Novel Approaches to Real-Time Interpretable Fuzzy Systems

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Outline

Problem formulation

- Mamdani fuzzy systems have easy interpretation but they are computationally more expensive and less efficient learning algorithms
- Takagi-Sugeno fuzzy systems have lower computational complexity and more efficient learning algorithms but they are less interpretable.
- Both Mamdani and Takagi-Sugeno fuzzy systems are universal approximators.
- Problem: Can we approximate a Mamdani fuzzy system with a Takagi-Sugeno fuzzy system and vice-versa?
- What is the accuracy of such approximation?
- Can we construct a combined Mamdani-Takagi-Sugeno system that would be both interpretable and fast and adaptable?

Problem formulation

- Mamdani Fuzzy Systems are based on "or" "and" interpretation of fuzzy if then rules. Łukasiewicz logic is better motivated from the theoretical point of view
- Mamdani system has a nice geometric interpretation. Does Łukasiewicz have a geometric interpretation
- Is a Łukasiewicz fuzzy system a universal approximation?
- Can we make neural-networks more interpretable?
- Can we construct a sigmoid based TS system from a neural network?

Mamdani Fuzzy System

- Consider a fuzzy system

$$\text{If } x \text{ is } A_i \text{ then } y \text{ is } B_i, i = 1, \ldots, n,$$

- Let $x \in [a, b]$ be a crisp input. The fuzzy output is calculated as

$$B'(y) = \bigvee_{i=1}^{n} A_i(x) \wedge B_i(y),$$

where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$

- The defuzzified output can be obtained as

$$COG(B') = \frac{\int_W B'(y) \cdot y \cdot dy}{\int_W B'(y) \cdot dy}.$$

Takagi-Sugeno Fuzzy Systems

- The rule base for Takagi-Sugeno fuzzy systems with linear outputs is

$$\text{if } x \text{ is } A_i \text{ then } z = a_i x + b_i, \ i = 1, \ldots, n$$

or in a two-dimensional case

$$\text{if } x \text{ is } A_i \text{ and } y \text{ is } B_i \text{ then } z = a_i x + b_i y + c_i, \ i = 1, \ldots, n$$

- Output of a Takagi-Sugeno (TS) fuzzy system is

$$TS(x, y) = \frac{\sum_{i=1}^{n} A_i(x) B_i(y) (a_i x + b_i y + c_i)}{\sum_{i=1}^{n} A_i(x) B_i(y)}.$$

Outline

First order TS approximation

- We consider a Mamdani System with the fuzzy rules as:

$$\text{if } x \text{ is } A_i \text{ then } y \text{ is } B_i,\ i = 1, \ldots, n$$

with $LR-$fuzzy membership functions $A_i = (t, x_{i-1}, x_i, x_{i+1})_{LR}$ and $B_i = (z, y_{i-1}, y_i, y_{i+1})_{LR}$

- Takagi-Sugeno approximation of these fuzzy rules is:

$$\text{if } x \text{ is } A_i \text{ then } z = a_i x + b_i,\ i = 1, \ldots, n$$

where

$$a_i = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}},\ b_i = y_i - a_i x_i$$

Example of function approximation

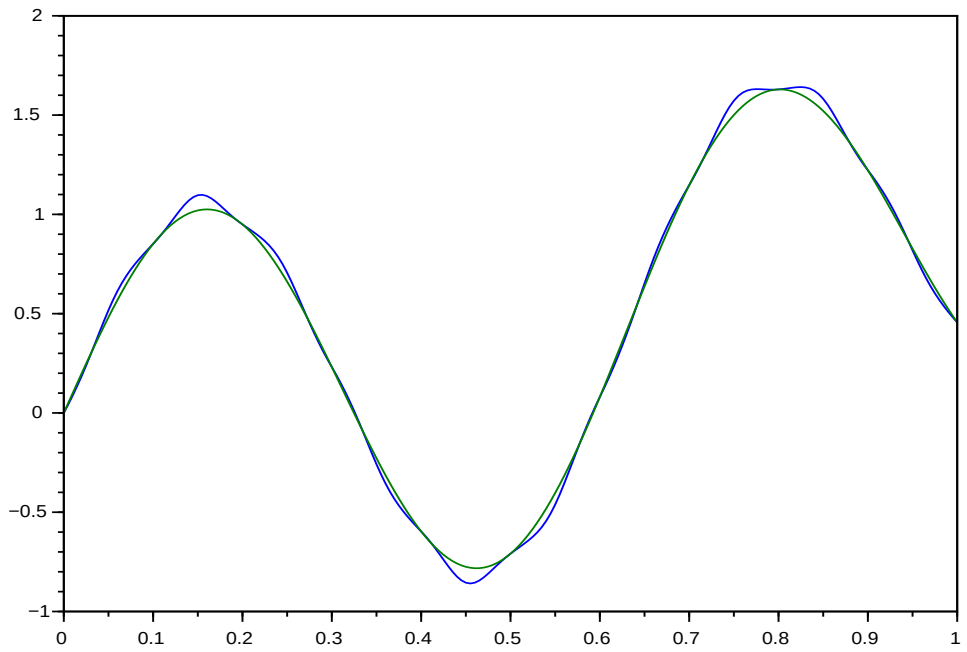


Figure: Function approximation using first order Takagi-Sugeno approximation of a Mamdani fuzzy system

Quadratic TS approximation

TS approximation with by a quadratic consequences

$$y_i = a_i x^2 + b_i x + c.$$

In order to calculate the unknowns a_i, b_i, c_i we used polynomial interpolation.
We consider the equation: $AX = B$, where

$$A = \begin{bmatrix} (x_{i-1})^2 & x_{i-1} & 1 \\ (x_i)^2 & x_i & 1 \\ (x_{i+1})^2 & x_{i+1} & 1 \end{bmatrix}, \; B = \begin{bmatrix} y_{i-1} \\ y_i \\ y_{i+1} \end{bmatrix}.$$

The solution matrix provides the coefficients:

$$x = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}.$$

Example of function approximation

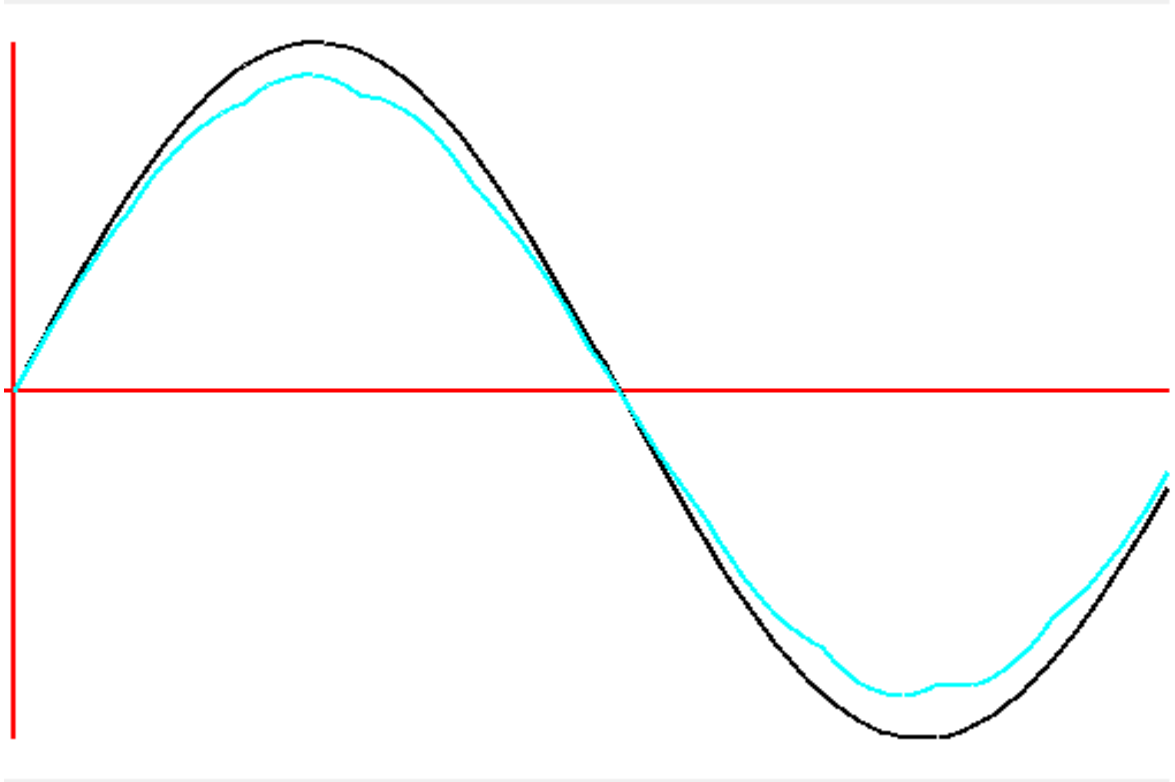


Figure: Function approximation using quadratic Takagi-Sugeno approximation of a Mamdani fuzzy system

