Regularization of Discontinuous Foliations: Blowing up and Sliding Conditions via Fenichel Theory

Subjects covered in the short course

Flows defined by ordinary differential equations

When we can not apply the existence and uniqueness theorem

Geometric singular perturbation theory

Flows defined by differential equations with discontinuous righthand side

Regularization of non–smooth systems

Global regularization

Singularly perturbed colleagues and collaborators.

Sotomayor $\to$ **Silva** $\to$ $\begin{cases} \textbf{Cardin} \\ \textbf{Lopes} \\ \textbf{Meza} \\ \textbf{Moraes} \end{cases}$ **Teixeira** $\to$ $\begin{cases} \textbf{Larrosa} \leftarrow \textbf{M.Seara} \\ \textbf{Novaes} \leftarrow \textbf{Llibre} \end{cases}$

$$
\begin{array}{ccccc}
 & & \textbf{Teixeira} & & \\
 & & \updownarrow & & \\
\textbf{Llibre} & \leftarrow & \textbf{Silva} & \rightarrow & \textbf{Buzzi} \\
 & \swarrow & \downarrow & \searrow & \\
\textbf{Medrado} & & \textbf{Panazzolo} & & \textbf{Novaes}
\end{array}
$$

Flows defined by ordinary differential equations

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set and $X : \mathcal{U} \to \mathbb{R}^n$ be a C^k-vector field. The trajectories of X are the solutions of the differential equation

$$x' = X(x).$$

- (existence and unicity) $\forall x \in \mathcal{U}$, $\exists I_x \ni 0$ and $\varphi_x : I_x \to \mathcal{U}$ such that $\varphi_x'(t) = X(\varphi_x(t))$ and $\varphi_x(0) = x$.
- (group structure) $y = \varphi_x(s), s \in I_x$, $I_y = I_x - s$ and $\varphi_y(0) = y, \varphi_y(t) = \varphi_x(t+s), \forall t \in I_y$.

The set

$$D = \{(t,x) : x \in \mathcal{U}, t \in I_x\}$$

is open and

$$\varphi : D \to \mathbb{R}^n, \quad \varphi(t,x) = \varphi_x(t)$$

is called flow of X.

If $X(p) \neq 0$ we say that p is a regular point and if $X(p) = 0$ we say that p is a singularity.

The orbit of $x \in \mathcal{U}$ is the set $\mathcal{O}(x) = \{\varphi(t,x) : t \in I_x\}$. There are 3 kinds of orbits:

- (homeomorphic to $\mathbb{R}$): $\varphi_x(t_1) \neq \varphi_x(t_2)$, $\forall t_1, t_2 \in I_x$ with $t_1 \neq t_2$;
- (only one point): $\varphi_x(t_1) = \varphi_x(t_2)$, $\forall t_1, t_2 \in I_x$;
- (homeomorphic to S^1): other cases (periodic orbits).

A limit cycle is an isolated periodic orbit. An equivalence relation on $\mathcal{U}$ is defined as follows: $p \sim q$ if and only if $p \in \mathcal{O}(q)$. The partition of $\mathcal{U}$ in equivalence classes is called phase portrait.

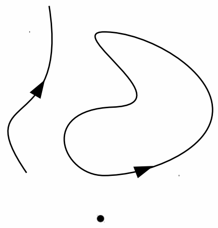


Figure 1: Kinds of orbits.

Main goal: to describe the phase portrait !!!!!

The simplest case: linear systems. In this case the phase portrait can be fully described.

$$X : \mathbb{R}^n \to \mathbb{R}^n, \quad X(x) = A.x, \quad A \in \mathcal{M}(n), \quad \varphi(t,x) = e^{tA}x.$$

A particularity of linear systems is the absence of limit cycles. The study reduces to the study of eigenvalues and eigenvectors of the matrix A. The origin is a singularity. There is a residual subset $\mathcal{H}(n) \subset \mathcal{M}(n)$, formed by the hyperbolic vector fields. The eigenvectors of $A \in \mathcal{H}(n)$ generate invariant directions on the phase portrait. The sign of the real part of the eigenvalues determines if the origin is attracting or repelling.

General case: For non-linear systems the description of the phase portrait depends on several local techniques. In the neighborhood of regular points, we use the flow box theorem to determine the phase portrait. Essentially the phase portrait is equivalent to the one of the constant field $Y = (1.0, ..., 0)$. In the neighborhood of hyperbolic singularities (those satisfying than $JX \in \mathcal{H}(n)$) we can use the Grobmann-Hartmann theorem, which says that the phase portrait of X is equivalent to one of the linear part JX in a neighborhood of the origin.

When $JX \notin \mathcal{H}(n)$ but has some nonzero eigenvalue we use the theorem of the central manifold. If $n = 2$ and the eigenvalues are both zero we use the process of blow up. For polynomial systems we also can analyze the global phase portrait using the Poincaré compactification.

When we can not apply the existence and uniqueness theorem

Some special systems:

- constrained systems or systems with impasse;
- implicit systems;
- slow-fast systems;
- non-smooth systems.

System with impasse. Consider $x \in \mathbb{R}^n$, a_{ij}, f_i of class C^r in $\mathbb{R}^n$, $i, j = 1, \ldots, n$. A system with impasse (or constrained system) is

$$\begin{cases} a_{11}(x)\dot{x}_1 + \cdots + a_{1n}(x)\dot{x}_n & = & f_1(x) \\ \vdots & & \vdots \\ a_{n1}(x)\dot{x}_1 + \cdots + a_{nn}(x)\dot{x}_n & = & f_n(x) \end{cases}$$

In matrix notation $A(x)\dot{x} = F(x)$ where $A = (a_{ij})$ and $F = (f_1, f_2, \ldots, f_n)$ is a vector field. The points of $\mathcal{I}_A = \{x \in \mathbb{R}^n : \det A(x) = 0\}$ are called impasse points.

