Regularization of Discontinuous Foliations: Blowing up and Sliding Conditions via Fenichel Theory

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1-dimensional oriented foliation

Let M be a smooth manifold (possibly with corners) and a non-flat smooth vector field defined on M.
(V, Y), where $V \subset M$ is an open set Y is a smooth vector field in V is a local vector field in M.
A 1-dimensional oriented foliation on M is a collection

$$\mathcal{F} = \{(U_i, X_i)\}_{i \in I}$$

of local vector fields such that

- $\{U_i\}$ is an open covering of M and
- For each pair $i, j \in I$, $X_i = \varphi_{ij} X_j$ for some strictly positive smooth function φ_{ij} defined on $U_i \cap U_j$.

Local generator of the foliation

We say that a local vector field (V, Y) is a generator of the foliation $\mathcal{F}$ if the augmented collection

$$\{(U_i, X_i)\}_{i\in I} \cup \{(V, Y)\}$$

also satisfies conditions 1. and 2. Let $\Phi : \widetilde{M} \to M$ be a smooth diffeomorphism from $\widetilde{M}$ to M. We say that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ defined respectively in M and $\widetilde{M}$ are related by Φ if for each local vector field $(\widetilde{U}, \widetilde{X})$ which is a generator of $\widetilde{\mathcal{F}}$, the push-forward of this local vector field under Φ, namely

$$(V, Y) = \left(\Phi(\widetilde{U}), \Phi_*\widetilde{X}\right),$$

is a generator of $\mathcal{F}$.

Discontinuous 1-foliation

A discontinuous 1-foliation on a manifold M is given by closed subset $\Sigma \subset M$ with empty interior and a 1-dimensional oriented foliation $\mathcal{F}$ defined in $M \setminus \Sigma$.
The set Σ is called the discontinuity locus of $\mathcal{F}$. We can write the decomposition

$$\Sigma = \Sigma^{\text{smooth}} \cup \Sigma^{\text{sing}}$$

where Σ^{smooth} denotes the subset of points where Σ locally coincides with a submanifold of M. We shall say that $\mathcal{F}$ has a smooth discontinuity locus if $\Sigma = \Sigma^{\text{smoth}}$.

Foliation blow-up smoothable

We will say that discontinuous 1-foliation $\mathcal{F}$ defined in M and with discontinuity locus Σ is blow-up smoothable if there exists a finite sequence of blowing-ups

$$M = M_0 \longleftarrow \cdots \longleftarrow M_k = \widetilde{M}$$

and a smooth 1-foliation $\widetilde{\mathcal{F}}$ defined in $\widetilde{M}$ such that:

1. The map $\Phi : \widetilde{M} \to M$ is a diffeomorphism outside Σ, and
2. $\widetilde{\mathcal{F}}$ and $\mathcal{F}$ are related by Φ, seen as a map from $\widetilde{M} \setminus \Phi^{-1}(\Sigma)$ to $M \setminus \Sigma$.

Piecewise smooth 1-foliations

Let $\mathcal{F}$ be a discontinuous 1-foliation defined on a manifold M and with discontinuity locus Σ. A local multi-generator of $\mathcal{F}$ is a pair $(U, \{X_1, \dots, X_k\})$ satisfying the following conditions:

1. We can write $U \setminus \Sigma$ as a finite disjoint union
$$U \setminus \Sigma = U_1 \sqcup \cdots \sqcup U_k \tag{1}$$
of open sets $U_1, .., U_k$.
2. For each $i = 1, .., k$, X_i is a smooth vector field defined in U and such that
$$(U_i, X_i|_{U_i})\text{is a local generator of } \mathcal{F}. \tag{2}$$

We will say that $\mathcal{F}$ is piecewise smooth if there exists a collection $\mathcal{C}$ of local multi-generators as above whose domain forms an open covering of Σ such that the following condition holds: For each two local multi-generators

$$(U, \{X_1, \ldots, X_k\}), \quad (V, \{Y_1, \ldots, Y_l\})$$

belonging to $\mathcal{C}$, there exists a strictly positive smooth function φ defined in $U \cap V$ such that

$$X_i = \varphi\, Y_j \quad \text{on} \quad U_i \cap V_j \tag{3}$$

for each pair of indices $i = 1, \ldots, k$ and $j = 1, \ldots, l$.

Theorem

Let $\mathcal{F}$ be a piecewise smooth 1-foliation on a manifold M whose discontinuity locus Σ is an smooth submanifold of codimension one. Then, $\mathcal{F}$ is blow-up smoothable.

We will say that $\mathcal{F}$ has an analytic discontinuity locus if the ambient space M is an analytic manifold and the discontinuous locus Σ is an analytic subset of M. In other words, we assume that Σ is locally defined (at each point of M) as the vanishing locus of an analytic function.