B-spline based TS approximation

B-splines as fuzzy rule outputs can be constructed as

$$Q_i(x) = (1-t)\,y_{i-1} + ty_i + t(1-t)\left(a_i(1-t) + b_i t\right),$$

with

$$t = \frac{x - x_{i-1}}{x_i - x_{i-1}}$$

$$a_i = k_{i-1}(x_i - x_{i-1}) - (y_i - y_{i-1})$$

$$b_i = -k_i(x_i - x_{i-1}) + (y_i - y_{i-1})$$

For example the values k_0, k_1, k_2 are found by solving the tridiagonal system

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

B-spline based TS approximation

The coefficients in the system above are given as

$$a_{11} = \frac{2}{x_1 - x_0}, a_{12} = \frac{1}{x_1 - x_0}, a_{21} = \frac{1}{x_1 - x_0},$$

$$a_{22} = 2\left(\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1}\right)$$

$$a_{23} = \frac{1}{x_2 - x_1}, a_{32} = \frac{1}{x_2 - x_1}, a_{33} = \frac{2}{x_2 - x_1}$$

$$b_1 = 3\frac{y_1 - y_0}{(x_1 - x_0)^2}, b_2 = 3\left(\frac{y_1 - y_0}{(x_1 - x_0)^2} + \frac{y_2 - y_1}{(x_2 - x_1)^2}\right),$$

$$b_3 = 3\frac{y_2 - y_1}{(x_2 - x_1)^2}$$

Examples of function approximation

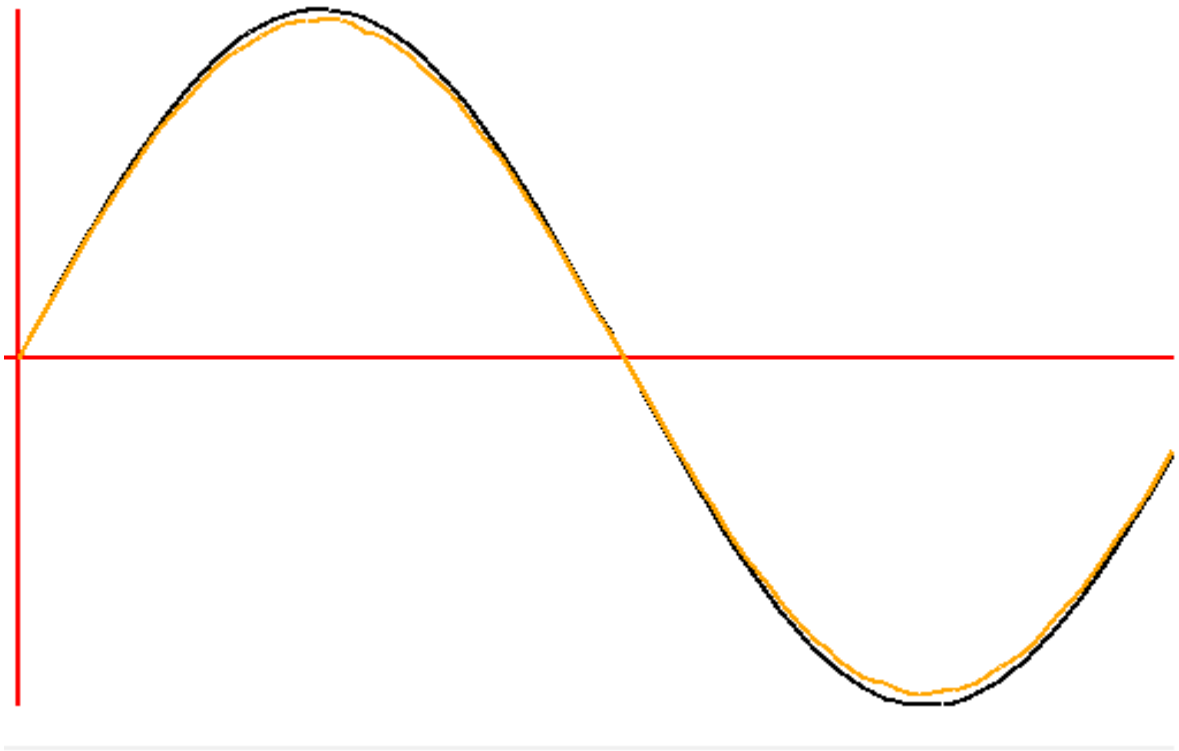


Figure: Function approximation using cubic B-spline based Takagi-Sugeno approximation of a Mamdani fuzzy system

Approximation theorem

Theorem

Let us consider a Mamdani fuzzy system given by the rule base

$$\text{if } x \text{ is } A_i \text{ then } y \text{ is } B_i,\ i = 1, \ldots, n$$

with antecedents and consequences satisfying the properties :
1) A_i *- continuous with* $supp(A_i) = [x_{i-1}, x_{i+1}]$ *and* $x_i \in core(A_i)$.
2) B_i*- integrable with* $supp(B_i) = [y_{i-1}, y_{i+1}]$ *and* $y_i \in core(B_i)$.
Then the Mamdani system can be approximated with arbitrary accuracy by a Takagi – Sugeno fuzzy system

$$TS(x) = \frac{\sum_{i=1}^{n} A_i(x) \cdot (a_i x + b_i)}{\sum_{i=1}^{n} A_i(x)}.$$

Approximation theorem (continued)

Theorem

The coefficients of the TS fuzzy system are

$$a_i = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}}, \; b_i = y_i - a_i x_i$$

Moreover the following error estimate holds true

$$|M(x) - TS(x)| \leq 3\delta_1 + c\delta_2$$

with $\delta_1 = \max_{i=1\ldots n} |y_{i+1} - y_i|$, $\delta_2 = \max_{i=1\ldots n} |x_{i+1} - x_i|$ *and* $c = \max_{i=1\ldots n} |a_i|$.

Comparison of various one dimensional approaches

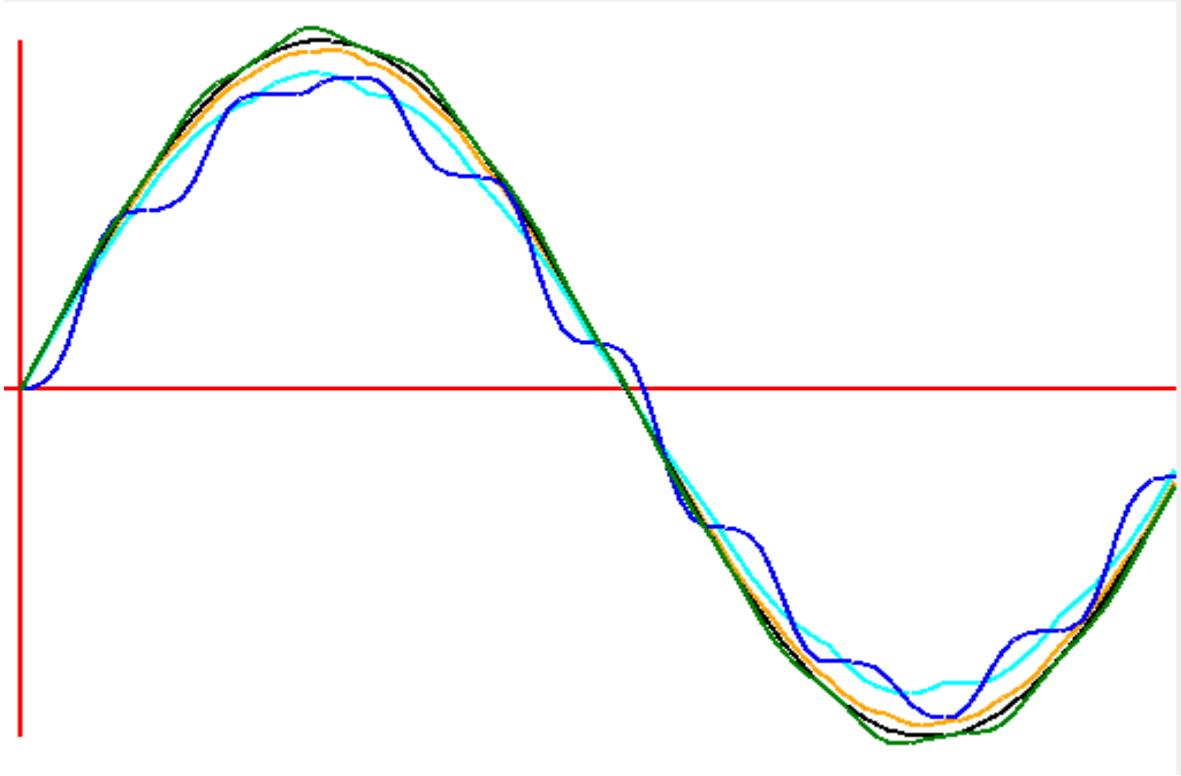


Figure: Comparison of various approaches to approximate the Mamdani system (blue) by various TS systems.

Outline

TS Approximation, two dimensions

Let us consider fuzzy rules of Mamdani type

$$\text{if } x \text{ is } A_i \text{ and } y \text{ is } B_j \text{ then } z \text{ is } C_{ij}$$

where $A_i(x) = (t, x_{i-1}, x_i, x_{i+1})_{LR}$, $B_j(y) = (t, y_{j-1}, y_j, y_{j+1})_{L'R'}$ and $C_{ij} = (t, l_{ij}, m_{ij}, r_{ij})_{L''R''}$

We will approximate the Mamdani system by a TS fuzzy system of the form

$$\text{if } x \text{ is } A_i \text{ and } y \text{ is } B_j \text{ then } z = a_{ij}x + b_{ij}y + c_{ij},$$

where

$$a_{ij} = \frac{r_{ij} - l_{ij}}{x_{i+1} - x_{i-1}}, \; b_{ij} = \frac{r_{ij} - l_{ij}}{y_{i+1} - y_{i-1}}$$

and

$$c_{ij} = m_{ij} - a_{ij}x_i - b_{ij}y_j.$$

Example of a Mamdani system for function approximation

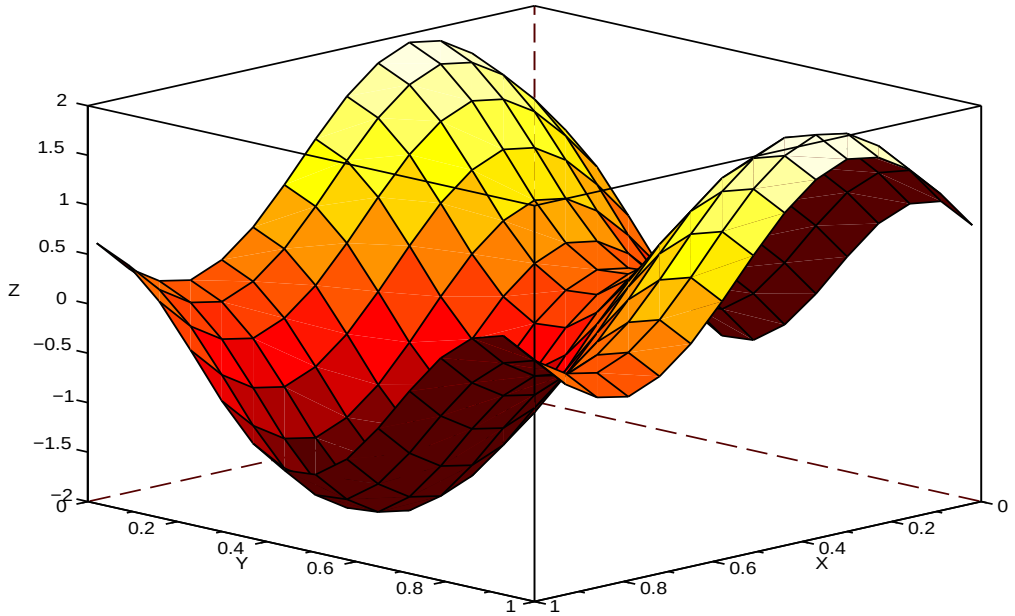


Figure: Mamdani system that approximates the function $f(x,y) = sin(6x) + cos(7y)$.