For example

$$\begin{pmatrix} -1 & 1/2 \\ 0 & y-x \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 \\ 2y \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{x-y}\begin{pmatrix} y-x & -1/2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2y \end{pmatrix}.$$

$$\begin{cases} x' &= -x \\ y' &= -2y \end{cases}, \quad \det A = 0 \Leftrightarrow x-y=0 \Leftrightarrow y = x.$$

The phase portrait can be obtained by the phase portrait of red system by removing from its orbits the impasse points and inverting the orientation along orbits on regions where $\det A$ is negative.

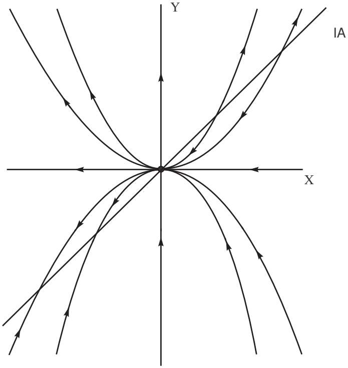


Figure 2: Phase portrait of a system with impasse

In the works of Sotomayor, Zhitomirskii and Llibre one can get the local phase portrait of a constrained system near impasse points in the generic case of class C^∞ systems. They also study structural stability of C^r and polynomial constrained systems and bifurcations of one-parameter families of constrained systems giving the stratification of the impasse surface for a generic family of constrained systems.

In the works of —,Cardin and Teixeira techniques of singular perturbation are used to analise these systems.

Implicit system.

$F : \mathbb{R}^3 \to \mathbb{R}$, C^r, $r \geq 1$, 0 regular value of F.

$$M = \{(x, y, p) \in \mathbb{R}^3; F(x, y, p) = 0\}$$

is a C^r- manifold. Here $p = \frac{dy}{dx}$.

Our interest is when the derivative $F_p(q) = 0$ at some $q \in M$.

M as above:

$$q_0 = (x_0, y_0, p_0) \in M,$$

The contact-plane is

$$CP_{q_0} = \{T = (x, y, p) \in \mathbb{R}^3 : dy = p_0 dx\}.$$

Assume CP_{q_0} intersects $T_{q_0}M$ in a line.

It defines a direction field on a neighborhood of $q_0 \in M$.

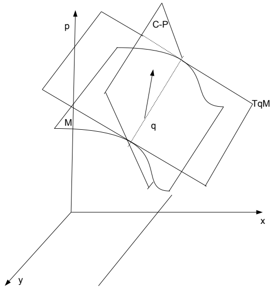


Figure 3: Directional field.

The integral curves of $F(x, y, p) = 0$ are the integral curves of this direction field. To solve this equation it is necessary to find these curves.

A direction field, as described above, can be obtained taking the vector field

$$\xi = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} - (F_x + pF_y) \frac{\partial}{\partial p}. \tag{1}$$

The direction of the p-axis in the space $\mathbb{R}^3$ is called vertical direction.

A point $q \in M$ is said to be regular if it is not a critical point of $\pi(x, y, p) = (x, y)$. In other words, a point of M is regular if the tangent plane at this point is not vertical. The other points of the surface M are said singular. The set of singular points, $\mathcal{C}$, is called criminant of M and its image, $\mathcal{D}$, via the application π, is called the discriminant. Note that if $q \in \mathcal{C}$ then $F(q) = F_p(q) = 0$. If $F_{pp}(q) \neq 0$ then q is a fold point of F, and if $F_{pp}(q) = 0$ and $F_{ppp}(q) \neq 0$ q it is a cusp point of F.

Consider the differential equation $p^2 = x$. In this case the surface M is a parabolic cylinder. The discriminant curve is the y-axis. In order to find the integral curves, we write down the conditions for dx, dy and dp at the point $q=(x,y,p)$ of the surface M :

$$\begin{cases} p^2 & = x, \quad \text{the condition } q \in M \\ 2pdp & = dx, \quad \text{the condition of tangence to } M \\ dy & = pdx, \quad \text{the condition of the contact plane} \end{cases}$$

Consequently, in coordinates (p,y), the integral curves are determined from the equation $dy = 2p^2 dp$.

Hence, the integral curves on M are given by the relations $y + C = \frac{2}{3}p^3$, $x = p^2$.

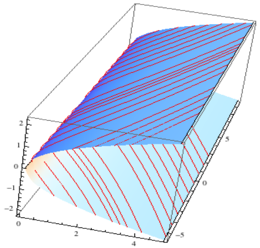


Figure 4: Integral curves on M.

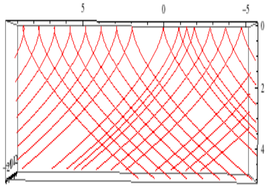


Figure 5: Projection of integral curves on the plane $(x{,}y)$.

Geometric singular perturbation theory

$\varepsilon_0 > 0$, $U \times V \subseteq \mathbb{R}^m \times \mathbb{R}^n$.

Slow System $\begin{cases} \varepsilon \dot{x} & = f(x, y, \varepsilon) \\ \dot{y} & = g(x, y, \varepsilon) \end{cases}$

with $(x, y, \varepsilon) \subseteq U \times V \times (-\varepsilon_0, \varepsilon_0)$, $f, g \in C^r, r \geq 1$.

Fast System $\begin{cases} x' & = f(x, y, \varepsilon) \\ y' & = \varepsilon g(x, y, \varepsilon) \end{cases}$

Reduced system: Put $\varepsilon = 0$ in the first equation!

$$\begin{cases} 0 & = f(x, y, \varepsilon) \\ \dot{y} & = g(x, y, \varepsilon) \end{cases}$$

The dimension problem is reduced : $m + n \to n$.

