Novel Approaches to Real-Time Interpretable Fuzzy Systems

Barnabas Bede

DigiPen Institute of Technology
Redmond, WA, USA

UNICAMP, Brazil
Outline

1. Introduction

2. Takagi-Sugeno Approximations of Mamdani fuzzy systems
   - One dimensional case
   - Two dimensional case
   - Mamdani Approximation of a TS fuzzy system

3. Łukasiewicz fuzzy system
   - Łukasiewicz fuzzy system and its geometric interpretation
   - Real-time Łukasiewicz Fuzzy System

4. Interpretable Neural Networks
   - Neural networks and TS fuzzy systems

5. Open problems
Problem formulation

- Mamdani fuzzy systems have easy interpretation but they are computationally more expensive and less efficient learning algorithms.
- Takagi-Sugeno fuzzy systems have lower computational complexity and more efficient learning algorithms but they are less interpretable.
- Both Mamdani and Takagi-Sugeno fuzzy systems are universal approximators.
- Problem: Can we approximate a Mamdani fuzzy system with a Takagi-Sugeno fuzzy system and vice-versa?
- What is the accuracy of such approximation?
- Can we construct a combined Mamdani-Takagi-Sugeno system that would be both interpretable and fast and adaptable?
Problem formulation

- Mamdani Fuzzy Systems are based on “or” “and” interpretation of fuzzy if then rules. Łukasiewicz logic is better motivated from the theoretical point of view.
- Mamdani system has a nice geometric interpretation. Does Łukasiewicz have a geometric interpretation?
- Is a Łukasiewicz fuzzy system a universal approximation?
- Can we make neural-networks more interpretable?
- Can we construct a sigmoid based TS system from a neural network?
Mamdani Fuzzy System

Consider a fuzzy system

If \( x \) is \( A_i \) then \( y \) is \( B_i \), \( i = 1, \ldots, n \),

Let \( x \in [a, b] \) be a crisp input. The fuzzy output is calculated as

\[
B'(y) = \bigvee_{i=1}^{n} A_i(x) \land B_i(y),
\]

where \( x \land y = \min\{x, y\} \) and \( x \lor y = \max\{x, y\} \)

The defuzzified output can be obtained as

\[
COG(B') = \frac{\int_{W} B'(y) \cdot y \cdot dy}{\int_{W} B'(y) \cdot dy}.
\]
The rule base for Takagi-Sugeno fuzzy systems with linear outputs is

\[
\text{if } x \text{ is } A_i \text{ then } z = a_i x + b_i, \quad i = 1, \ldots, n
\]

or in a two-dimensional case

\[
\text{if } x \text{ is } A_i \text{ and } y \text{ is } B_i \text{ then } z = a_i x + b_i y + c_i, \quad i = 1, \ldots, n
\]

Output of a Takagi-Sugeno (TS) fuzzy system is

\[
\text{TS}(x, y) = \frac{\sum_{i=1}^{n} A_i(x) B_i(y) (a_i x + b_i y + c_i)}{\sum_{i=1}^{n} A_i(x) B_i(y)}.
\]
Outline

1. Introduction

2. Takagi-Sugeno Approximations of Mamdani fuzzy systems
   - One dimensional case
   - Two dimensional case
   - Mamdani Approximation of a TS fuzzy system

3. Łukasiewicz fuzzy system
   - Łukasiewicz fuzzy system and its geometric interpretation
   - Real-time Łukasiewicz Fuzzy System

4. Interpretable Neural Networks
   - Neural networks and TS fuzzy systems

5. Open problems
First order TS approximation

We consider a Mamdani System with the fuzzy rules as:

\[
\text{if } x \text{ is } A_i \text{ then } y \text{ is } B_i, \ i = 1, \ldots, n
\]

with \( LR \)-fuzzy membership functions \( A_i = (t, x_{i-1}, x_i, x_{i+1})_{LR} \) and \( B_i = (z, y_{i-1}, y_i, y_{i+1})_{LR} \)