TS Approximation of the above Mamdani system

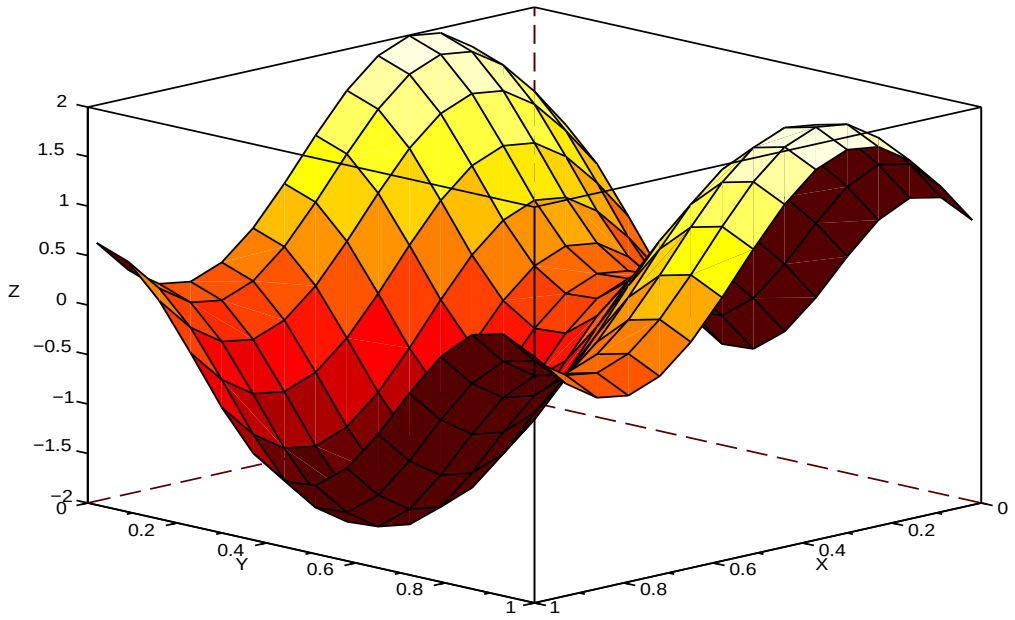


Figure: TS approximation of the Mamdani system.

Linguistic example

Example

We consider a problem with Lt. Columbo. The problem was proposed in the paper [Dvorak-Novak]:
"Mr. John Smith has been shot dead in his house. He was found by his friend, Mr. Carry. Lt. Columbo suspects Mr. Carry to be the murderer. Mr. Carry's testimony is the following: I have started from my home at about 6:30, arrived to John's house at about 7, found John dead and went immediately to the phone box to call police. They told me to wait and came immediately. Lt. Columbo has found the following evidence about dead Mr. Smith: He had high quality suit with broken wristwatch stopped at 5:45. No evidence of strong strike on his body. Lt. Columbo touched engine of Mr. Carry's car and found it to be more or less cold."

Linguistic example

We have to implement commonsense knowledge:

1. If drive duration is big and time since stopped is small then engine is hot.
2. If drive duration is small then engine is cold.
3. If time since stopped is big then engine is cold.

Drive Duration

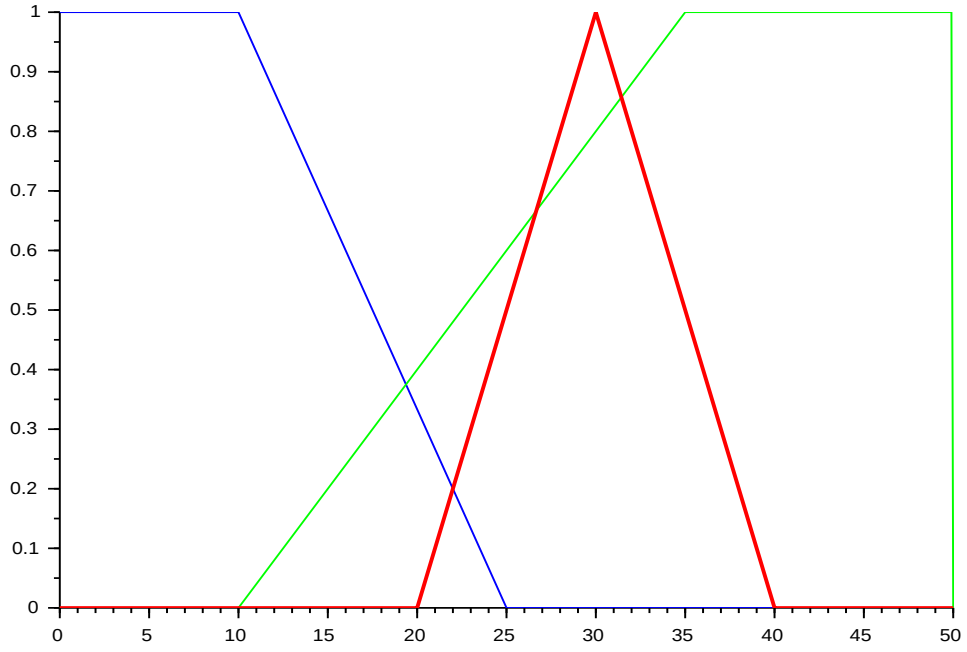


Figure: Drive Duration

Comparison of various one dimensional approaches

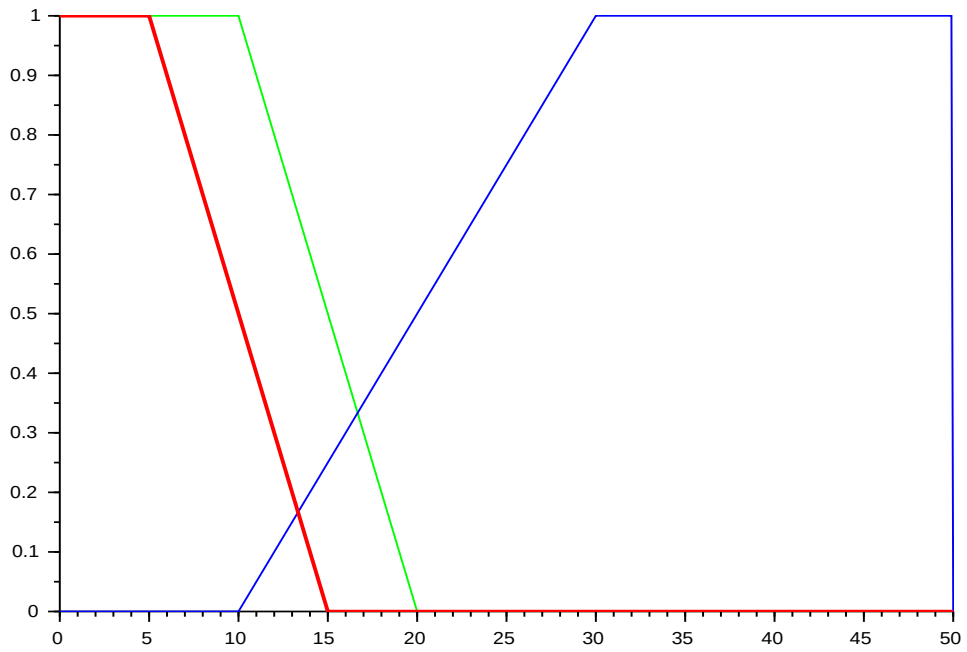


Figure: Comparison of various approaches to approximate the Mamdani system (blue) by various TS systems.

Engine Temperature

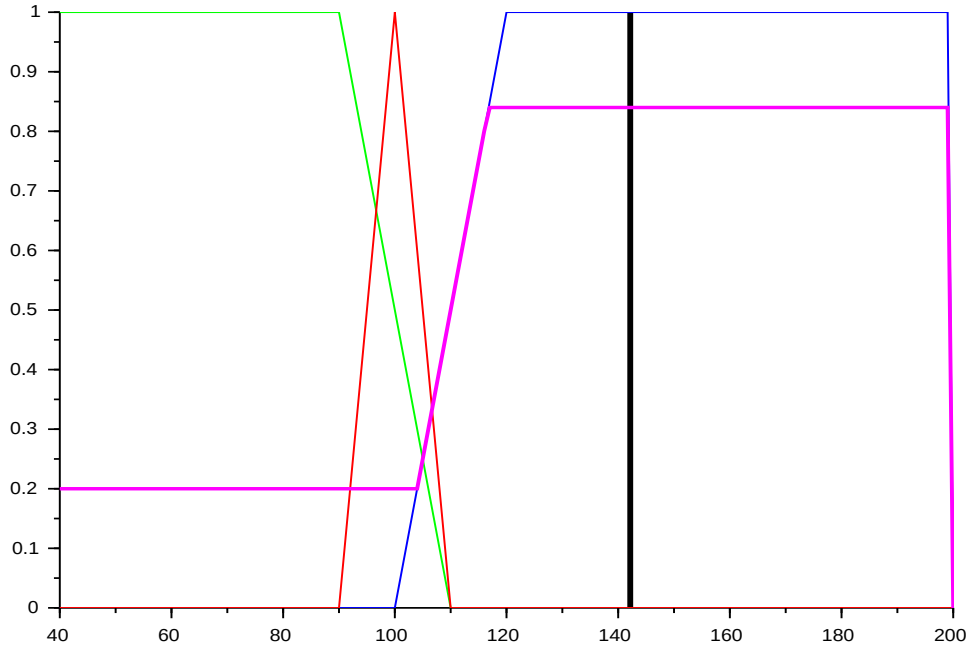


Figure: Engine Temperature

Linguistic example

Another set of fuzzy rules concerns the wristwatch quality.

1. If suit quality is high then wristwatch quality is high.
2. If suit quality is low then wristwatch quality is low.

The last set of fuzzy rules concerns how likely the wristwatch is broken.

1. If wristwatch quality is high and strike is unlikely then broken is unlikely.
2. If wristwatch quality is low and strike is likely then broken is likely.
3. If strike is likely then broken is more or less likely.

Should the watch be broken?