Another way to get a smaller dimension problem is considering $\varepsilon = 0$ in the second equation. In this case we get a problem with dimension m.

$$\begin{cases} x' & = f(x, y, \varepsilon) \\ y' & = 0 \end{cases}$$

The set

$$\mathcal{M}_0 = \{(x, y, 0) \in U \times V \times \{0\} : f(x, y, 0) = 0\}$$

is called slow manifold.

If the rank of $D_x f(x, y, 0)$ is m we have that $\mathcal{M}_0$ is a graphic $x = \psi(y)$ and the reduced system becomes

$$x = \psi(y), \quad \dot{y} = g(\psi(y), y, 0).$$

ε can be considered as an additional variable:

$$S_\varepsilon : \begin{cases} x' &= f(x,y,\varepsilon) \\ y' &= \varepsilon g(x,y,\varepsilon) \\ \varepsilon' &= 0. \end{cases}$$

$(p_0, 0) \in \mathcal{M}_0$. The linear part of the above system, with $\varepsilon = 0$, has the following matrix:

$$\begin{bmatrix} f_x & f_y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda = 0$ is the trivial eigenvalue with algebraic multiplicity $n+1$. The remaining eigenvalues are called non–trivial. The number of non-trivial eigenvalues with real part negative, zero or positive is denoted by $k^s, \quad k^c, \quad k^u$.

We say that p_0 is normally hyperbolic if all non–trivial eigenvalues are non–zero.

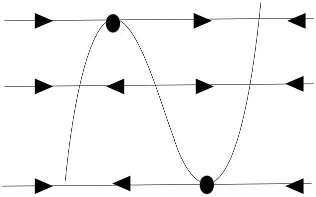


Figure 6: Fast flow and normally hyperbolic points.

Assuming normal hyperbolicity, Fenichel proved that all equilibrium points and invariant compact sets are preserved by singular perturbation. For example, suppose $m = 1, n = 2$, and reduced problem with a saddle occuring in a normally hyperbolic point p with $\frac{\partial f}{\partial x}(p) > 0$. Then for $\varepsilon \sim 0$ there exists an equilibrium point p_ε with stable dimension 1 and with unstable dimension 2 (one from the saddle and another from the fast flow)

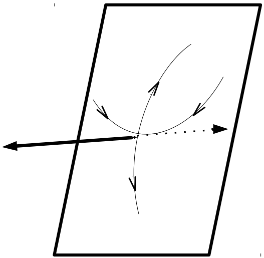


Figure 7: Fast and slow dynamics.

Theorem. (**Fenichel**) *Let $\mathcal{N}$ be a j-dimensional compact invariant manifold on the normally hyperbolic part of the slow manifold. Suppose that the stable and unstable manifolds of $\mathcal{N}$, with respect to the reduced system, have dimensions $j + j^s$ and $j + j^u$, respectively. Then there exists a family of invariant manifolds $\{\mathcal{N}_\varepsilon : \varepsilon \sim 0\}$ such that $\mathcal{N}_0 = \mathcal{N}$ and $\mathcal{N}_\varepsilon$ with stable and unstable manifolds with dimensions $(j + j^s + k^s)$ and $(j + j^u + k^u)$.*

Example. Fitzhugh–Nagumo equation.

$$\begin{cases} x_1' = x_2 \\ x_2' = cx_2 - f(x_1) + y \\ y' = \dfrac{\varepsilon}{c}(x_1 - \gamma y) \end{cases}$$

with $f(x_1) = x_1(x_1 - a)(x_1 - 1), 0 < a < \frac{1}{2}, c > 0$ and $0 < \varepsilon \ll 1$.
We have slow system

$$x_2 = 0, \quad y = f(x_1), \quad \dot{y} = \frac{1}{c}(x_1 - \gamma y).$$

Take γ and a such that the intersection of $x_1 = \gamma y$ with $y = f(x_1)$ occurs in three points: $0, P$ and Q. The reduced system has 5 equilibrium: $0, P, Q, S$ and I. The signal of $\dot{y}$ determines if the equilibrium are attracting or repelling .

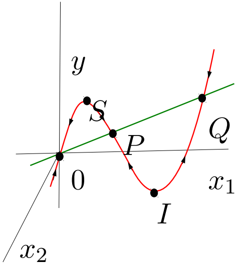


Figure 8: Slow flow of the Fitzhugh–Nagumo equation

The fast system is obtained taking $\varepsilon = 0$ in the original system

$$x_1' = x_2, \quad x_2' = cx_2 - f(x_1) + y, \quad y' = 0.$$

The jacobian matrix is

$$\begin{bmatrix} 0 & 1 & 0 \\ -f'(x_1) & c & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

with eigenvalues 0, $\dfrac{c + \sqrt{c^2 - 4f'(x_1)}}{2}$ and $\dfrac{c - \sqrt{c^2 - 4f'(x_1)}}{2}$.

There exist 5 equilibria but only 3 unfold for $\varepsilon > 0$:

$$0 = (0,0,0), \quad P = (p, 0, f(p)), \quad Q = (q, 0, f(q)).$$

The equilibra $0, P$ and Q are normally hyperbolic and

$$S = (s, 0, f(s)) \quad I = (i, 0, f(i))$$

don't. More precislly, $f'(x_1) = 0$ for $x_1 = s$ and $x_1 = i$ and thus the jacobian matrix has two zero eigenvalues.

