

ME 414 - Gabarito da Lista 3 ¹

1. a. Sabemos que $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^0 0dx + \int_0^1 kx^2 dx + \int_1^{\infty} 0dx \\ &= \int_0^1 kx^2 dx = k \int_0^1 x^2 dx = k \frac{x^3}{3} \Big|_0^1 \\ &= k \left(\frac{1}{3} \right) = 1 \\ \Rightarrow k &= 3\end{aligned}$$

b.

$$\begin{aligned}P(1/4 < X < 1/2) &= \int_{1/4}^{1/2} 3x^2 dx = 3 \left(\frac{x^3}{3} \right) \Big|_{1/4}^{1/2} \\ &= \left(\frac{1}{2^3} - \frac{1}{4^3} \right) = 0.1093.\end{aligned}$$

c.

$$\begin{aligned}E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ E(X) &= \int_0^1 x(3x^2)dx = 3 \int_0^1 x^3 dx \\ E(X) &= 3 \left(\frac{x^4}{4} \right) \Big|_0^1 = 3/4 = 0.75.\end{aligned}$$

Para a variança utilizamos

$$V(X) = E(X^2) - E(X)^2.$$

Vamos a calcular $E(X^2)$

$$\begin{aligned}E(X^2) &= \int_0^1 x^2 f(x)dx = \int_0^1 x^2 (3x^2) dx \\ E(X^2) &= 3 \int_0^1 x^4 dx \\ E(X^2) &= 3 \left(\frac{x^5}{5} \right) \Big|_0^1 \\ E(X^2) &= 3/5.\end{aligned}$$

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Assim

$$V(X) = E(X^2) - E(X)^2 = 3/5 - 0.75^2 = 0.0375$$

2. Seja $X \sim N(5, 16)$ assim $\mu = 5$ e $\sigma = 4$

a.

$$\begin{aligned} P(X \leq 13) &= P\left(Z \leq \frac{13 - \mu}{\sigma}\right) \\ &= P\left(Z \leq \frac{8}{4}\right) = P(Z \leq 2) \\ &= 0.977 \end{aligned}$$

b.

$$\begin{aligned} P(X > 1) &= P\left(Z > \frac{1 - \mu}{\sigma}\right) = P\left(Z > \frac{-4}{4}\right) \\ &= P(Z > -1) = P(Z \leq 1) = 0.8413 \end{aligned}$$

c.

$$\begin{aligned} P(X \leq a) &= P\left(Z \leq \frac{a - \mu}{\sigma}\right) = P\left(Z \leq \frac{a - 5}{4}\right) = 0.04 \\ &\Rightarrow \phi\left(\frac{a - 5}{4}\right) = 0.04 \\ &\Rightarrow \frac{a - 5}{4} = -1.75 \\ &\Rightarrow a = 4(-1.75) + 5 = -2 \end{aligned}$$

3.

a. Mostraremos que $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} \frac{xe^{-x/2}}{4} dx.$$

Para demostrar isso utilizaremos integração por partes.

$$\begin{aligned}
u = x \Rightarrow du = dx \text{ e } dv = e^{-x/2}dx \Rightarrow v = -2e^{-x/2} \\
\int_0^\infty f(x)dx &= uv - \int_0^\infty vdu \\
&= \frac{1}{4} \left\{ x(-2e^{-x/2}) - \int_0^\infty -2e^{-x/2}dx \right\} \\
&= \frac{1}{4} \left\{ x(-2e^{-x/2}) + 2 \int_0^\infty e^{-x/2}dx \right\} \\
&= \frac{1}{4} \left\{ x(-2e^{-x/2}) + 2(-2e^{-x/2}) \right\} \Big|_0^\infty \\
&= 1
\end{aligned}$$

note que para substituir os valores é preciso saber $e^{-\infty/2} = \frac{1}{e^{\infty/2}} \rightarrow 0$.

b. 6 meses equivale a meio ano (1/2)

$$\begin{aligned}
p(X \leq 1/2) &= \int_0^{1/2} f(x)dx = \int_0^{1/2} \frac{xe^{-x/2}}{4}dx \\
&= \frac{1}{4} \left(-2xe^{-x/2} - 4e^{-x/2} \right) \Big|_0^{1/2} \\
&= \frac{1}{4} (-5e^{-1/4} + 4) = 2.65\%
\end{aligned}$$

4. X: Peso de um determinado produto tal que $X \sim N(\mu, 20^2 g^2)$

a.

$$\begin{aligned}
P(X < 500) &= P(Z < \frac{500 - \mu}{20}) = 0.01 \\
&\Rightarrow \phi\left(\frac{500 - \mu}{20}\right) = 0.01 \\
&\Rightarrow \left(\frac{500 - \mu}{20}\right) = -1.28 \\
&\Rightarrow \mu = 525.6
\end{aligned}$$

b. Sejam X_1, X_2, X_3, X_4 pesos dos 4 pacotes da amostra.