Takagi-Sugeno approximation of these fuzzy rules is:

\[
\text{if } x \text{ is } A_i \text{ then } z = a_i x + b_i, \ i = 1, \ldots, n
\]

where

\[
a_i = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}}, \quad b_i = y_i - a_i x_i
\]
Example of function approximation

Figure: Function approximation using first order Takagi-Sugeno approximation of a Mamdani fuzzy system
Quadratic TS approximation

TS approximation with by a quadratic consequences

\[ y_i = a_i x^2 + b_i x + c. \]

In order to calculate the unknowns \( a_i, b_i, c_i \) we used polynomial interpolation.

We consider the equation: \( AX = B \), where

\[
A = \begin{bmatrix}
(x_i-1)^2 & x_i-1 & 1 \\
(x_i)^2 & x_i & 1 \\
(x_i+1)^2 & x_i+1 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
y_{i-1} \\
y_i \\
y_{i+1}
\end{bmatrix}.
\]

The solution matrix provides the coefficients:

\[
x = \begin{bmatrix}
a_i \\
b_i \\
c_i
\end{bmatrix}.
\]
Example of function approximation

**Figure**: Function approximation using quadratic Takagi-Sugeno approximation of a Mamdani fuzzy system
B-spline based TS approximation

B-splines as fuzzy rule outputs can be constructed as

\[ Q_i(x) = (1 - t)y_{i-1} + ty_i + t(1 - t)(a_i(1 - t) + b_it), \]

with

\[ t = \frac{x - x_{i-1}}{x_i - x_{i-1}} \]

\[ a_i = k_{i-1}(x_i - x_{i-1}) - (y_i - y_{i-1}) \]

\[ b_i = -k_i(x_i - x_{i-1}) + (y_i - y_{i-1}) \]

For example the values \( k_0, k_1, k_2 \) are found by solving the tridiagonal system

\[
\begin{bmatrix}
a_{11} & a_{12} & 0 \\
a_{21} & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
k_0 \\
k_1 \\
k_2
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]
B-spline based TS approximation

The coefficients in the system above are given as

\[ a_{11} = \frac{2}{x_1 - x_0}, \quad a_{12} = \frac{1}{x_1 - x_0}, \quad a_{21} = \frac{1}{x_1 - x_0}, \]

\[ a_{22} = 2\left(\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1}\right) \]

\[ a_{23} = \frac{1}{x_2 - x_1}, \quad a_{32} = \frac{1}{x_2 - x_1}, \quad a_{33} = \frac{2}{x_2 - x_1} \]

\[ b_1 = 3\frac{y_1 - y_0}{(x_1 - x_0)^2}, \quad b_2 = 3\left(\frac{y_1 - y_0}{(x_1 - x_0)^2} + \frac{y_2 - y_1}{(x_2 - x_1)^2}\right), \]

\[ b_3 = 3\frac{y_2 - y_1}{(x_2 - x_1)^2} \]
Examples of function approximation

Figure: Function approximation using cubic B-spline based Takagi-Sugeno approximation of a Mamdani fuzzy system
Approximation theorem

**Theorem**

Let us consider a Mamdani fuzzy system given by the rule base

\[
\text{if } x \text{ is } A_i \text{ then } y \text{ is } B_i, \ i = 1, \ldots, n
\]

with antecedents and consequences satisfying the properties:

1) \( A_i \) - continuous with \( \text{supp}(A_i) = [x_{i-1}, x_{i+1}] \) and \( x_i \in \text{core}(A_i) \).
2) \( B_i \) - integrable with \( \text{supp}(B_i) = [y_{i-1}, y_{i+1}] \) and \( y_i \in \text{core}(B_i) \).