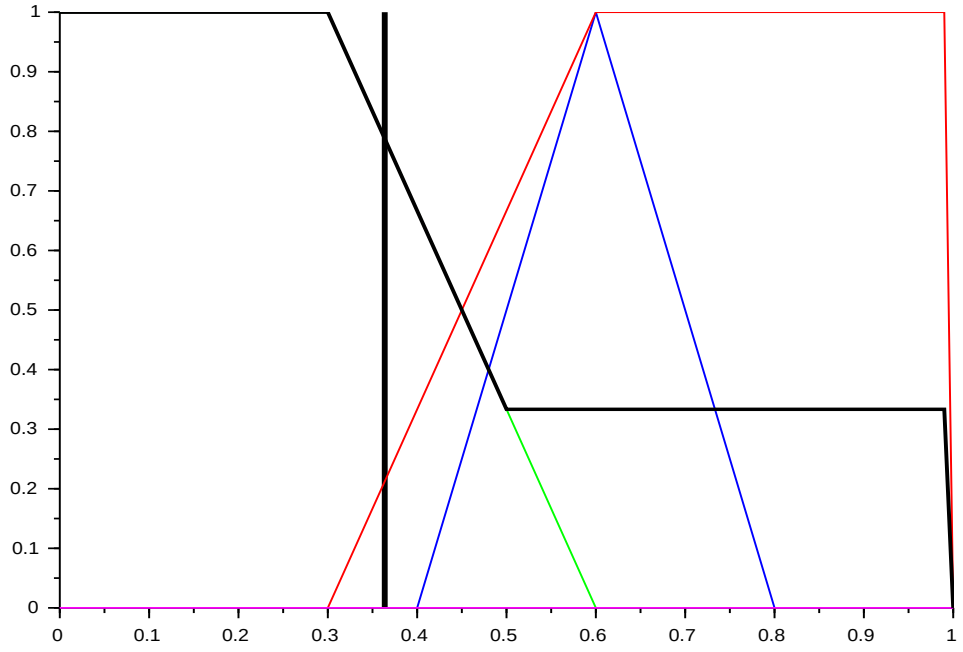


Figure: Fuzzy system to decide whether the watch should be broken.

Linguistic example

Recall:

1. If drive duration is big and time stopped is small then engine is hot.
2. If drive duration is small then engine is cold.
3. If time stopped is big then engine is cold.

Mamdani system for a linguistic example

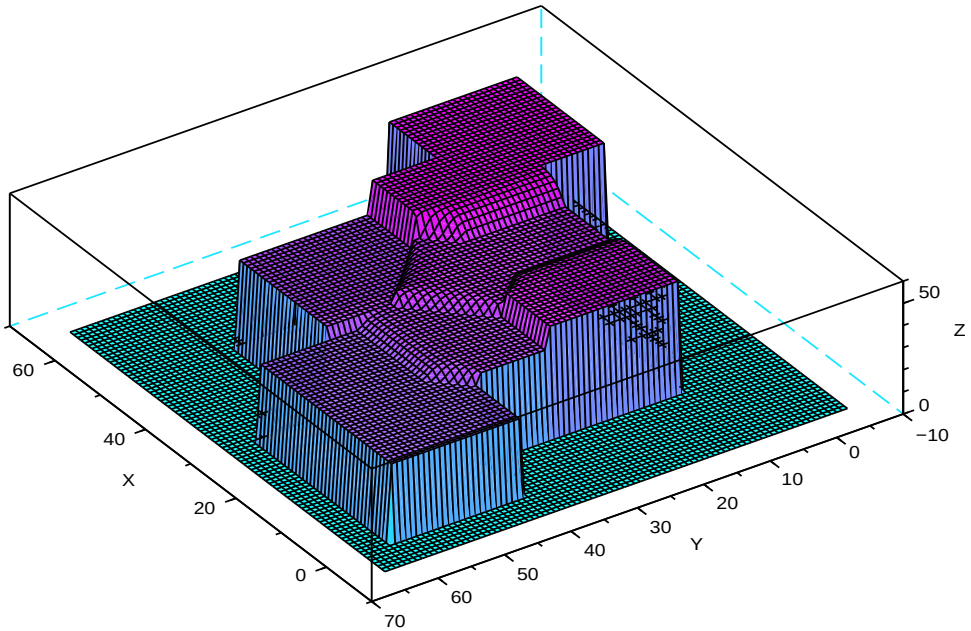


Figure: Mamdani system that models engine temperature dependence on drive duration and the time since the car has stopped.

TS Approximation for the linguistic example

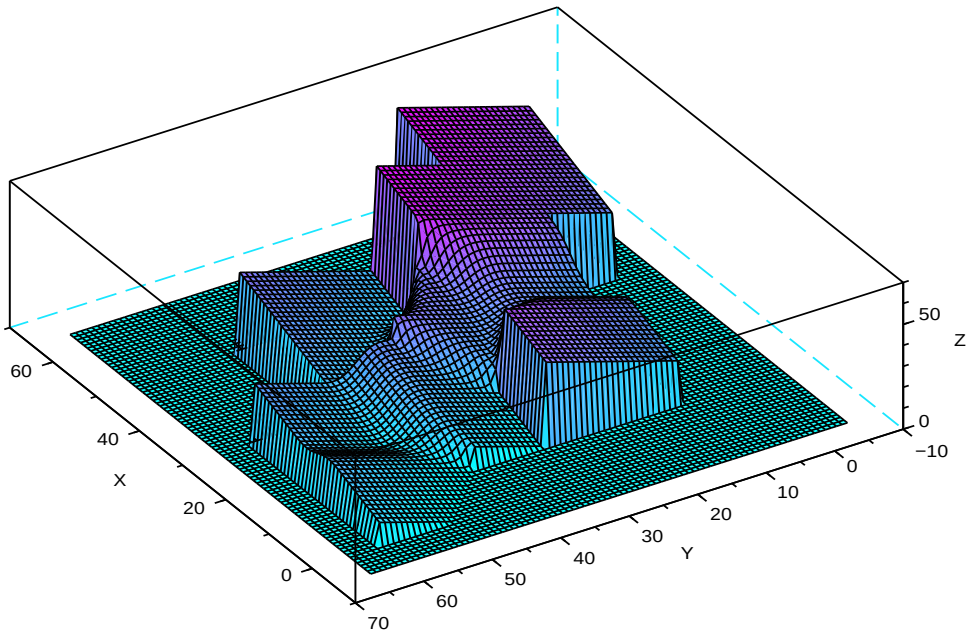
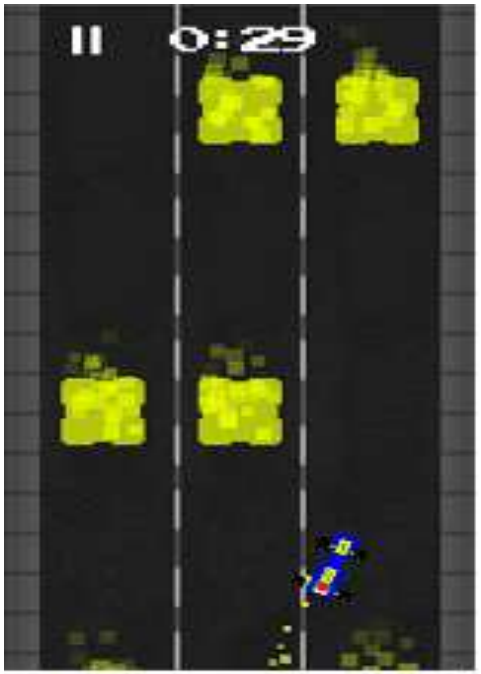


Figure: TS approximation of the above Mamdani system.

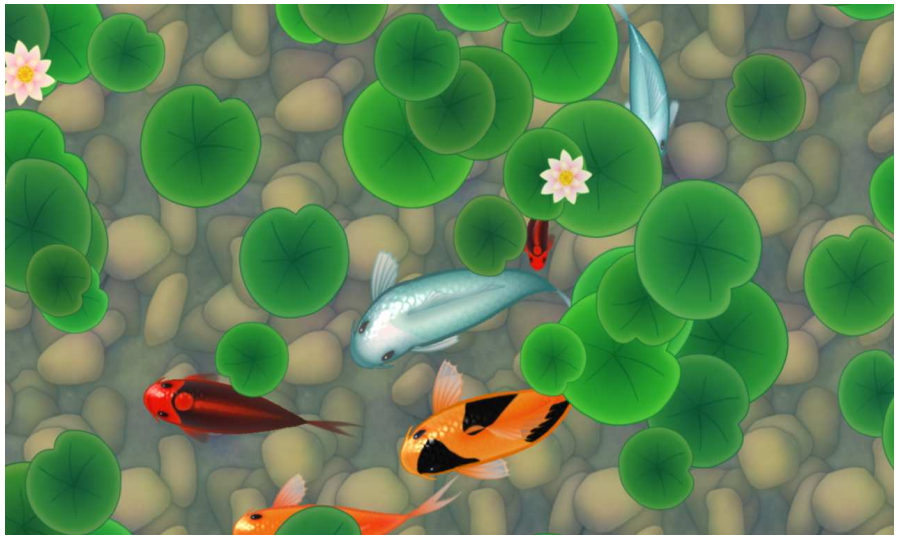
Applications

A fuzzy system was used in order to control the difficulty of a videogame DigiRacer based on the distance between the row of cubes and the reaction time of the player and how fast the car is moving. (Salem Bacha, DigiPen)



Applications

Another real time fuzzy system was developed by Matthew P. Peterson, DigiPen, LU-fuzzy systems



Outline

Mamdani Approximation of a TS fuzzy system

Theorem

Let us consider a TS-fuzzy system given by the rule base

$$\text{if } x \text{ is } A_i \text{ then } y = a_i x + b_i, \ i = 1, \ldots, n,$$

with A_i - continuous with $supp(A_i) = [x_{i-1}, x_{i+1}]$ and $x_i \in core(A_i)$:

$$TS(x) = \frac{\sum_{i=1}^{n} A_i(x) \cdot (a_i x + b_i)}{\sum_{i=1}^{n} A_i(x)}.$$

Approximation theorem (continued)

Theorem

Let

$$y_i^- = \min\{TS(x_j), j = i-1, i, i+1\}$$

$$y_i = TS(x_j),$$

$$y_i^+ = \max\{TS(x_j), j = i-1, i, i+1\}.$$

Let B_i*- integrable with* $supp(B_i) = [y_i^-, y_i^+]$ *and* $y_i \in core(B_i)$.