Estão pedindo a probabilidade do peso total menor que 2kg, isto é, $\sum_{i=1}^4 X_i < 2000g$ e é o mesmo que, $\bar{X} < \frac{2000}{4}$, assim a média amostral

$$\bar{X} \sim N\left(525.6g, \frac{20^2}{4}g^2\right) = N\left(525.6g, \left(\frac{20}{2}g\right)^2\right)$$

então

$$\begin{aligned} P(\bar{X} < 500) &= P\left(Z < \frac{500 - 525.6}{10}\right) \\ &= P(Z < -2.56) = 0.0052 \end{aligned}$$

5. a.

$$\bar{X} \sim N\left(525.6g, \left(\frac{20}{2}g\right)^2\right)$$

A probabilidade de ser feita uma parada desnecessária é dada por:

$$\begin{aligned} P(\bar{X} < 495.6 \cup \bar{X} > 555.6) &= P(\bar{X} < 495.6) + P(\bar{X} > 555.6) \\ &= P\left(Z < \frac{495.6 - 525.6}{10}\right) + P\left(Z > \frac{555.6 - 525.6}{10}\right) \\ &= P(Z < -3) + P(Z > 3) \\ &= 2P(Z > 3) = 2(1 - P(Z < 3)) \\ &= 2(1 - 0.9987) = 0.0026 \end{aligned}$$

b.

$$\bar{X} \sim N\left(510g, \left(\frac{20}{2}g\right)^2\right)$$

$$\begin{aligned} P(495.6 < \bar{X} < 555.6) &= P(\bar{X} < 555.6) - P(\bar{X} < 495.6) \\ &= P\left(Z < \frac{555.6 - 510}{10}\right) - P\left(Z < \frac{495.6 - 510}{10}\right) \\ &= P(Z < 4.56) + P(Z < -1.44) = 1 - 0.0749 = 0.9251 \end{aligned}$$

6. X: Quantidade de ácido xanturênico excretado na urina.

$$X \sim N\left(\frac{4.8mg}{15ml}, \left(\frac{2mg}{15ml}\right)^2\right)$$

a. Para fines do cálculo desconsideramos o denominador dos dados de entrada.

$$\begin{aligned}
P(2.8 < X < 7) &= P\left(\frac{2.8 - 4.8}{2} < Z < \frac{7 - 4}{2}\right) \\
&= P(-1 < Z < 1.5) \\
&= \phi(1.5) - \phi(-1) = 0.7745
\end{aligned}$$

b.

$$\begin{aligned}
P(X \leq x) &= P\left(Z < \frac{x - 4.8}{2}\right) = 0.1 \\
&= \phi\left(\frac{x - 4.8}{2}\right) = 0.1 \\
\Rightarrow \frac{x - 4.8}{2} &= -1.28 \\
\Rightarrow x &= 2.24
\end{aligned}$$

Deve ter uma quantidade de ácido xanturênico de 2.24mg/15ml.

c. Y: Número de pessoas com quantidade de ácido xanturênico anormal.

Veja que a probabilidade da quantidade de ácido xanturênico anormal é determinada a partir do item a.

$$p = 1 - P(2.8 < X < 7) = 1 - 0.775 = 0.225$$

Então para $n = 10$, $p = 0.225$ e $Y \sim \mathcal{B}(10, 0.225)$ temos:

$$\begin{aligned}
P(Y \leq 2) &= P(Y = 0) + P(Y = 1) + P(Y = 2) \\
&= \binom{10}{0}(0.225)^0(0.775)^{10} + \binom{10}{1}(0.225)(0.775)^9 \\
&\quad + \binom{10}{2}(0.225)^2(0.775)^8 \\
&= 0.601576
\end{aligned}$$

7. Seja $X \sim U([0, 5])$, nosso interesse é $Y = X^2$, então

$$G(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F(\sqrt{y}) - F(-\sqrt{y})$$

derivando $G'(y)$ obtemos a densidade

$$\begin{aligned}
G'(y) &= F(\sqrt{y})' - F(-\sqrt{y})' \\
&= F'(\sqrt{y}) \frac{1}{2\sqrt{y}} - F'(-\sqrt{y}) \frac{1}{2\sqrt{y}} \\
&= \frac{1}{2\sqrt{y}}(f(\sqrt{y}) - f(-\sqrt{y}))
\end{aligned}$$

Se $f(x) = \frac{1}{5}$ para $0 < x < 5$, então

$$g(y) = \frac{1}{10\sqrt{y}}$$

Assim

$$\begin{aligned} E(Y) &= \int_0^5 y \frac{1}{10\sqrt{y}} dy = \frac{1}{10} \int_0^5 \sqrt{y} dy \\ &= \frac{1}{15} y^{3/2} \Big|_0^5 = 0.745 \end{aligned}$$