Then the Mamdani system can be approximated with arbitrary accuracy by a Takagi – Sugeno fuzzy system

\[
\text{TS}(x) = \frac{\sum_{i=1}^{n} A_i(x) \cdot (a_i x + b_i)}{\sum_{i=1}^{n} A_i(x)}.
\]
Approximation theorem (continued)

**Theorem**

The coefficients of the TS fuzzy system are

\[ a_i = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}}, \quad b_i = y_i - a_i x_i \]

Moreover the following error estimate holds true

\[ |M(x) - TS(x)| \leq 3\delta_1 + c\delta_2 \]

with \( \delta_1 = \max_{i=1\ldots n} |y_{i+1} - y_i|, \quad \delta_2 = \max_{i=1\ldots n} |x_{i+1} - x_i| \) and

\[ c = \max_{i=1\ldots n} |a_i|. \]
Comparison of various one dimensional approaches

**Figure:** Comparison of various approaches to approximate the Mamdani system (blue) by various TS systems.
Outline

1 Introduction

2 Takagi-Sugeno Approximations of Mamdani fuzzy systems
   - One dimensional case
   - Two dimensional case
   - Mamdani Approximation of a TS fuzzy system

3 Łukasiewicz fuzzy system
   - Łukasiewicz fuzzy system and its geometric interpretation
   - Real-time Łukasiewicz Fuzzy System

4 Interpretable Neural Networks
   - Neural networks and TS fuzzy systems

5 Open problems
TS Approximation, two dimensions

Let us consider fuzzy rules of Mamdani type:

\[
\text{if } x \text{ is } A_i \text{ and } y \text{ is } B_j \text{ then } z \text{ is } C_{ij}
\]

where \( A_i(x) = (t, x_{i-1}, x_i, x_{i+1})_{LR} \), \( B_j(y) = (t, y_{j-1}, y_j, y_{j+1})_{L'R'} \) and \( C_{ij} = (t, l_{ij}, m_{ij}, r_{ij})_{L''R''} \).

We will approximate the Mamdani system by a TS fuzzy system of the form:

\[
\text{if } x \text{ is } A_i \text{ and } y \text{ is } B_j \text{ then } z = a_{ij}x + b_{ij}y + c_{ij},
\]

where

\[
a_{ij} = \frac{r_{ij} - l_{ij}}{x_{i+1} - x_{i-1}}, \quad b_{ij} = \frac{r_{ij} - l_{ij}}{y_{i+1} - y_{i-1}}
\]

and

\[
c_{ij} = m_{ij} - a_{ij}x_i - b_{ij}y_j.
\]
Example of a Mamdani system for function approximation

**Figure:** Mamdani system that approximates the function
\[ f(x, y) = \sin(6x) + \cos(7y). \]
TS Approximation of the above Mamdani system

Figure: TS approximation of the Mamdani system.
Linguistic example

Example

We consider a problem with Lt. Columbo. The problem was proposed in the paper [Dvorak-Novak]:
“Mr. John Smith has been shot dead in his house. He was found by his friend, Mr. Carry. Lt. Columbo suspects Mr. Carry to be the murderer. Mr. Carry’s testimony is the following: I have started from my home at about 6:30, arrived to John’s house at about 7, found John dead and went immediately to the phone box to call police. They told me to wait and came immediately. Lt. Columbo has found the following evidence about dead Mr. Smith: He had high quality suit with broken wristwatch stopped at 5:45. No evidence of strong strike on his body. Lt. Columbo touched engine of Mr. Carry’s car and found it to be more or less cold.”
We have to implement commonsense knowledge:

1. If drive duration is big and time since stopped is small then engine is hot.
2. If drive duration is small then engine is cold.
3. If time since stopped is big then engine is cold.
Drive Duration

Figure: Drive Duration
Comparison of various approaches to approximate the Mamdani system (blue) by various TS systems.
Engine Temperature

Figure: Engine Temperature
Another set of fuzzy rules concerns the wristwatch quality.

1. If suit quality is high then wristwatch quality is high.
2. If suit quality is low then wristwatch quality is low.

The last set of fuzzy rules concerns how likely the wristwatch is broken.