Approximation theorem (continued)

Theorem

Then the TS fuzzy system

$$\text{if } x \text{ is } A_i \text{ then } y = a_i x + b_i, \ i = 1, \ldots, n,$$

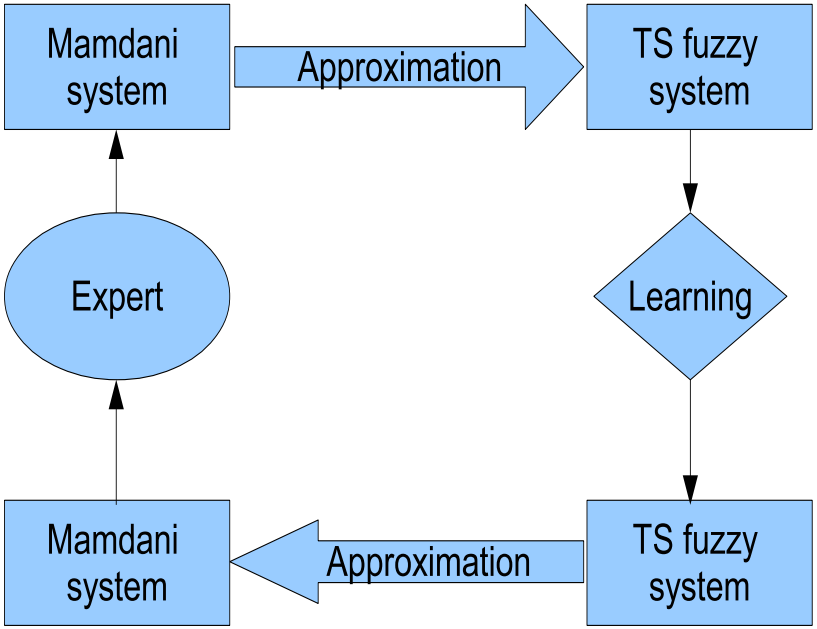
can be approximated by a Mamdani system with fuzzy system given by the rule base

$$\text{If } x \text{ is } A_i \text{ then } y \text{ is } B_i, i = 1, \ldots, n,$$

with arbitrary accuracy.
Moreover the same error estimate holds true, i.e.,

$$|M(x) - TS(x)| \leq 3\delta_1 + c\delta_2.$$

Combined Mamdani Takagi-Sugeno Approximation



Outline

Mamdani Fuzzy System

We can see that in the Mamdani rule base

$$\text{If } x \text{ is } A_i \text{ then } y \text{ is } B_i, i = 1, \ldots, n,$$

"*and*" is replaced by $\bigvee$, "*then*" was replaced by $\bigwedge$
It should be a fuzzy relation

$$x \text{ is } A_1 \text{ and } y \text{ is } B_1$$

or ...

$$x \text{ is } A_n \text{ and } y \text{ is } B_n$$

Mamdani Fuzzy System

To model the rules as a proper generalization of classical logic's Modus Ponens we should use a conjunction and an inplication [Novak]

$$\text{If } x \text{ is } A_1 \text{ then } y \text{ is } B_1$$

and ...

$$\text{If } x \text{ is } A_n \text{ then } y \text{ is } B_n$$

We model the output of the fiuzzy rule base as

$$B'(y) = (\wedge)_{i=1}^{n}(A_i(x) \rightarrow B_i(y)).$$

- Advantage: Theory has deep roots in classical logic [Łukasiewicz]
- Disadvantages: Geometric interpretation, Real-time implementation

Łukasiewicz t-norm and Implication

- Łukasiewicz t-norm defined as

$$x \wedge_L y = \max\{x + y - 1, 0\}$$

  and it is aimed to model the properties of a conjunction between different fuzzy rules.
- Łukasiewicz implication is defined as

$$x \rightarrow_L y = \min\{1 - x + y, 1\}$$

  and it is used to describe a fuzzy cause effect relationship.

Łukasiewicz fuzzy system

- Given the fuzzy rule base

$$\text{if } x \text{ is } A_i \text{ then } y \text{ is } B_i,\ i = 1, \ldots, n.$$

- For a fixed single input x, modus ponens can be generalized in this case by the relation

$$B'(y) = (\wedge_L)_{i=1}^{n} (A_i(x) \rightarrow_L B_i(y)).$$

Łukasiewicz fuzzy system with MOM defuzzification

- Center of Gravity defuzzification

$$COG(B') = \frac{\int_c^d B'(y) \cdot y \cdot dy}{\int_c^d B'(y) dy}.$$

- MOM defuzzification. If the 1-level set is an interval then

$$MOM(B') = \frac{(B')_1^- + (B')_1^+}{2}.$$

Geometric Interpretation

- Antecedents ($n = 3$)

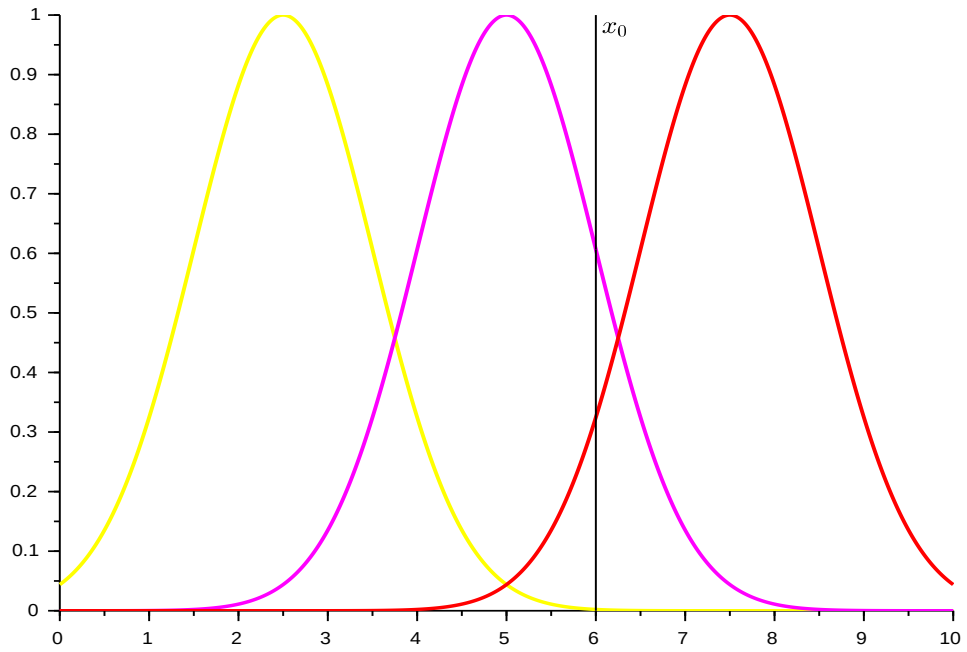


Figure: Antecedents of a Lukasiewicz fuzzy system

Geometric interpretation

- Individual output of the second rule

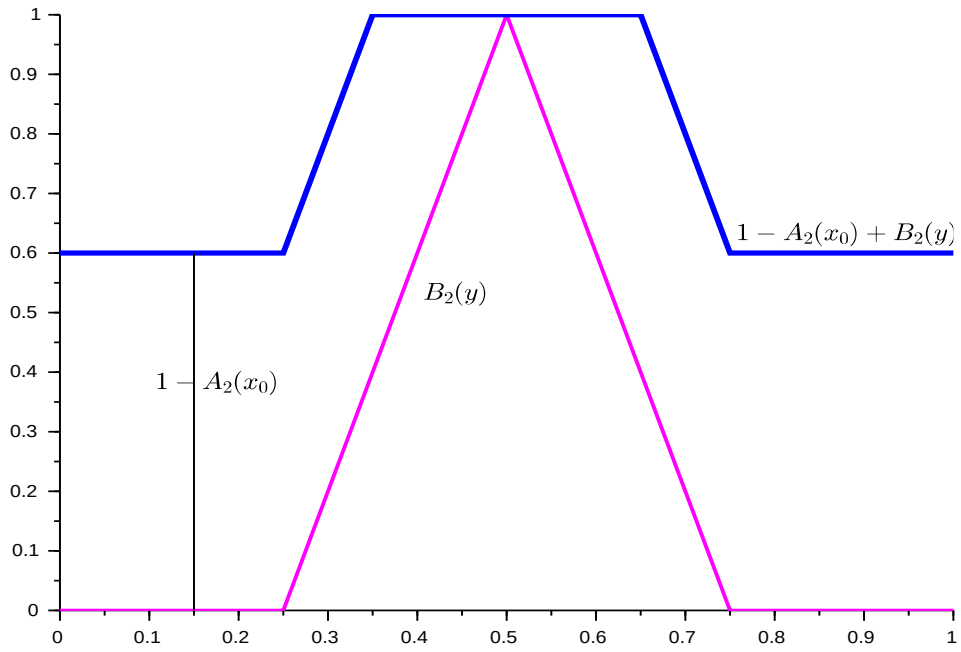


Figure: The consequence and the output of one rule in a Łukasiewicz fuzzy system

Geometric interpretation

- Output of the Łukasiewicz Fuzzy System

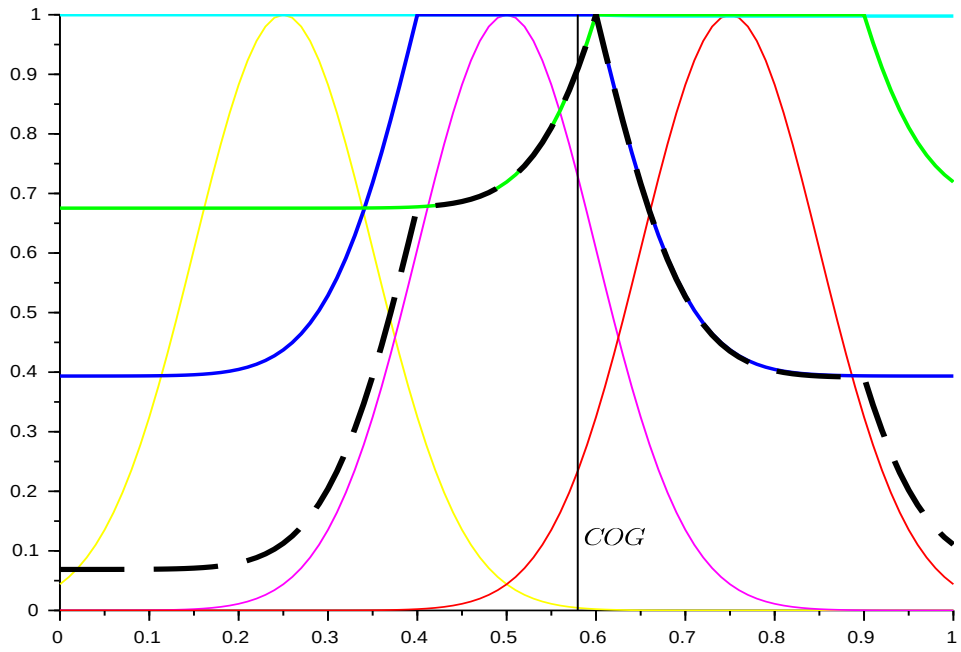


Figure: Consequences, the fuzzy output (dashed line) and the COG defuzzification of the fuzzy output.

