Non Smooth Dynamical Systems (NSDS)

Reflections and Guidelines

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Introduction. In this note I intend to discuss in very general terms what is currently occurring with an emerging structure theory of geometric and qualitative nature in NSDS theory. Currently the great interest in such theory is displayed by the rapid growth in the number of publications and specialized meetings in the area in recent decades. I believe that the main challenge in the study of NSDS is to understand and clarify some of the often very complicated dynamical behaviors and establish a precise mathematical framework for the problems encountered herein. This subject has been tainted with the reputation of being lax, mainly because there is an endless list of experimental research in genuine applied sciences. Due to the explosion of scientific developments in the smooth theory in the last century, NSDS has not attracted an expressive number of theoretical mathematicians. In my own research I have experienced many challenges and extreme mathematical difficulties to elucidate a lot of problems in NSDS. On the other hand, I may say that some technical difficulties in general NSDS are rather formidable. My first motivation to study this field was the theoretical paper of Ekeland [13] where the main problem in Calculus of Variations was discussed via piecewise smooth systems. Personal discussions with V. Arnold, H. Sussman, I. Kupka and D. Anosov were also estimulating. In what follows, I briefly indicate directions in which the field can be developed as well as two very natural questions that are raised in this context: “How does the dynamical mathematical community react to these developments? Does the study of non smooth systems have to be motivated by real world applications?”.

0) Some words from Mauricio Peixoto.

In the early 1970s I was a PhD student at USP-São Paulo and had been invited by J. Sotomayor to give a talk at IMPA on my master's thesis “On Stratified Sets”. Mauricio Peixoto was a professor at USP – São Paulo and was constantly using the Rio-São Paulo air-shuttle. That day we traveled back together to São Paulo and of course being considered him (together with Leopoldo Nachbin) as the greatest Brazilian mathematician. I was anxious to have his intake on mathematical research. Inside the plane, at one point I naively asked him about a subject that intrigued me. I said: "Professor Maurício, something I find curious in Mathematics is an area that studies Differential Equations with Complex Time, since as I see it does not exist in the real world.” He replied, *"In Mathematics what matters in any problem is to have mathematically consistent statements that are, non trivial and to have correct proofs of what is asserted. Keep this advice always and be careful in saying whether something in mathematics is important or not."* His philosophical view on mathematical development made quite an impression on me and I have treasured it and always follow the words of the ***Master.”***

a) Why Filippov?

It is clear that in studying the phase portrait of a differential equation the first object that comes to mind is the behavior of the solutions. I emphasize that the concept of a solution of a differential equation with second discontinuous member is not universal. Usually it obeys a certain convention that is, a priori, stipulated or it is defined ad-hoc according to the problem. The lack of uniqueness of solutions requires of course extra attention. For simplicity, we have chosen as the basis of our theoretical study the Filippov Convention due to its essentially geometric character although I understand that the Caratheodory Convention would be the most natural choice. Nowadays the book of Filippov seems to be unanimously accepted as an important contribution to dynamical systems theory. On the other hand some recent results have motivated me to better understand the Utkin sliding mode convention (see [5], [9], [14] and references therein).

b) Take any heuristically "proved" result and try to give a rigorous mathematical proof.

Obviously there may be controversy over what is actually a heuristically proved result. Observe that it is common to find heuristic results for specific models and build upon them for generalization. In my point of view and in the strictly mathematical world, the above procedure has, philosophically speaking, a high scientific value. In short, our task would be to give formal mathematical justifications to the conclusions.

c) Try to give non-smooth versions to classical results of the smooth world.

It is evident that this scenario is purely theoretical and in each case the first step would be to detect whether such a procedure is trivial or not. Hypotheses and "new" statements must be highly clear and new techniques and / or methods are welcome. To exemplify, in the literature we find results of Classical Averaging Theory very well settled when one tries to detect limit cycles in NSDS (see [13] and references there in).

d) Look for problems, for which there are no counterparts in the smooth universe.