Denote j^u and j^s the number of eigenvalues with positive and negative real parts, respectively, considering the slow system. Moreover denote by k^u and k^s the number of eigenvalues with positive and negative real parts, respectively, considering the fast system. Thus $0, P$ and Q satisfy

- $j^u(0) = 0, j^s(0) = 1, k^u(0) = 2, k^s(0) = 0;$
- $j^u(P) = 0, j^s(P) = 1, k^u(P) = 2, k^s(P) = 0;$
- $j^u(Q) = 0, j^s(Q) = 1, k^u(Q) = 1, k^s(Q) = 1$.

For $\varepsilon > 0$ the perturbations $0_\varepsilon = 0, P_\varepsilon = P$ and $Q_\varepsilon = Q$ satisfy

- the local stable manifold of 0_ε has dimension 1 and the local unstable manifold of 0_ε has dimension 2.
- the local stable manifold of P_ε has dimension 1 and the local unstable manifold of Q_ε has dimension 2.
- the local stable manifold of Q_ε has dimension 2 and the local unstable manifold of P_ε has dimension 1.

Theorem. $\exists\ \varepsilon_1 > 0$ *and a smooth function* $c = c(\varepsilon), \varepsilon \in (0, \varepsilon_1)$ *such that the Fitzhugh–Nagumu equation has a heteroclinic orbit connecting* 0_ε *and* Q_ε *for* $0 < \varepsilon < \varepsilon_1$.

We say singular orbit any orbit with three parts

- *one part in the stable manifold of p_1 of the reduced system;*
- *one part in the unstable manifold of p_2 of the reduced system;*
- *one orbit of the fast system connecting the two parts above.*

Denote $\mathcal{W}_1^u$ the unstable manifold of p_1, for the slow system. Denote $\mathcal{N}_1^u$ the unstable manifold, via fast flow (joining the orbits from $\mathcal{W}_1^u$). Anagously, we define $\mathcal{W}_2^s$ and $\mathcal{N}_2^s$.

Theorem (Szmolyan) *If p_1 and p_2 are normally hyperbolic and $\mathcal{N}_1^u \pitchfork \mathcal{N}_2^s$ then there exists one orbit of S_ε, $\varepsilon \sim 0$, connecting p_1^ε and p_2^ε.*

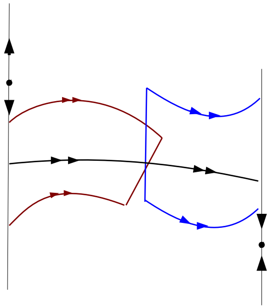


Figure 9: Singular orbit.

Dumortier and Roussarie studied, via GSP-theory, the canard phenomenon in Van der Pol's equation

$$\varepsilon\ddot{x} + (x^2 + x)\dot{x} + x - a = 0.$$

Essentially the phenomenon is the rapid growth of a limit cycle that was created in a Hopf bifurcation. Consider the change of variable

$$y = \varepsilon\dot{x} + \int_0^x (\xi + \xi^2)d\xi.$$

$$X_{\varepsilon,a} = \begin{cases} x' &= y - \dfrac{x^2}{2} - \dfrac{x^3}{3} \\ y' &= \varepsilon(a - x) \end{cases}$$

For $\varepsilon = 0$ the vector field $X_{0,a}$ has a curve $L = \{y = \dfrac{x^2}{2} + \dfrac{x^3}{3}\}$ of singularities and out of L the flow is horizontal. Excluding $n = (-1, \frac{1}{6})$ and $s = (0, 0)$, all singularities on L are normally hyperbolic. The bifurcation diagram of $X_{\varepsilon,a}$ with $\varepsilon > 0$: At $H = \{a = 0\}$ and $H' = \{a = -1\}$ occur Hopf bifurcations. Between H and H', $X_{\varepsilon,a}$ has an unstable singularity $(a, \dfrac{a^2}{2} + \dfrac{a^3}{3})$ and an attracting limit cycle $\Gamma_{\varepsilon,a}$ around it. Outside this region the system has a stable singularity $(a, \dfrac{a^2}{2} + \dfrac{a^3}{3})$.

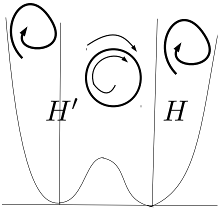


Figure 10: Bifurcation diagram of the Van der Pol's equation

Theorem. There exists a curve $C_0 = \{a = c_0(\varepsilon)\}$ with $c_0(\varepsilon) = \sqrt{\varepsilon}\overline{a}(\sqrt{\varepsilon})$ and $\overline{a} \in C^\infty$ with $\overline{a}'(0) = -1$ such that for a continuous curve $C = \{a = c(\varepsilon)\}$ with $c(\varepsilon) \leq 0$ and $c(0) \in [-\frac{1}{2}, 0]$ we have:

$$a)\lim_{\varepsilon\to 0} \Gamma_{\varepsilon,c(\varepsilon)} = \Gamma_0 \iff \text{for small } \varepsilon > 0 : c(\varepsilon) > c_0(\varepsilon), \quad \text{and}$$

$$\overline{\lim}(-\varepsilon \log(c(\varepsilon) - c_0(\varepsilon))) \leq 0;$$

$$b)\lim_{\varepsilon\to 0} \Gamma_{\varepsilon,c(\varepsilon)} = \Gamma_B \iff \text{for small } \varepsilon > 0 : c(\varepsilon) < c_0(\varepsilon), \quad \text{and}$$

$$\overline{\lim}(-\varepsilon \log(c_0(\varepsilon) - c(\varepsilon))) \leq 0.$$

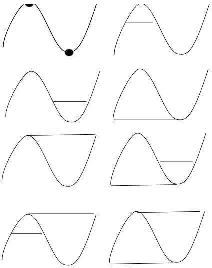


Figure 11: What happens with $\Gamma_{\varepsilon,a}$ when $\varepsilon \to 0$? The first is s and we denote Γ_0. The last is the big , Γ_B. The intermediaries are the canards

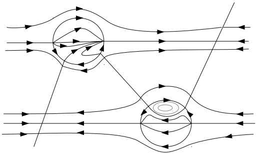


Figure 12: Blowing-up the non-normally hyperbolic points.