Under these conditions, it follows that the singular part Σ^{sing} of Σ is a closed analytic subset, which moreover lies in the closure of the smooth part Σ^{smooth}.

From the Theorem of Resolution of Singularities we conclude that there exists an analytic proper map $\Phi : N \to M$ defined by a finite sequence of blowing-ups such that

1. Φ is a diffeomorphism outside Σ^{sing}.
2. $D = \Phi^{-1}(\Sigma^{\text{sing}})$ is a finite union of boundary components
$$D_1, \cdots, D_k \subset \partial N$$
of codimension one.
3. The closure of $\Phi^{-1}(\Sigma^{\text{smooth}})$ is a smooth submanifold $\Omega \subset N$.

The next result states that, under the above conditions, the foliation $\mathcal{F}$ pulls-back to a discontinuous foliation in N which has a smooth discontinuity locus.

Theorem

Let $\mathcal{F}$ is a piecewise smooth 1-foliation with analytic discontinuity locus. Then, using the above notation, there is a piecewise smooth 1-foliation $\mathcal{G}$ defined in N, which is related to $\mathcal{F}$ by Φ, and whose discontinuity locus is Ω .

Corollary

Under the assumptions of the Theorem 2, suppose further that the discontinuity locus of $\mathcal{F}$ has codimension one. Then, $\mathcal{F}$ is blow-up smoothable.

Regularization of discontinuous 1-foliation

Let $\mathcal{F}$ be a discontinuous 1-foliation on a manifold M, with discontinuity locus Σ. A regularization of $\mathcal{F}$ (with p-parameters) is a discontinuous 1-foliation $\mathcal{F}^{\mathbf{r}}$ defined in the product manifold

$$M \times (\mathbb{R}^p, 0)$$

which satisfies the three following conditions:

1. $\mathcal{F}^{\mathbf{r}}$ is tangent to the fibers of the canonical projection
$$\pi : M \times (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$$
2. The restriction $\mathcal{F}^{\mathbf{r}}_0$ of $\mathcal{F}^{\mathbf{r}}$ to the fiber $\pi^{-1}(0)$ coincides with $\mathcal{F}$,
3. The discontinuity locus $\Sigma^{\mathbf{r}}$ of $\mathcal{F}^{\mathbf{r}}$ is a subset of $\Sigma \times \left\{ \prod_i \varepsilon_i = 0 \right\}$, where $(\varepsilon_1, .., \varepsilon_p)$ are the coordinates in $(\mathbb{R}^p, 0)$.

Suppose that the discontinuity locus of $\mathcal{F}$ is a smooth submanifold $\Sigma \subset M$ of codimension one.

1. $f : N\Sigma \to M$, which maps $N\Sigma$ diffeomorphically to an open neighborhood $W = f(N\Sigma)$ of Σ;
2. A smoothly varying metric $|\cdot|$ on the fibers of the bundle $N\Sigma \to \Sigma$ (such that $|p| = 0$ iff $p \in \Sigma$).
3. A monotone transition function $\phi : \mathbb{R}^+ \to [-1, 1]$

Using the map f, we pull-back $\mathcal{F}$ to a discontinuous 1-foliation $\mathcal{G}$ on $N\Sigma$, with discontinuity locus given by the zero section $\Sigma \subset N\Sigma$.

Without loss of generality, we can assume that $N\Sigma$ is covered by local charts

$$\begin{array}{ccc} V \times \mathbb{R} & \longrightarrow & V \\ (x, y) & \longmapsto & x \end{array}$$

for some open set $V \subset \Sigma$, and that $\mathcal{F}$ has a local multi-generator of the form $(V \times \mathbb{R}, \{X_+, X_-\})$, where X_+ (resp. X_-) is a smooth vector field in $V \times \mathbb{R}$ which generate $\mathcal{G}$ on $U_+ = \{y > 0\}$ (resp. $U_- = \{y < 0\}$).
For each $\varepsilon > 0$, we now define a smooth vector field X_ε in $V \times \mathbb{R}$ as follows

$$X_\varepsilon \stackrel{\text{def}}{=} \frac{1}{2}\left(1 + \phi\left(\frac{y}{\varepsilon}\right)\right) X_+ + \frac{1}{2}\left(1 - \phi\left(\frac{y}{\varepsilon}\right)\right) X_-$$

Notice that, by construction

$$X_\varepsilon(x,y) = \begin{cases} X_+(x,y) & y \geq \varepsilon, \\ X_-(x,y) & y \leq -\varepsilon, \end{cases}$$

Moreover, if we choose another multi-generator of $\mathcal{G}$, say $(V \times \mathbb{R}, \{Y_+, Y_-\})$ then we define a family Y_ε exactly as above but replacing $X_\pm$ by $Y_\pm$, we conclude that $Y_\varepsilon = \varphi X_\varepsilon$, $\forall \varepsilon > 0$. In other words, the X_ε and Y_ε define precisely a same smooth 1-foliation in the domain $V \times \mathbb{R}$.