In this item, the best argument is to cite the existence of the elliptical fold-fold singularity in high dimensions. There is no phenomenon in the smooth universe that is a counterpart of this singularity. Moreover, the proof of its stability / instability is indeed very complicated and use several non trivial techniques. Finding objects without smooth counterparts is a hard task, perhaps with arduous abstraction power. It seems that an analysis of the robustness of typical minimal sets would be highly encouraging.

e) Approximating a NSDS by smooth systems (regularization).

The regularization process of a non-smooth system Z_0, known as Sotomayor-Teixeira regularization, consists in approximating this system by m-parameter families Z_k ($k = (k_1, k_2, ... k_m)$) of smooth systems. It is worthwhile to cite two directions:
i) depending on the characteristic of Z_0, each Z_k can have a very interesting intrinsic behavior under which deserves a deep analysis.
ii) Sometimes, information about the behavior of Z_k can contribute to the understanding of the dynamics of Z_0. This can be observed in works involving averaging theory and also in the bridge established between NSDS and Singular Perturbation Theory.

It would seem to me absolutely essential to reflect on the scope of general regularizations.

f) Miscellaneous in Geometric and Qualitative Theory in NSDS:

i) Perturbative results are inherent to the methodology of structural stability and bifurcation theory. So it is very important to specify the topological space to which the systems belong.

ii) Some titles may be borrowed from the smooth universe whose contents can be successfully exploited: generic bifurcation (Local and Global), stability theorems, normal forms, ergodic theory (in certain classes, Lyapunov exponents could be, rigorously, extended to non-smooth systems), new trends in hyperbolic theory, generalization of Melnikov's method, Conley index, piecewise continuous mappings, synchronization, singular perturbation theory, integrability (piecewise hamiltonian theory), NSDS tangent to continuous foliations, symmetries in NSDS (including Refractive Systems), stochastic differential equations etc....

iii) Another point that also deserves reflection is one in which a smooth vector field is approximated by non-smooth systems. This assertion comes from the fact that many continuous phenomena, for purely technical reasons, are modeled by differential equations with discontinuous second member. In this direction I recall the phenomenon named “Pinching”(see [6] and references therein).

iv) Interesting problems appear when the switching set is not a differentiable manifold (see [7], [8], [11] and references therein).

v) I confess that I tried to understand, , the substance of the most practical models presented in various congress by engineers and physicists and I was unsuccessful. I would like to really understand something of the machinery but unfortunately cannot recognize a “good mathematical model”.

vi) Why study piecewise smooth systems? One finds in real life and in various branches of science distinguished phenomena whose mathematical models are expressed by discontinuous systems and deserve a systematic analysis. However, sometimes they are far from the usual techniques or methodologies found in the smooth universe. This might be a good time to reflect on the role of the discussions found in [1], [2] and [3]. In what follows we reproduce the abstract of [1]:

“Abstract - Despite the aphorism “Nature does not make jumps” (attributed to Newton, Leibniz, Linnaeus,. . .!!) *it is frequently useful to work, either descriptively or prescriptively, with simplified models which involve switching between different modes of evolution. We describe a variety of examples of such modeling with particular attention to some situations in which the interpretation of the reduced model is a matter of concern.”*

In [2] and [3] one finds arguments that invite us to a discussion about the (non) use of a complete presentation of cases, sub-cases and sub-cases that appear in the general theory of NSDS.

The following paragraphs were selected from a P. Glendining's talk:

“To paraphrase Mike Field:

1. Although most of dynamical systems theory for the last fifty years has been smooth, the real technological innovations and applications (computers, control, mechanics, some biological *models) are not smooth.*

2. If you want a car to stop when the brakes are applied, don't choose an analytic or smooth system!!!"

g) Conclusion

The present work looks primarily to the future. It is mainly concerned with the intrinsic significance of the classification results in NSDS and to its range of applicability in other areas of science. Moreover, the importance of providing readily accessible proofs of the statements is assessed. Finally, I hope this text will help young researchers to face challenges in developing and performing consistent research projects in NSDS.

h) References:

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