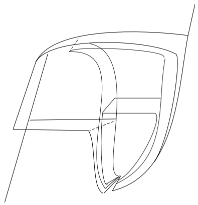


Figure 13: Saturing by the flow.

Flows defined by differential equations with discontinuous righthand side

From now on we will consider only discontinuous vector fields. We refer it also as Nonsmooth Dynamical System or Piecewise Smooth System.

One of the most important researchers on this subject is Teixeira. He introduced, joint with Sotomayor, a regularization process. Moreover, they made a systematic study, inspired by Peixoto's Theorem, of the structural stability of these systems.

To fix our ideas, we suppose that our equation is as follows

$$\dot{p} = X(p), \quad p \in \mathbb{R}^n$$

with switching on $\Sigma = \{F = 0\}$, 0 being a regular value of F.

- $\Sigma_+ = \{F > 0\} \quad \Sigma_- = \{F < 0\}$
- $X = X_+$ in Σ_+ and $X = X_-$ in Σ_-.

Sliding occurs when for any initial condition near Σ the corresponding solution trajectories are attracted to Σ.

Given $p \in \Sigma_i, i = +, -$, the orbit through p is formed by the orbit of X_i through p on Σ_i and if the orbit intersects Σ then we follow the Fillipov convention, whith the regions in Σ given by classified as:

- *Sliding Region*: $\Sigma^{sl} = \{p \in \Sigma : X_+.F < 0, X_-F > 0\}$. Any orbit which meets Σ^{sl} remains tangent to Σ for positive time.
- *Escaping Region*: $\Sigma^{es} = \{p \in \Sigma : X_+.F > 0, X_-F < 0\}$. Any orbit which meets Σ^{es} remains tangent to Σ for negative time.
- *Sewing Region*: $\Sigma^{sw} = \{p \in \Sigma : (X_+.F)(X_-F) > 0\}$. Any orbit which meets Σ^{sw} crosses Σ.

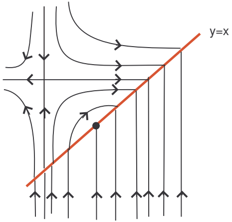


Figure 14: Flow of a discontinuous differential equations

On $\Sigma^{sl} \bigcup \Sigma^{es}$ the flow slides on Σ; it follows X^{Σ} called sliding vector field . The sliding is on the convex combination of X_+ and X_- and it is tangent to Σ.

Regularization of non–smooth systems

Example. $X^- = (x+1, -y+1)$ and $X^+ = (0,1)$ in $\mathbb{R}^2$. $X = X^-$ if $y > x$, and $X = X^+$ if $y < x$. On $\Sigma = \{(x,x); x \in \mathbb{R}\}$ X is bivaluated.

$(x,x) \in \Sigma$ with $x < 0$ are sewing points and $(x,x) \in \Sigma$ with $x > 0$ are sliding points.

Rotation of angle $\pi/4$:

$$Y^+ = \frac{\sqrt{2}}{2}(-1,1), \quad x > 0; \quad Y^- = \frac{\sqrt{2}}{2}(x+y, x-y+2), \quad x < 0.$$

$\varphi : \mathbb{R} \to (-1,1)$ given by $\varphi(s) = \frac{2}{\pi}\arctan(s)$ satifies $\varphi'(s) > 0$ for $s \in \mathbb{R}$ and $\lim\limits_{s\to\pm\infty} \varphi(s) = \pm 1$. On $\widetilde{\Sigma} = \{x = 0\}$ we apply the regularization

$$((\dot{x},\dot{y}),\dot{\varepsilon}) = (Y_\varepsilon, 0)$$

where

$$Y_\varepsilon = \frac{Y^+ + Y^-}{2} + \varphi\left(\frac{x}{\varepsilon}\right)\frac{Y^+ - Y^-}{2}.$$

With a blow up we get a singular perturbation problem

$$x = r\cos\theta, \quad \varepsilon = r\sin\theta, \quad r \geq 0, \theta \in [0,\pi].$$

$$\begin{aligned} r\dot{\theta} &= \tfrac{\sqrt{2}}{4}\sin\theta\left(1 - r\cos\theta - y + \varphi(\cot\theta)(1 + r\cos\theta + y)\right), \\ \dot{y} &= \tfrac{\sqrt{2}}{4}\left(3 + r\cos\theta - y + \varphi(\cot\theta)(-1 - r\cos\theta + y)\right). \end{aligned}$$

$\lambda(\theta) = \varphi(\cot\theta)$ is a decreasing continuous function connecting $(\theta, \lambda) = (0, 1)$ and $(\theta, \lambda) = (\pi, -1)$. With $r = 0$ in the first equation we get $\varphi(\cot\theta) = \frac{y-1}{y+1}$, connecting $(\theta, y) = (0, 0)$ and $(\theta, y) = (\pi, \infty)$. The slow flow is

$$\dot{y} = (3 - y) + \frac{y - 1}{y + 1}(y - 1) > 0.$$

The fast flow is $\theta' = \frac{\sqrt{2}}{4}\sin\theta \cdot \left(1 - y + \varphi(\cot\theta)(1 + y)\right).$

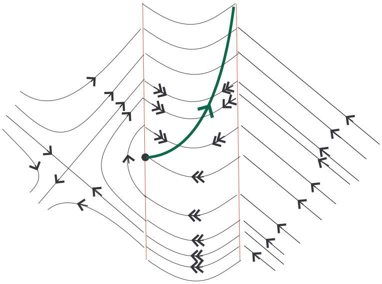


Figure 15: Phase portrait on the singular set.