Approximation properties

Theorem

Let us consider a continuous function $f : [a, b] \to \mathbb{R}$ *and* $y_i = f(x_i)$, $i = 0, ..., n+1$. *Then* f *can be uniformly approximated by a Łukasiewicz fuzzy system*

$$F(f, x) = COG\left[(\wedge_L)_{i=1}^{n}(A_i(x) \to_L B_i(y))\right]$$

with membership functions for the antecedents and consequences $A_i, B_i, i = 1, ..., n$ *satisfying*

Approximation properties

Theorem

(i) $(A_i)_0 \subseteq [x_{i-1}, x_i, x_{i+1}]$
(ii) $\sum_{i=1}^{n} A_i(x) = 1.$
(iii) $(B_i)_0 \subseteq [\min\{y_{i-1}, y_i, y_{i+1}\}, \max\{y_{i-1}, y_i, y_{i+1}\}]$
(iv) $(B_i)_1 \neq \emptyset.$
Moreover the following error estimate holds true

$$\|F(f,x) - f(x)\| \leq 3\omega(f,\delta),$$

with $\delta = \max_{i=1,\dots,n}\{x_i - x_{i-1}\}.$

Approximation properties

Theorem

Let us consider a continuous function $f : [a, b] \to \mathbb{R}$ *and* $y_i = f(x_i)$, $i = 0, ..., n+1$. *Then* f *can be uniformly approximated by a Łukasiewicz fuzzy system*

$$F(f, x) = MOM\,[(\wedge_L)_{i=1}^{n}(A_i(x) \to_L B_i(y))]$$

with membership functions for the antecedents and consequences $A_i, B_i, i = 1, ..., n$ *satisfying*

Approximation properties

Theorem

(i) $(A_i)_0 \subseteq [x_{i-1}, x_i, x_{i+1}]$
(ii) $\sum_{i=1}^{n} A_i(x) \geq 1$. *(with COG the requirement was that the sum is exactly 1)*
(iii) $(B_i)_0 \subseteq [\min\{y_{i-1}, y_i, y_{i+1}\}, \max\{y_{i-1}, y_i, y_{i+1}\}]$
(iv) $(B_i)_1 \neq \emptyset$.
Moreover the following error estimate holds true

$$\|F(f,x) - f(x)\| \leq 3\omega(f,\delta),$$

with $\delta = \max_{i=1,\dots,n}\{x_i - x_{i-1}\}$.

Approximation properties

Theorem

Any continuous function $f : [a, b] \to [c, d]$ *can be approximated by the Łukasiewicz fuzzy system with MOM defuzzification*

$$F(f, x) = MOM\left[(\wedge_L)_{i=1}^{n}(A_i(x) \to_L B_i(y))\right]$$

with continuous antecedents and consequences $A_i, B_i, i = 1, \ldots, n$ *such that there exist* $0 < \varepsilon$, $r \in \mathbb{N}$, $r < n$, *such that*

Approximation properties

Theorem

(i) $(A_i)_0 \subseteq [x_i, x_{i+r}]$, $i = 1, \ldots, n$;
(ii) $\sum_{i=1}^{n} A_i(x) \geq 1$.
(iii) $(B_i)_0 \subseteq [\min_{j=i,\ldots,i+r}\{y_j\}, \max_{j=i,\ldots,i+r}\{y_j\}]$.
(iv) $(B_i)_1 \neq \emptyset$.
Moreover the following error estimate holds true

$$\|F(f,x) - f(x)\| \leq (r+1)\omega(f,\delta)$$

with

$$\delta = \max_{i=1,\ldots,n}\{x_i - x_{i-1}\}$$

Outline

Real time MOM defuzzification

- Let us assume that consequences B_j are $L-R$ fuzzy sets

$$B_j(y) = \begin{cases} L\left(\frac{y-a}{b-a}\right) & y \in [a,b) \\ 1 & y \in [b,c] \\ R\left(\frac{d-y}{d-c}\right) & y \in (c,d] \\ 0 & otherwise \end{cases} .$$

- Maximum point satisfies $B_j(y) \geq A_j(x)$ and $B_{j+1}(y) \geq A_{j+1}(x)$.
- If we solve the equations $B_j(y) = A_j(x)$ and $B_{j+1}(y) = A_{j+1}(x)$ we obtain left and right endpoints of the maximal interval.

Real time MOM defuzzification

- Left endpoints

$$(y_l)_i = y_{i-1} + L^{-1}(A_i(x))(y_i - y_{i-1}),$$

- Right Endpoints

$$(y_r)_i = y_{i+1} - R^{-1}(A_{i+1}(x))(y_{i+1} - y_i), \; i = j, j+1$$

- We calculate the intersection

$$[y_l, y_r] = [(y_l)_j, (y_r)_j] \cap [(y_l)_{j+1}, (y_r)_{j+1}]$$

- We get the MOM defuzzification as

$$MOM = \frac{y_l + y_r}{2}.$$

Examples

- $f : [0,1] \to [0,1]$, $f(x) = \sin(6x)$ with $x \in [0,1]$
- Antecedents:

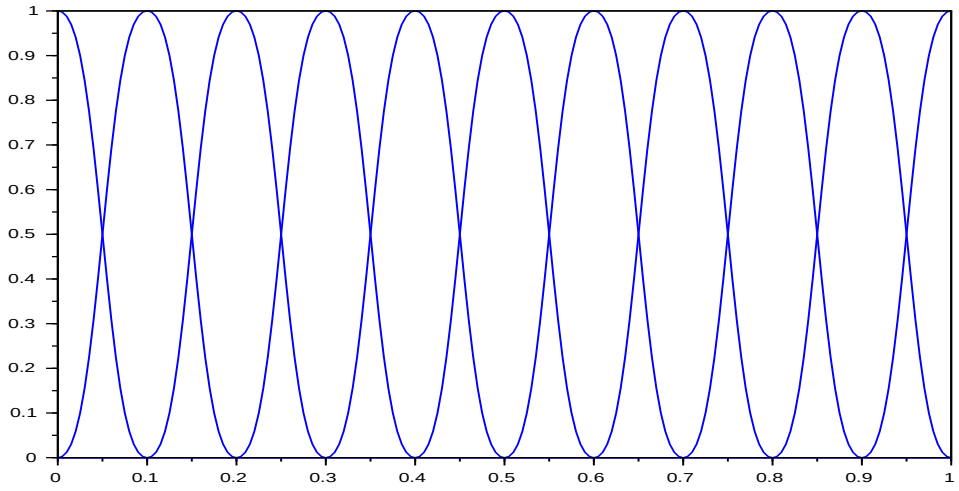


Figure: Antecedents for a Lukasiewicz fuzzy systems for function approximation

Examples

- Consequences

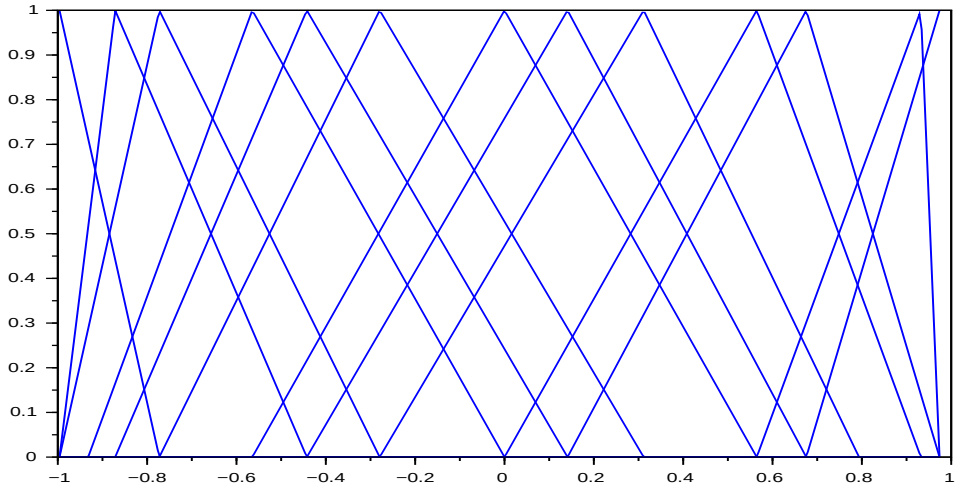


Figure: Consequences for our fuzzy rule base.

Examples

- Output

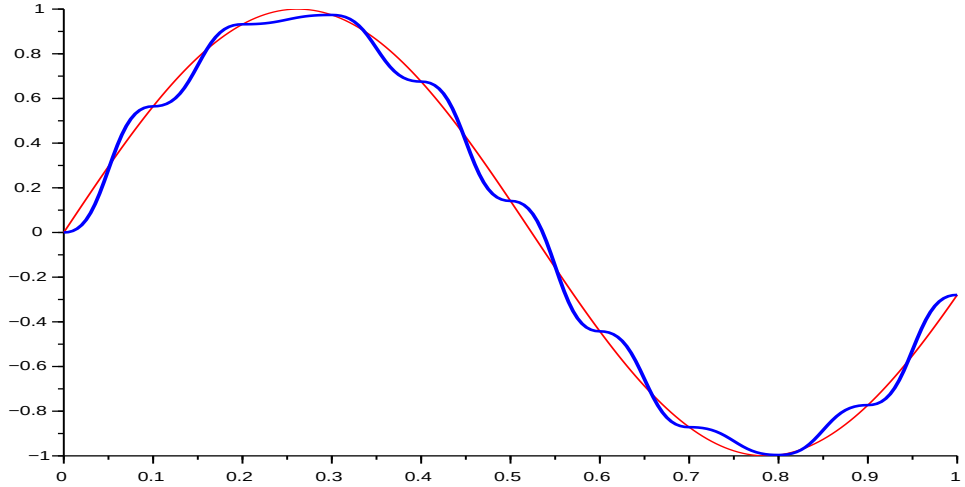


Figure: Function $f(x) = \sin(6x)$(red) is approximated by a Mamdani (green) and a Lukasiewicz (blue) fuzzy system.