8. a. $P(X \leq 10) = 1 - e^{-10} = 0.999$
 - b. $P(5 < X < 15) = P(X < 15) - P(X < 5) = e^{-5} - e^{-15} = 0.0067$
 - c. $P(X > t) = 0.01 \Leftrightarrow e^{-t} = 1/100 \Leftrightarrow t = 4.605$
9. Seja $X \sim \exp(3)$, nosso interesse é $Y = X^3$, então

$$G(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq \sqrt[3]{y}) = F(\sqrt[3]{y})$$

derivando $G'(y)$ obtemos a densidade

$$\begin{aligned} G'(y) &= F'(\sqrt[3]{y})' \\ &= F'(\sqrt[3]{y}) \left(\frac{1}{3} y^{-2/3} \right) \\ g(y) &= f(\sqrt[3]{y}) \left(\frac{1}{3} y^{-2/3} \right) \end{aligned}$$

Se $f(x) = \frac{1}{3}e^{-x/3}$ para $x \geq 0$, então

$$g(y) = \frac{1}{9} e^{-\frac{\sqrt[3]{y}}{3}} \left(y^{-2/3} \right)$$

Assim

$$\begin{aligned} E(Y) &= \frac{1}{9} \int_0^\infty y \left(e^{-\frac{\sqrt[3]{y}}{3}} (y^{-2/3}) \right) dy \\ &= \frac{1}{9} \int_0^\infty y^{1/3} e^{-\frac{\sqrt[3]{y}}{3}} dy \\ &= 162 \end{aligned}$$

A solução pode ser conferida no link:

https://www.wolframalpha.com/input/?i=integrate+281%2F9%29*x%5E%281%2F3%29*e%5E%28-%28x%5E%281%2F3%29%29%2F3%29

10. Seja $T \sim \exp(2)$

$$\begin{aligned} P(X = 1) &= P(0 \leq T < 1) = P(T < 1) - P(T \leq 0) \\ &= (1 - e^{-1/2}) - (1 - e^{-0/2}) = 1 - e^{-1/2} \end{aligned}$$

$$\begin{aligned} P(X = 1) &= P(1 \leq T < 2) = P(T < 2) - P(T \leq 1) \\ &= (1 - e^{-1}) - (1 - e^{-1/2}) = e^{-1/2} - e^{-1} \end{aligned}$$

$$\begin{aligned} P(X = 2) &= P(T \geq 2) = 1 - P(T \leq 2) \\ &= 1 - (1 - e^{-1}) = e^{-1} \end{aligned}$$

11. Seja $X \sim (\mu, \sigma^2)$

a.

$$\begin{aligned} P(X \leq \mu + 2\sigma) &= P\left(\frac{X - \mu}{\sigma} \leq 2\right) \\ &= P(Z \leq 2) = 0.9772 \end{aligned}$$

b. Utilize a propriedade

$$\begin{aligned} P(|X - \mu| \leq \sigma) &= P(-\sigma \leq X - \mu \leq \sigma) \\ &= P\left(-1 \leq \frac{X - \mu}{\sigma} \leq 1\right) \\ &= P(-1 \leq Z \leq 1) \\ &= P(Z \leq 1) - P(Z \leq -1) \\ &= 0.8413 - 0.1587 = 0.6826 \end{aligned}$$

c.

$$\begin{aligned} P(-a\sigma + \mu \leq X \leq a\sigma + \mu) &= P(-a\sigma \leq X - \mu \leq a\sigma) \\ &= P\left(-a \leq \frac{X - \mu}{\sigma} \leq a\right) \\ &= P(-a \leq Z \leq a) = 0.99 \\ &= P(Z \leq a) - P(Z \leq -a) \\ &= 1 - 2P(Z \leq -a) = 0.99 \\ &= 1 - 2(1 - P(Z \leq a)) \\ &= 2P(Z \leq a) - 1 = 0.99 \\ &\Rightarrow P(Z \leq a) = 0.995 \\ &\Rightarrow \phi(a) = 0.995 \\ &\Rightarrow a = 2.56 \end{aligned}$$

d.

$$\begin{aligned} P(X > a) &= 1 - P(X < a) = 0.90 \\ \Rightarrow P(X < a) &= 0.01 \\ \Rightarrow P\left(\frac{X - \mu}{\sigma} < \frac{a - \mu}{\sigma}\right) &= 0.01 \\ \Rightarrow P\left(Z < \frac{a - 1}{\sqrt{2}}\right) &= 0.01 \\ \Rightarrow \phi\left(\frac{a - 1}{\sqrt{2}}\right) &= 0.01 \\ \Rightarrow \frac{a - 1}{\sqrt{2}} &= -2.32 \\ \Rightarrow a &= -2.2809 \end{aligned}$$