Local analysis when discontinuities occur on an algebraic variety.

First case: regular switching. Consider $\dot{p} = X(p), p \in \mathbb{R}^3$ with

$$\Sigma = \{z = 0\}, \Sigma_+ = \{z > 0\}, \Sigma_- = \{z < 0\},$$

$$X_+ = (f_1, g_1, h_1), \quad X_- = (f_2, g_2, h_2)$$

X^Σ defined by

$$X^\Sigma = \frac{1}{h_1 - h_2}(h_1 f_2 - h_2 f_1, h_1 g_2 - h_2 g_1). \tag{2}$$

Theorem (Regular sliding) *There exists a singular perturbation problem*

$$r\dot{\theta} = \alpha(x, y, \theta, r), \quad \dot{x} = \beta(x, y, \theta, r), \quad \dot{y} = \gamma(x, y, \theta, r) \tag{3}$$

with $x, y \in \mathbb{R}, \quad \theta \in (0, \pi), \quad r \geq 0,$ *such that the slow manifold*

$$\mathcal{S} = \{\alpha(x, y, \theta, 0) = 0\}$$

and $\Sigma^{sl} \bigcup \Sigma^{es}$ *are homeomorphic and the reduced problem obtained considering* $r = 0$ *in* (3) *and the sliding vector field* (2) *are topologically equivalent.*

Proof.

- Step 1. A transition function $\varphi : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying that $\varphi(s) = 1$, for $s \geq 1$; $\varphi(s) = -1$ for $s \leq -1$ and $\varphi'(s) > 0$ $\forall s \in (-1, 1)$.
- Step 2. Regularization process:

$$X_\varepsilon = \left[\frac{1}{2} + \frac{1}{2}\varphi\left(\frac{z}{\varepsilon}\right)\right] X_+ \left[\frac{1}{2} - \frac{1}{2}\varphi\left(\frac{z}{\varepsilon}\right)\right] X_-.$$

- Step 3. Blow up

$$z = r\cos\theta, \quad \varepsilon = r\sin\theta, \quad r \geq 0, \quad \varepsilon \in [0, \pi].$$

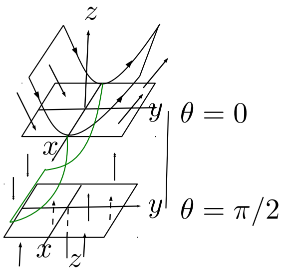


Figure 16: Slow-fast system. Double arrow represents fast flow. The green surface is the slow manifold which is homeomorphic to the sliding region Σ^{sl}.

The geometric interpretation of our result is as follows. By means of a polar blow up we may replace the discontinuity Σ by the cylinder $\mathbb{R}^2 \times [0, \pi]$. In this cylinder we draw the phase portrait of the fast-slow system (3), which is composed by a slow manifold and a vertical fast flow. On the slow manifold we have the phase portrait of the reduced system. The projection of the slow manifold on the surface Σ coincides with the sliding region and the reduced system has the same phase portrait as the sliding system.

The sliding vector field idealized by Filippov can not be uniquely extended for a self–intersecting switching manifold. However the following theorems say that for each double, triple, cone or Whitney descontinuity, we are able to, after a finite number of blow–ups, reduce the study to the regular case. Consequently we have a sliding vector field well defined.

Second case: critical switching.

- X defined in $\mathcal{U} \subset \mathbb{R}^3$ with switching set Σ.
- $\tilde{\mathcal{U}}$ neighborhood of $0 \in \mathbb{R}^3$ and a C^∞– diffeomorphism $\phi : \tilde{\mathcal{U}} \to \mathcal{U}$.
- ϕ induces $\tilde{X} = (\tilde{X_+}, \tilde{X_-})$ on $\tilde{\mathcal{U}}$ with $\tilde{X}_i(\tilde{p}) = d\phi^{-1} X_i(p), i = +, -$.

The ϕ-induced vector fields are determined by $\tilde{X_+}$ and $\tilde{X_-}$ on $\phi^{-1}(\Sigma_+)$ and on $\phi^{-1}(\Sigma_-)$, respectively. Besides, the switching manifold is $\tilde{\Sigma} = \phi^{-1}(\Sigma)$.

We restrict the degeneracy of the singularity so as to admit only those which appear when the regularity conditions in the definition of smooth surfaces of $\mathbb{R}^3$ in terms of implicit functions and immersions are broken in a stable manner. In this case Σ is locally diffeomorphic to one of the following sets

- $\mathcal{D} = \{(x, y, z) \in \mathbb{R}^3; xy = 0\}$ (double crossing);
- $\mathcal{T} = \{(x, y, z) \in \mathbb{R}^3; xyz = 0\}$ (triple crossing);
- $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3; z^2 - x^2 - y^2 = 0\}$ (cone) or
- $\mathcal{W} = \{(x, y, z) \in \mathbb{R}^3; zx^2 - y^2 = 0\}$ (Whitney's umbrella).

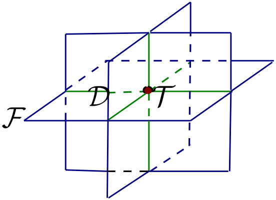


Figure 17: Regular (blue), double (green) and triple (bold) crossing switching points

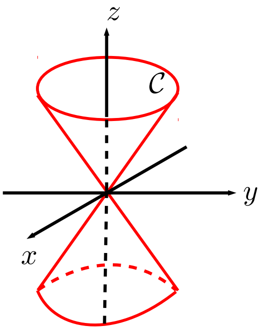


Figure 18: Cone ($\mathcal{C}$) switching manifold.