Examples

- Comparisons

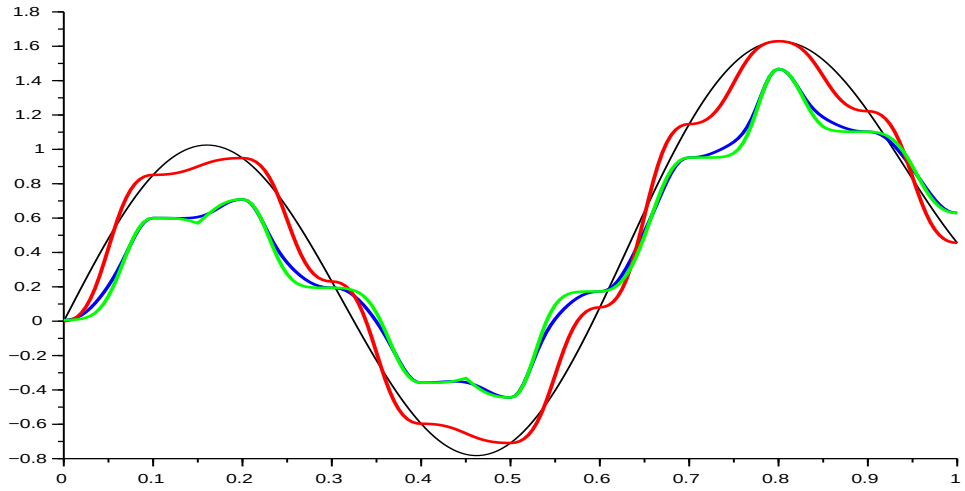


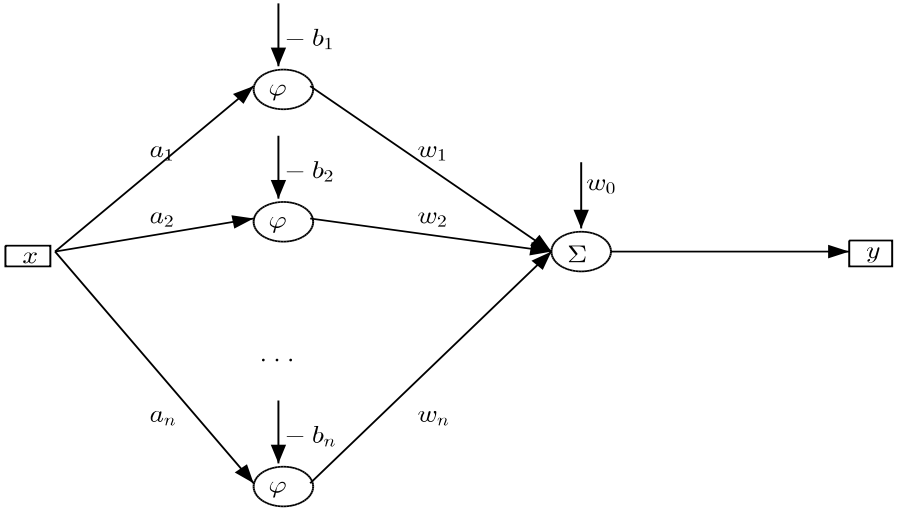
Figure: Function $f(x) = x^2 + \sin(10x)$ (black) is approximated by a Lukasiewicz (red) fuzzy system with MOM defuzzification, Łukasiewicz (blue) fuzzy system with COG defuzzification and Mamdani (green) fuzzy system with COG defuzzification.

Outline

Neural Network

Let us consider a neural network with sigmoid activation function $\varphi(x) = \frac{1}{1+e^{-x}}$

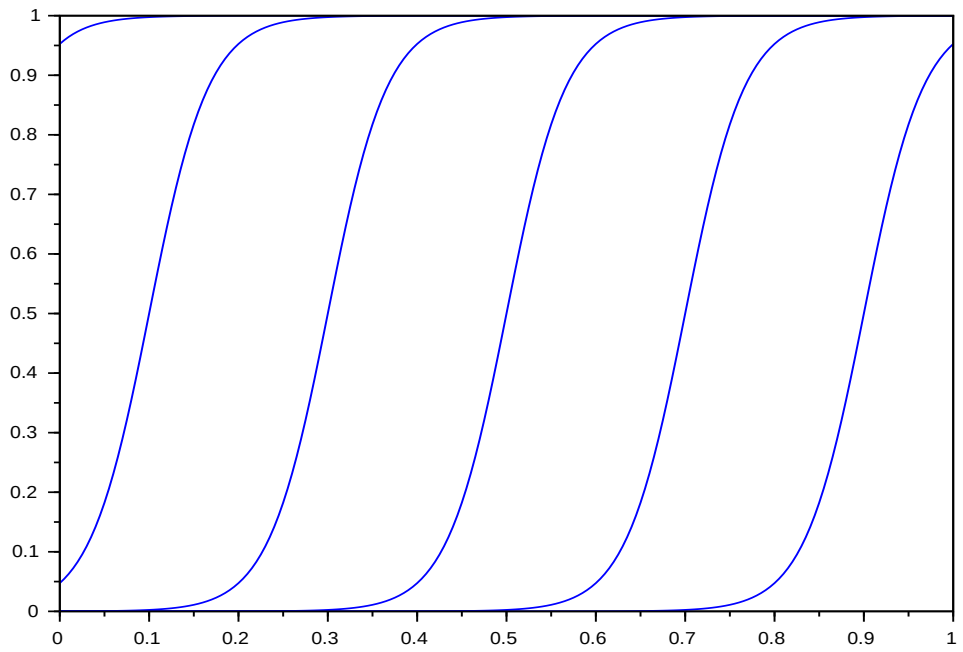
$$NN(x) = w_0 + \sum_{i=1}^{n} w_i \varphi(a_i(x - b_i)). \tag{1}$$



Neural Network

This type of neural network can be interpreted as a fuzzy system

$$\text{if } x \text{ is } A_i \text{ then } y = w_i, \ i = 0, \ldots, n.$$



Takagi Sugeno Fuzzy System with Sigmoid based membership functions

We construct a Takagi-Sugeno Fuzzy System with sigmoid-based membership functions.
Let us start with a sequence real numbers $b_1 < b_2 < ... < b_n$. Also consider positive numbers $a_1, ..., a_n$. We consider a fuzzy system with sigmoid shaped antecedents, i.e.,

$$A_0(x) = 1 - \varphi(a_1(x - b_1)),$$

$$A_i(x) = \varphi(a_i(x - b_i)) - \varphi(a_{i+1}(x - b_{i+1})), \ i = 1, ..., n-1$$

$$A_n(x) = \varphi(a_n(x - b_n)).$$

Takagi Sugeno Fuzzy System with Sigmoid based membership functions

First we observe that the system gives a fuzzy partition as

$$\sum_{i=0}^{n} A_i(x) = 1.$$

TS fuzzy rules with piecewise constant consequences can be written as

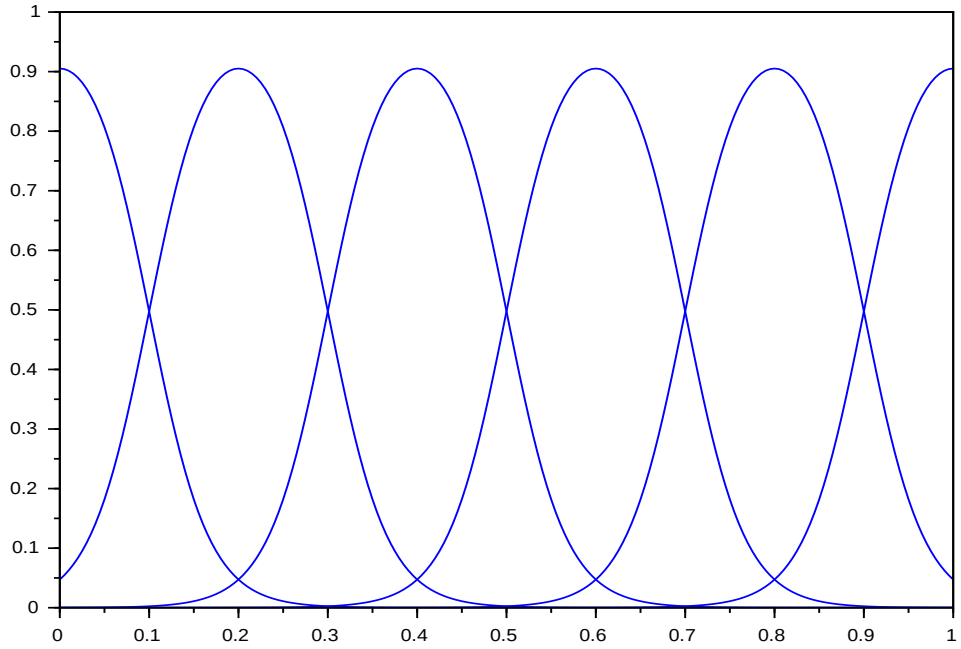
$$\text{if } x \text{ is } A_i \text{ then } y = y_i, \ i = 0, \ldots, n.$$

The Takagi-Sugeno fuzzy system associated to these rules can be expressed as

$$TS(x) = \sum_{i=0}^{n} A_i(x) \cdot y_i. \tag{2}$$

Takagi Sugeno Fuzzy System with Sigmoid based membership functions

Sigmoid based membership functions



Takagi Sugeno Fuzzy System with Sigmoid based membership functions

Theorem

The neural network and the TS fuzzy system descried above are equivalent in the sense that
i) Given a TS fuzzy system as in (2), it can be written as a neural network as in (1), with weights

$$w_0 = y_0,$$

$$w_i = -y_{i-1} + y_i, \ i = 1, \ldots, n.$$

Takagi Sugeno Fuzzy System with Sigmoid based membership functions

Theorem

ii) Given a Neural Network as in (1), it can be written as a TS fuzzy system as in (2), with weights

$$y_i = \sum_{j=0}^{i} w_j, \ i = 0, 1, ..., n.$$

TS Fuzzy System as an Interpretable Neural Network

Proof.