1. If wristwatch quality is high and strike is unlikely then broken is unlikely.
2. If wristwatch quality is low and strike is likely then broken is likely.
3. If strike is likely then broken is more or less likely.
Should the watch be broken?

Figure: Fuzzy system to decide whether the watch should be broken.
Linguistic example

Recall:

1. If drive duration is big and time stopped is small then engine is hot.
2. If drive duration is small then engine is cold.
3. If time stopped is big then engine is cold.
Mamdani system for a linguistic example

Figure: Mamdani system that models engine temperature dependence on drive duration and the time since the car has stopped.
**TS Approximation for the linguistic example**

**Figure**: TS approximation of the above Mamdani system.
Applications

A fuzzy system was used in order to control the difficulty of a videogame DigiRacer based on the distance between the row of cubes and the reaction time of the player and how fast the car is moving. (Salem Bacha, DigiPen)
Applications

Another real time fuzzy system was developed by Matthew P. Peterson, DigiPen, LU-fuzzy systems
Outline

1 Introduction

2 Takagi-Sugeno Approximations of Mamdani fuzzy systems
   - One dimensional case
   - Two dimensional case
   - Mamdani Approximation of a TS fuzzy system

3 Łukasiewicz fuzzy system
   - Łukasiewicz fuzzy system and its geometric interpretation
   - Real-time Łukasiewicz Fuzzy System

4 Interpretable Neural Networks
   - Neural networks and TS fuzzy systems

5 Open problems
Theorem

Let us consider a TS-fuzzy system given by the rule base

\[ \text{if } x \text{ is } A_i \text{ then } y = a_i x + b_i, \; i = 1, \ldots, n, \]

with \( A_i \) - continuous with \( \text{supp}(A_i) = [x_{i-1}, x_{i+1}] \) and \( x_i \in \text{core}(A_i) \):

\[ TS(x) = \frac{\sum_{i=1}^{n} A_i(x) \cdot (a_i x + b_i)}{\sum_{i=1}^{n} A_i(x)}. \]
Approximation theorem (continued)

Theorem

Let

\[ y_i^- = \min \{ TS(x_j), j = i - 1, i, i + 1 \} \]

\[ y_i = TS(x_j), \]

\[ y_i^+ = \max \{ TS(x_j), j = i - 1, i, i + 1 \}. \]

Let \( B_i \) be integrable with \( \text{supp}(B_i) = [y_i^-, y_i^+] \) and \( y_i \in \text{core}(B_i) \).
Approximation theorem (continued)

**Theorem**

*Then the TS fuzzy system*

\[
\text{if } x \text{ is } A_i \text{ then } y = a_i x + b_i, \ i = 1, \ldots, n,
\]

*can be approximated by a Mamdani system with fuzzy system given by the rule base*

\[
\text{If } x \text{ is } A_i \text{ then } y \text{ is } B_i, \ i = 1, \ldots, n,
\]

*with arbitrary accuracy.*

*Moreover the same error estimate holds true, i.e.,*

\[
|M(x) - TS(x)| \leq 3\delta_1 + c\delta_2.
\]
Combined Mamdani Takagi-Sugeno Approximation

- Mamdani system
- Approximation
- TS fuzzy system
- Expert
- Learning
- Mamdani system
- TS fuzzy system
- Approximation

Bede Real-Time Interpretable Fuzzy Systems
Outline

1. Introduction

2. Takagi-Sugeno Approximations of Mamdani fuzzy systems
   - One dimensional case
   - Two dimensional case
   - Mamdani Approximation of a TS fuzzy system

3. Łukasiewicz fuzzy system
   - Łukasiewicz fuzzy system and its geometric interpretation
   - Real-time Łukasiewicz Fuzzy System

4. Interpretable Neural Networks
   - Neural networks and TS fuzzy systems

5. Open problems
Mamdani Fuzzy System

We can see that in the Mamdani rule base

\[ \text{If } x \text{ is } A_i \text{ then } y \text{ is } B_i, \quad i = 1, \ldots, n, \]