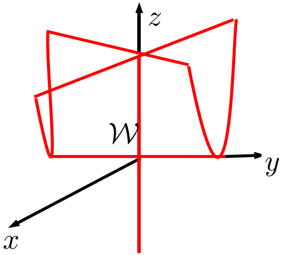


Figure 19: Whitney's umbrella ($\mathcal{W}$) switching manifold.

Theorem. *Suppose that* $\Sigma = \mathcal{D}$. *The map*

$$\phi : \mathbb{S}^1 \times [0, +\infty) \times \mathbb{R} \to \mathbb{R}^3$$

given by $\phi(\theta, r, z) = (r\cos\theta, r\sin\theta, z)$ *induces a vector field* $\tilde{X}$ *satisfying that any discontinuity* $q \in \tilde{\Sigma}$ *is regular.*

Theorem. *Suppose that* $\Sigma = \mathcal{T}$. *The map* $\phi : (0, \pi) \times (0, 2\pi) \times [0, +\infty) \to \mathbb{R}^3$ *given by*

$$\phi(q) = \phi(\theta_1, \theta_2, r) = (r\sin\theta_1\cos\theta_2, r\sin\theta_1\sin\theta_2, r\cos\theta_1)$$

induces a non–smooth vector field $\tilde{X}$ *satisfying that any discontinuity* $q \in \tilde{\Sigma}$ *is either regular or a double crossing.*

Theorem. *Suppose that* $\Sigma = \mathcal{C}$*. The map* $\phi : (0,\pi) \times (0,2\pi) \times [0,+\infty) \to \mathbb{R}^3$ *given by*

$$\phi(q) = \phi(\theta_1, \theta_2, r) = (r \sin\theta_1 \cos\theta_2, r \sin\theta_1 \sin\theta_2, r\cos\theta_1)$$

induces a non–smooth vector field $\tilde{X}$ *satisfying that any discontinuity* $q \in \tilde{\Sigma}$ *with* $\phi(q) \neq 0$ *is regular. Moreover the switching manifold on* $(0,\pi) \times (0,2\pi) \times \{0\}$ *is homeomorphic to two non–intersecting circles.*

Theorem. *Suppose that* $\Sigma = \mathcal{W}$*. The map*

$$\phi : \mathbb{R} \setminus \{0\} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$$

given by

$$\phi(u, v, w) = (u, uv, w)$$

induces a non–smooth vector field $\tilde{X}$ satisfying that any discontinuity $q \in \tilde{\Sigma}$ with $u \neq 0$ is regular. Moreover, if $q \in \tilde{\Sigma}$ is a discontinuity with $u^2 + w^2 = 0$ then q is a double crossing.

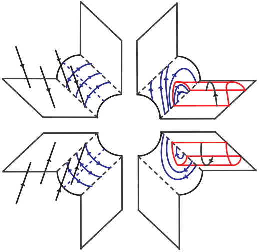


Figure 20: Switching curves on S^2 after blow up - case $\mathcal{D}$.

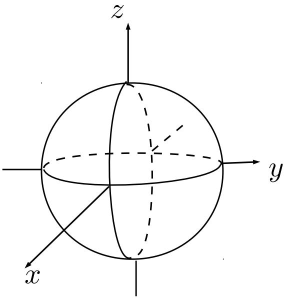


Figure 21: Switching curves on S^2 after blow up - case $\mathcal{T}$.

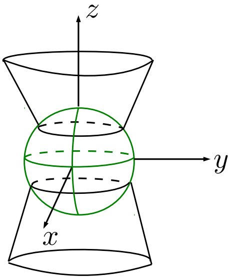


Figure 22: Switching manifold after blow up - case $\mathcal{C}$.

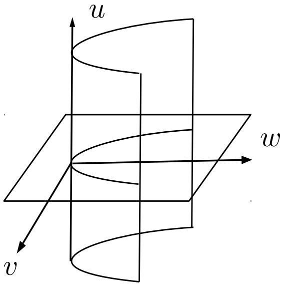


Figure 23: Switching manifold after blow up - case $\mathcal{W}$.

Local analysis considering non–regular regularization (M, Σ, X) is a PSVF with M being an open set on $\mathbb{R}^n$, $\Sigma = h^{-1}(0)$ for some C^r $h : M \to \mathbb{R}$.
$[X_-(p), X_+(p)]$ is the convex combination of $X_-(p)$ and $X_+(p)$:

$$[X_-, X_+] = \{\left(\frac{1}{2} + \frac{\lambda}{2}\right) X_+ + \left(\frac{1}{2} - \frac{\lambda}{2}\right) X_- : \quad \lambda \in [-1, 1]\}.$$

A continuous combination of X_- and X_+ is a 1–parameter family of vector fields $\widetilde{X}(\lambda, p)$, $C^r, r \geq 1$, with $(\lambda, p) \in [-1, 1] \times M,$ and satisfying that $\widetilde{X}(-1, p) = X_-(p), \quad \widetilde{X}(1, p) = X_+(p)$. We denote

$$[X_-, X_+]^c = \{\widetilde{X}(\lambda, p), \lambda \in [-1, 1]\}.$$

Let X^ε be a regularization of X.

(a) We say that X^ε is of the kind Filippov if $X^\varepsilon(p) \in [X_-(p), X_+(p)]$, for any $p \in M$.

(b) We say that X^ε is of the kind Nonlinear if there exists a continuous combination such that $[X_-, X_+]^c \neq [X_-, X_+]$ and $X^\varepsilon(p) \in [X_-(p), X_+(p)]^c$, for any $p \in M$.