We observe that $TS(x)$ and $NN(x)$ are linear combinations of the same functions $\{1, \varphi(a_0(x-b_0)), \varphi(a_1(x-b_1)), ..., \varphi(a_n(x-b_n))\}$. Let us consider the equation

$$TS(x) = NN(x), \forall x \in \mathbb{R}.$$

We obtain

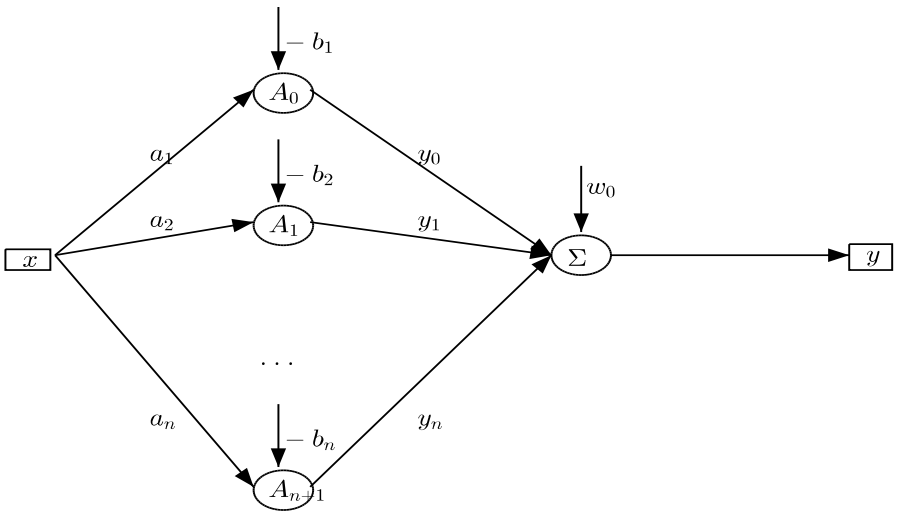
$$\sum_{i=0}^{n} A_i(x) \cdot y_i = w_0 + \sum_{i=1}^{n} w_i \varphi(a_i(x-b_i)).$$

□

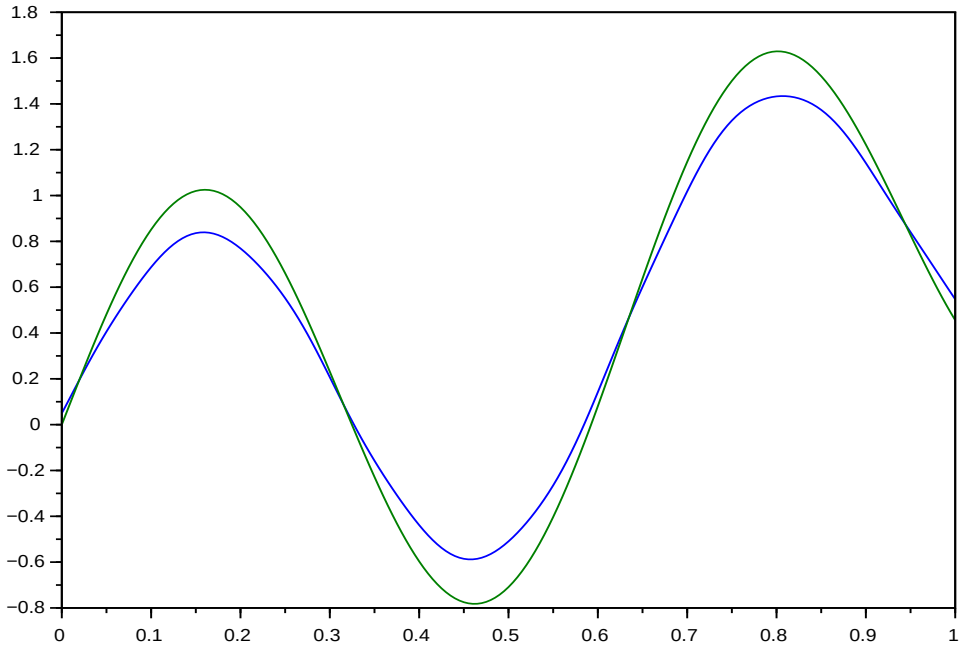
TS Fuzzy System as an Interpretable Neural Network

We can illustrate the TS fuzzy system as below

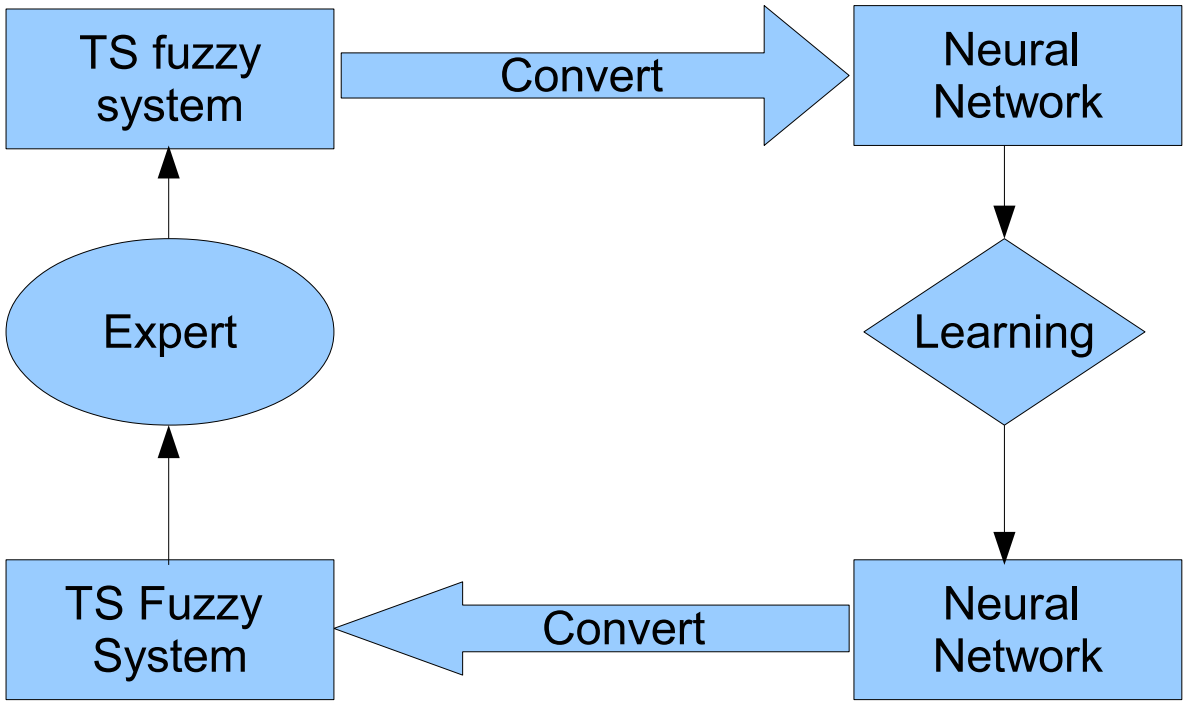
$$TS(x) = \sum_{i=0}^{n} A_i(x) \cdot y_i.$$



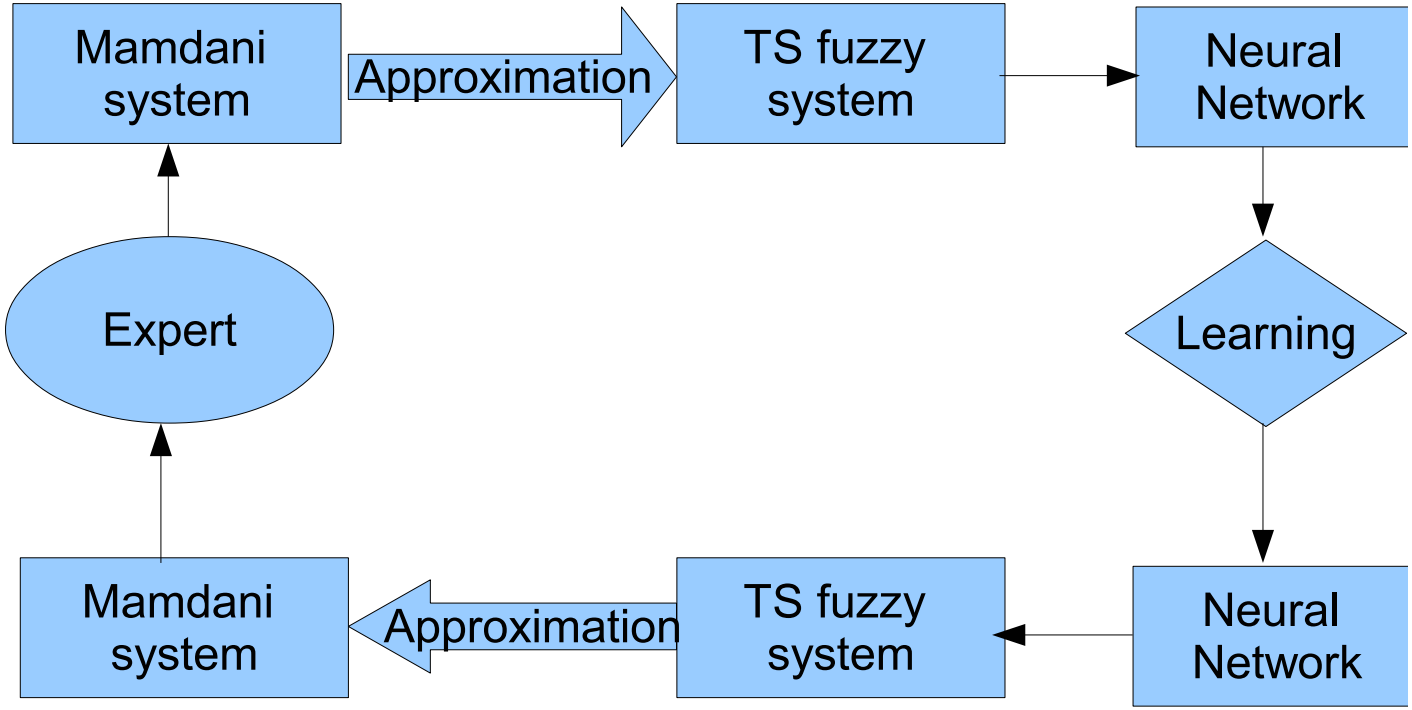
TS Fuzzy System as an Interpretable Neural Network



TS Fuzzy System as an Interpretable Neural Network



Mamdani Systems and Interpretable Neural Networks



Open problems

- Backpropagation for TS fuzzy systems
- Multiple hidden layers
- New proof of Cybenko's theorem with error estimates
- Connecting with Łukasiewicz fuzzy systems
- Real-time applications such as simulations and videogames
- Data Analysis applications
- Combine interpretability with adaptivity and learning