We define, for each $\varepsilon > 0$, a smooth foliation $\mathcal{G}_\varepsilon$. By construction, such foliation coincides which the original foliation $\mathcal{G}$ outside the region $\{p \in N\Sigma : |p| < \varepsilon\}$.

The Sotomayor-Teixeira regularization of $\mathcal{F}$ is the discontinuous 1-foliation $\mathcal{F}^{\mathbf{r}}$ defined in the product space $M \times (\mathbb{R}^+, 0)$ as follows: For $\varepsilon = 0$, we let $\mathcal{F}_0^{\mathbf{r}} = \mathcal{F}$. For $\varepsilon > 0$, we consider the foliation in M given by

$$\mathcal{F}_\varepsilon^{\mathbf{r}} = \begin{cases} \mathcal{F} & \text{on } M \setminus W \\ f_*\mathcal{G}_\varepsilon & \text{on } W \end{cases}$$

This defines a globally smooth 1-foliation in M.

Regularization of transition type

More generally, under the same assumptions of the previous example, we can define regularization of $\mathcal{F}$ by dropping the assumption of monotonicity and x-independence of the transition function. Namely, by replacing the choice of function ϕ in item 3. by the choice of a smooth function

$$\psi : \Sigma \times \mathbb{R}_+ \to [-1, 1] \tag{4}$$

such that $\psi(x, t) = -1$ if $t \leq -1$ and $\psi(x, t) = 1$ if $t \geq 1$. Correspondingly, we replace the expression of X_ε given above by

$$X_\varepsilon \stackrel{\text{def}}{=} \frac{1}{2}\left(1 + \psi\left(x, \frac{y}{\varepsilon}\right)\right) X_+ + \frac{1}{2}\left(1 - \psi\left(x, \frac{y}{\varepsilon}\right)\right) X_- \tag{5}$$

The resulting regularization will be called a regularization of transition type.

Sliding regions

Let $\mathcal{F}$ be a discontinuous 1-foliation defined on a manifold M and with discontinuity locus Σ. Consider a p-parameter regularization $\mathcal{F}^{\mathbf{r}}$ of $\mathcal{F}$. We will say that point $p \in \Sigma$ is a point of sliding for $\mathcal{F}^{\mathbf{r}}$ if there exists an open neighborhood $U \subset M$ of p and a family of smooth manifolds

$$S_\varepsilon \subset U$$

defined for all $\varepsilon \in ((\mathbb{R}^\star)^p, 0)$ such that:

1. For each ε, S_ε is invariant by the restriction of $\mathcal{F}^{\mathbf{r}}_\varepsilon$ to U.
2. For each compact subset $K \subset U$, the sequence $S_\varepsilon \cap K$ converges to $\Sigma \cap K$ as ε goes to zero for some given Hausdorff metric d_H on compact sets of M

The set of sliding points for $\mathcal{F}^{\mathbf{r}}$ is a relatively open subset of Σ, which we denote by $\mathrm{Slide}(\mathcal{F}^{\mathbf{r}})$.
Assume that the discontinuity locus Σ is an analytic set of dimension d. Then, we can define a more refined notion of sliding by considering different strata of Σ
More precisely, under the above hypothesis, there exists an unique filtration by analytic sets

$$\Sigma^0 \subset \Sigma^1 \subset \cdots \subset \Sigma^d = \Sigma$$

where, for each $k = 1, \ldots, d$, the set $\Sigma^k \setminus \Sigma^{k-1}$ is a smooth manifold of dimension k.

Using this decomposition, we say that point $p \in \Sigma^k \setminus \Sigma^{k-1}$ is a stratified point of sliding for $\mathcal{F}^{\mathbf{r}}$ if the conditions 1. and 2. of the above definition holds, when we replace the convergence condition in 2. by

$$d_H(S_\varepsilon \cap K, \Sigma^k \cap K) \to 0$$

as ε goes to zero. The set of all points $\Sigma^k \setminus \Sigma^{k-1}$ satisfying the above condition is called sliding region of dimension k, and denoted by $\mathrm{Slide}^k(\mathcal{F}^{\mathbf{r}})$.

Regularizations of transition type: blowing-up and conditions for sliding

In this section, we consider piecewise smooth 1-foliations whose discontinuity set are smooth submanifold of codimension one. Our main goal is to describe conditions which guarantee that a point belongs to the sliding region of given a regularization of transition type.