(a) We say that p is a *c–sewing* point and denote $p \in \Sigma_c^{sw}$ if

$$(\widetilde{X}.h)(p) \neq 0, \quad \forall \lambda \in [-1,1].$$

(b) We say that p is a *c–slidding* point and denote $p \in \Sigma_c^{sl}$ if

$$\exists \lambda \in [-1,1], \quad (\widetilde{X}.h)(p) = 0.$$

Proposition 1. *$\Sigma_c^{sw} \subseteq \Sigma^{sw}$ and $\Sigma^{sl} \subseteq \Sigma_c^{sl}$.*

For each $p \in \Sigma_c^{sl}$ there exists $\lambda(p) \in [-1,1]$ such that $(\widetilde{X}.h)(p) = 0$. We say that $\widetilde{X}(\lambda(p), p)$ is a *c–sliding vector field* .

Example. $h(x, y) = y$, $X_+ = (1, 1-x)$ and $X_- = (1, 3-x)$.

$$\widetilde{X} = (-1 + 2\lambda^2, -x - \lambda + 2\lambda^2).$$

Two possible sliding vector fields

$$X^{\lambda_1} = \left(\frac{-3 + 4x + \sqrt{1+8x}}{4}, 0\right) \quad X^{\lambda_2} = \left(\frac{-3 + 4x - \sqrt{1+8x}}{4}, 0\right).$$

$$\Sigma^{sl} = (1, 3), \quad \Sigma^{sl}_c = (-\frac{1}{8}, 1) \cup (1, 3).$$

In $(-\frac{1}{8}, 1)$ are defined two c–sliding vector fields (X^{λ_1} and X^{λ_2}) and on $(1, 3)$ is defined only X^{λ_2}.

A nonlinear regularization of X_- and X_+ is the 1–parameter family

$$X^\varepsilon = \widetilde{X}(\varphi\left(\frac{h}{\varepsilon}\right), p).$$

Note that if $h > \varepsilon$ then $\varphi\left(\frac{h}{\varepsilon}\right) = 1$ and $X^\varepsilon = X_+$; and if $h < -\varepsilon$ then $\varphi\left(\frac{h}{\varepsilon}\right) = -1$ and $X^\varepsilon = X_-$. The regularization of the the kind nonlinear does not depend of the transition function considered.

Teorema 2. *There exists a singular perturbation problem*

$$r\dot{\theta} = \alpha(r,\theta,p), \quad \dot{p} = \beta(r,\theta,p), \tag{4}$$

with $r \geq 0, \theta \in [0,\pi], p \in \Sigma_c^{sl}$ and slow manifold $\mathcal{S}$ satisfying that the following.

(a) For any $p \in \Sigma_c^{sl}$ there exist homeomorphic neighborhoods $p \in I_p$ and $\mathcal{S}_p \subset \mathcal{S}$. A sliding vector field $\widetilde{X}(\lambda(p),p)$ is defined in I_p and it is $C^r - equivalent$ to the slow flow on $\mathcal{S}_p \subset \mathcal{S}$.

(b) For any $p \in \Sigma_c^{sl}$ consider $\ell = \#\{\theta \in (0,\pi) : (\theta,p) \in \mathcal{S}\}$. There exist ℓ choices of sliding vector fields defined in p.

(c) Iff all points on $\mathcal{S}$ are normally hyperbolic and $\mathcal{S}$ has only one conneted component then there exists only one choice for the sliding vector field in Σ_c^s.

Example. $X = (X_+, X_-)$ as in previous Example.

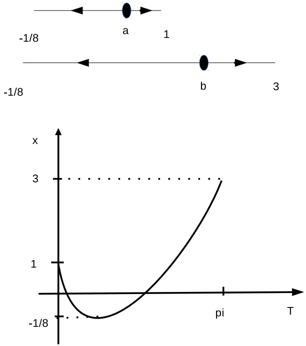
$$X^\varepsilon = \widetilde{X}(\varphi\left(\frac{y}{\varepsilon}\right), x, y).$$

The trajectories of X^ε satisfies the system

$$x' = -1 + 2\varphi\left(\frac{y}{\varepsilon}\right)^2, \quad y' = -x - \varphi\left(\frac{y}{\varepsilon}\right) + 2\varphi\left(\frac{y}{\varepsilon}\right)^2.$$

Consider the blow up $y = r\cos\theta$ and $\varepsilon = r\sin\theta$ with $r \geq 0$ and $\theta \in [0, \pi]$. Thus, denoting $\psi(\theta) = \varphi(\cot\theta)$, the system becomes

$$r\dot{\theta} = -x - \psi(\theta) + 2\psi(\theta)^2, \quad \dot{x} = -1 + 2\psi(\theta)^2.$$



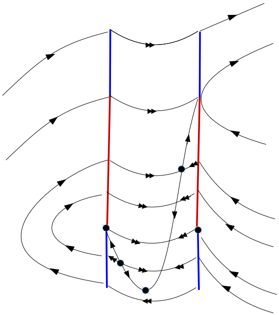


Figure 24: Sewing (blue) and sliding (red) regions.

The slow manifold is given by

$$x = -\psi(\theta)(2\psi(\theta) - 1).$$

x is a smooth curve connecting $(\theta, x) = (0, 1)$ and $(\theta, x) = (\pi, 3)$. $x' = \psi'(4\psi - 1)$ and it is zero if $\psi = \frac{1}{4}$. In this case $x = -\frac{1}{8}$. The slow flow is given by

$$x' = \frac{-3 + 4x \pm \sqrt{1 + 8x}}{4}$$

which is exactly the same expression of the sliding X^{λ_1} and X^{λ_2}.

Global regularization

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