"and" is replaced by \( \lor \), "then" was replaced by \( \land \)

It should be a fuzzy relation

\[ x \text{ is } A_1 \text{ and } y \text{ is } B_1 \]

or ...

\[ x \text{ is } A_n \text{ and } y \text{ is } B_n \]
Mamdani Fuzzy System

To model the rules as a proper generalization of classical logic’s Modus Ponens we should use a conjunction and an implication [Novak]

\[
\text{If } x \text{ is } A_1 \text{ then } y \text{ is } B_1
\]

and ...

\[
\text{If } x \text{ is } A_n \text{ then } y \text{ is } B_n
\]

We model the output of the fuzzy rule base as

\[
B'(y) = (\wedge)_{i=1}^{n}(A_i(x) \rightarrow B_i(y)).
\]

- Advantage: Theory has deep roots in classical logic [Łukasiewicz]
- Disadvantages: Geometric interpretation, Real-time implementation
Łukasiewicz t-norm and Implication

- Łukasiewicz t-norm defined as
  \[ x \land_L y = \max\{x + y - 1, 0\} \]
  and it is aimed to model the properties of a conjunction between different fuzzy rules.

- Łukasiewicz implication is defined as
  \[ x \rightarrow_L y = \min\{1 - x + y, 1\} \]
  and it is used to describe a fuzzy cause effect relationship.
Łukasiewicz fuzzy system

- Given the fuzzy rule base

  \[
  \text{if } x \text{ is } A_i \text{ then } y \text{ is } B_i, \ i = 1, \ldots, n.
  \]

- For a fixed single input $x$, modus ponens can be generalized in this case by the relation

  \[
  B'(y) = (\land_L)_{i=1}^{n} (A_i(x) \rightarrow_L B_i(y)).
  \]
Łukasiewicz fuzzy system with MOM defuzzification

- Center of Gravity defuzzification

\[
COG(B') = \frac{\int_{c}^{d} B'(y) \cdot y \cdot dy}{\int_{c}^{d} B'(y) dy}.
\]

- MOM defuzzification. If the 1-level set is an interval then

\[
MOM(B') = \frac{(B')_{1}^{-} + (B')_{1}^{+}}{2}.
\]
Geometric Interpretation

- Antecedents \((n = 3)\)

**Figure:** Antecedents of a Łukasiewicz fuzzy system
Geometric interpretation

- Individual output of the second rule

**Figure:** The consequence and the output of one rule in a Łukasiewicz fuzzy system
Geometric interpretation

- Output of the Łukasiewicz Fuzzy System

**Figure:** Consequences, the fuzzy output (dashed line) and the COG defuzzification of the fuzzy output.
Theorem

Let us consider a continuous function \( f : [a, b] \rightarrow \mathbb{R} \) and \( y_i = f(x_i) \), \( i = 0, \ldots, n + 1 \). Then \( f \) can be uniformly approximated by a Łukasiewicz fuzzy system

\[
F(f, x) = \text{COG}[(\land_i A_i(x) \rightarrow_L B_i(y))]\]

with membership functions for the antecedents and consequences \( A_i, B_i, i = 1, \ldots, n \) satisfying
Approximation properties

**Theorem**

(i) \((A_i)_0 \subseteq [x_{i-1}, x_i, x_{i+1}]\)

(ii) \(\sum_{i=1}^{n} A_i(x) = 1\).

(iii) \((B_i)_0 \subseteq [\min\{y_{i-1}, y_i, y_{i+1}\}, \max\{y_{i-1}, y_i, y_{i+1}\}]\)

(iv) \((B_i)_1 \neq \emptyset\).