To fix the notation, we choose a piecewise smooth 1-foliation $\mathcal{F}$ defined in a manifold M, and whose discontinuity locus is a smooth submanifold $\Sigma \subset M$ of codimension one. According to the definition at each point $p \in \Sigma$ we can choose local coordinates $(x_1, \ldots, x_{n-1}, y)$ and two smooth vector fields X_+ and X_- such that $\Sigma = \{y = 0\}$ and X_+ and X_- are generators of $\mathcal{F}$ on the sets $\{y > 0\}$ and $\{y < 0\}$, respectively.
First of all, we prove a result which will allow to use some powerful tools from the theory of smooth dynamical systems to study such regularization.

Theorem

Let $\mathcal{F}^{\mathbf{r}}$ be a regularization of transition type of $\mathcal{F}$. Then, $\mathcal{F}^{\mathbf{r}}$ is blow-up smoothable.

We will show that a single blowing-up suffices to obtain a smooth foliation. More precisely, consider the blowing up

$$\Phi : N \to M \times (\mathbb{R}^{+}, 0)$$

with center on Σ. We claim that there exists a smooth foliation in $\mathcal{G}$ in N which is related to $\mathcal{F}^{\mathbf{r}}$ by Φ.

Now, we will study the sliding regions. The criterion that we are going to describe needs one additional definition: Using the notation introduced above, the height function of $\mathcal{F}^{\mathbf{r}}$ is the smooth function $h^{\mathbf{r}}$ with domain $(x, t) \in \Sigma \times \mathbb{R}$ defined by

$$h^{\mathbf{r}} = \psi \, \mathcal{L}_{(X_+ - X_-)}(y) + \mathcal{L}_{(X_+ + X_-)}(y)$$

where $\psi(x, t)$ is the transition function and $\mathcal{L}_X(f)$ denotes the Lie derivative of the function f with respect to the vector field X. We remark that that the Lie derivative of $X_+ - X_-$ and $X_+ + X_-$ needs to be evaluated only at points of Σ.

More explicitly, if we write X_+ and X_- in terms of the local trivializing coordinates (x, y) described above as

$$X_\pm = a_\pm \frac{\partial}{\partial y} + \sum_{i=1}^{n-1} b_{i,\pm} \frac{\partial}{\partial x_i}$$

(for some smooth functions $a_\pm$ and $b_{i,\pm}$) then the height function is given by

$$h^{\mathbf{r}}(x,t) = \psi(x,t)\left(a_+(x,0) - a_-(x,0)\right) + \left(a_+(x,0) + a_-(x,0)\right).$$

Based on this function, define the following subsets in $\Sigma \times \mathbb{R}$:

$$\mathbf{Z^r} = \left\{ h^{\mathbf{r}}(x,t) = 0 \right\}$$

$$\mathbf{W^r} = \left\{ \frac{\partial h^{\mathbf{r}}}{\partial t}(x,t) \neq 0 \right\}$$

$$\mathbf{NH^r} = \mathbf{Z^r} \cap \mathbf{W^r}$$

Theorem

Let $\mathcal{F}^{\mathbf{r}}$ be a regularization of transition type of $\mathcal{F}$, defined by a transition function ψ as above. Then,

$$\pi(\mathbf{NH}^{\mathbf{r}}) \subset \mathrm{Slide}(\mathcal{F}^{\mathbf{r}}) \subset \pi(\mathbf{Z}^{\mathbf{r}}).$$

where $\pi : \Sigma \times \mathbb{R} \to \Sigma$ is the canonical projection.

Let us now describe the behavior of a regularization in the complement of the sliding set. For this, we introduce the so-called sewing region.

Keeping the above notation, we will say that a point $p \in \Sigma$ is a point of sewing for the regularization $\mathcal{F}^{\mathbf{r}}$ if there exists an open neighborhood $U \subset M$ of p and local coordinates (x, y) defined in U such that

1. $\Sigma = \{y = 0\}$ and,
2. For each sufficiently small $\varepsilon > 0$, the vertical vector field $\frac{\partial}{\partial y}$ is a generator of $\mathcal{F}^{\mathbf{r}}_{\varepsilon}$ in U.

We will denote the set of all sewing points by $\mathrm{Sew}(\mathcal{F}^{\mathbf{r}})$.

Theorem

Let $\mathcal{F}^{\mathbf{r}}$ be a regularization of transition type of $\mathcal{F}$, defined by a transition function ψ. Then,

$$\pi(\mathbf{Z}^{\mathbf{r}})^{\complement} \subset \mathrm{Sew}(\mathcal{F}^{\mathbf{r}})$$

where $\pi(\mathbf{Z}^{\mathbf{r}})^{\complement}$ denotes the complement of $\pi(\mathbf{Z}^{\mathbf{r}})$ in Σ.