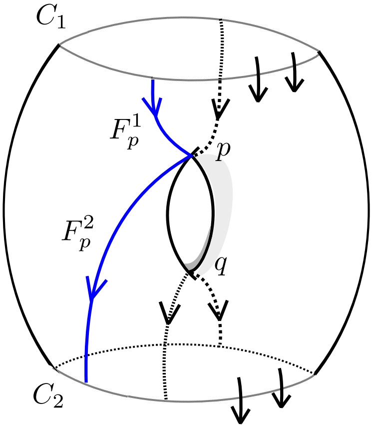
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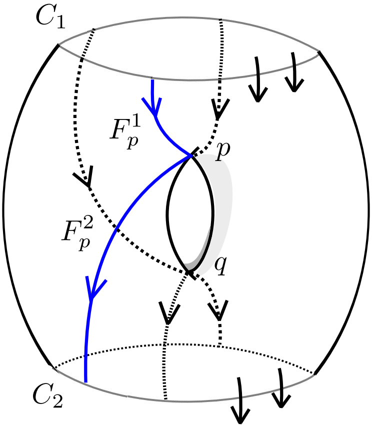
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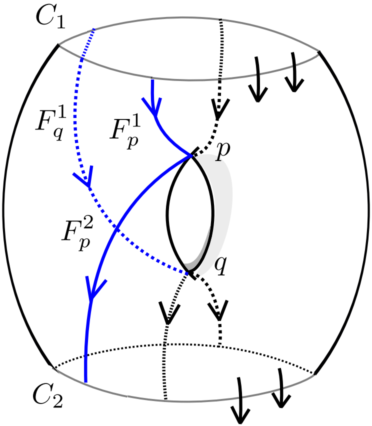
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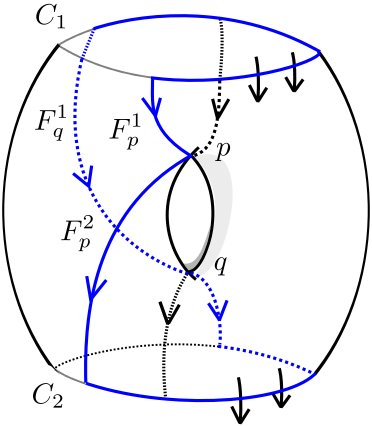

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C_2

Global solvability X Local solvability —Morse case

The operator $\mathbb{L} = d_t + ib(t)\partial_x$ is said to be locally solvable at $p = (t, x) \in M \times \mathbb{S}^1$ if any given neighborhood U of p contains another neighborhood V of p such that for every $f \in C^\infty(U, \Lambda^{1,0})$ with $\mathbb{L}f = 0$ there is $u \in C^\infty(V)$ satisfying $\mathbb{L}u = f$ on V.

Theorem [Treves, 1976]

The operator $\mathbb{L}$ is locally solvable at (t_j, x) if and only if the index of the critical point t_j is not 1 neither $n - 1$.

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