Moreover the following error estimate holds true

\[
\| F(f, x) - f(x) \| \leq 3\omega(f, \delta),
\]

with \(\delta = \max_{i=1,\ldots,n}\{x_i - x_{i-1}\}\).
Theorem

Let us consider a continuous function $f: [a, b] \rightarrow \mathbb{R}$ and $y_i = f(x_i)$, $i = 0, \ldots, n + 1$. Then $f$ can be uniformly approximated by a Łukasiewicz fuzzy system

$$F(f, x) = \text{MOM}[(\land_i^n (A_i(x) \rightarrow_L B_i(y)))]$$

with membership functions for the antecedents and consequences $A_i, B_i, i = 1, \ldots, n$ satisfying
Approximation properties

**Theorem**

1. \((A_i)_0 \subseteq [x_{i-1}, x_i, x_{i+1}]\)
2. \(\sum_{i=1}^{n} A_i(x) \geq 1.\) *(with COG the requirement was that the sum is exactly 1)*
3. \((B_i)_0 \subseteq [\min\{y_{i-1}, y_i, y_{i+1}\}, \max\{y_{i-1}, y_i, y_{i+1}\}]\)
4. \((B_i)_1 \neq \emptyset.\)

Furthermore, the following error estimate holds true

\[ \|F(f, x) - f(x)\| \leq 3\omega(f, \delta), \]

*with \(\delta = \max_{i=1,...,n}\{x_i - x_{i-1}\}.\)
Approximation properties

**Theorem**

Any continuous function \( f : [a, b] \rightarrow [c, d] \) can be approximated by the Łukasiewicz fuzzy system with MOM defuzzification

\[
F(f, x) = \text{MOM} \left[ (\wedge_L)^n_{i=1} (A_i(x) \rightarrow_L B_i(y)) \right]
\]

with continuous antecedents and consequences \( A_i, B_i, i = 1, \ldots, n \) such that there exist \( 0 < \varepsilon, r \in \mathbb{N}, r < n, \) such that
Approximation properties

Theorem

(i) \((A_i)_0 \subseteq [x_i, x_{i+r}], \ i = 1, \ldots, n;\)
(ii) \(\sum_{i=1}^{n} A_i(x) \geq 1.\)
(iii) \((B_i)_0 \subseteq [\min_{j=i,\ldots,i+r} \{y_j\}, \max_{j=i,\ldots,i+r} \{y_j\}].\)
(iv) \((B_i)_1 \neq \emptyset.\)

Moreover the following error estimate holds true

\[ \| F(f, x) - f(x) \| \leq (r + 1) \omega(f, \delta) \]

with

\[ \delta = \max_{i=1,\ldots,n} \{x_i - x_{i-1}\} \]
Outline

1 Introduction

2 Takagi-Sugeno Approximations of Mamdani fuzzy systems
   • One dimensional case
   • Two dimensional case
   • Mamdani Approximation of a TS fuzzy system

3 Łukasiewicz fuzzy system
   • Łukasiewicz fuzzy system and its geometric interpretation
   • Real-time Łukasiewicz Fuzzy System

4 Interpretable Neural Networks
   • Neural networks and TS fuzzy systems

5 Open problems
Let us assume that consequences $B_j$ are $L-R$ fuzzy sets

$$B_j(y) = \begin{cases} 
L\left(\frac{y-a}{b-a}\right) & y \in [a, b) \\
1 & y \in [b, c] \\
R\left(\frac{d-y}{d-c}\right) & y \in (c, d] \\
0 & \text{otherwise}
\end{cases}$$

Maximum point satisfies $B_j(y) \geq A_j(x)$ and $B_{j+1}(y) \geq A_{j+1}(x)$.

If we solve the equations $B_j(y) = A_j(x)$ and $B_{j+1}(y) = A_{j+1}(x)$ we obtain left and right endpoints of the maximal interval.
Real time MOM defuzzification

- **Left endpoints**
  \[(y_l)_i = y_{i-1} + L^{-1}(A_i(x))(y_i - y_{i-1}),\]

- **Right Endpoints**
  \[(y_r)_i = y_{i+1} - R^{-1}(A_{i+1}(x))(y_{i+1} - y_i), \quad i = j, j + 1\]

- We calculate the intersection
  \[[y_l, y_r] = [(y_l)_j,(y_r)_j] \cap [(y_l)_{j+1},(y_r)_{j+1}]\]

- We get the MOM defuzzification as
  \[MOM = \frac{y_l + y_r}{2}.\]
Examples

- $f : [0, 1] \rightarrow [0, 1]$, $f(x) = \sin(6x)$ with $x \in [0, 1]$
- Antecedents:

![Figure: Antecedents for a Łukasiewicz fuzzy systems for function approximation](image)

**Figure:** Antecedents for a Łukasiewicz fuzzy systems for function approximation
Examples

- Consequences

Figure: Consequences for our fuzzy rule base.
Examples

Output

**Figure:** Function $f(x) = \sin(6x)$ (red) is approximated by a Mamdani (green) and a Łukasiewicz (blue) fuzzy system.
Examples

- Comparisons

**Figure:** Function $f(x) = x^2 + \sin(10x)$ (black) is approximated by a Łukasiewicz (red) fuzzy system with MOM defuzzification, Łukasiewicz (blue) fuzzy system with COG defuzzification and Mamdani (green) fuzzy system with COG defuzzification.
Let us consider a neural network with sigmoid activation function

\[ \varphi(x) = \frac{1}{1 + e^{-x}} \]

\[ \text{NN}(x) = w_0 + \sum_{i=1}^{n} w_i \varphi(a_i(x - b_i)). \] (1)
This type of neural network can be interpreted as a fuzzy system

\[
\text{if } x \text{ is } A_i \text{ then } y = w_i, \quad i = 0, \ldots, n.
\]
We construct a Takagi-Sugeno Fuzzy System with sigmoid-based membership functions. Let us start with a sequence real numbers $b_1 < b_2 < \ldots < b_n$. Also consider positive numbers $a_1, \ldots, a_n$. We consider a fuzzy system with sigmoid shaped antecedents, i.e.,

\[
A_0(x) = 1 - \varphi(a_1(x - b_1)),
\]

\[
A_i(x) = \varphi(a_i(x - b_i)) - \varphi(a_{i+1}(x - b_{i+1})), \quad i = 1, \ldots, n-1
\]

\[
A_n(x) = \varphi(a_n(x - b_n)).
\]
Takagi Sugeno Fuzzy System with Sigmoid based membership functions

First we observe that the system gives a fuzzy partition as

\[ \sum_{i=0}^{n} A_i(x) = 1. \]

TS fuzzy rules with piecewise constant consequences can be written as

if \( x \) is \( A_i \) then \( y = y_i, \ i = 0, \ldots, n \).

The Takagi-Sugeno fuzzy system associated to these rules can be expressed as

\[ TS(x) = \sum_{i=0}^{n} A_i(x) \cdot y_i. \] (2)
Takagi Sugeno Fuzzy System with Sigmoid based membership functions

Sigmoid based membership functions
Takagi Sugeno Fuzzy System with Sigmoid based membership functions

Theorem

The neural network and the TS fuzzy system described above are equivalent in the sense that

i) Given a TS fuzzy system as in (2), it can be written as a neural network as in (1), with weights

\[ w_0 = y_0, \]

\[ w_i = -y_{i-1} + y_i, \quad i = 1, \ldots, n. \]
**Theorem**

\( \text{ii) Given a Neural Network as in (1), it can be written as a TS fuzzy system as in (2), with weights} \)

\[ y_i = \sum_{j=0}^{i} w_j, \quad i = 0, 1, ..., n. \]
Proof.

We observe that \( TS(x) \) and \( NN(x) \) are linear combinations of the same functions \( \{1, \varphi(a_0(x - b_0)), \varphi(a_1(x - b_1)), ..., \varphi(a_n(x - b_n))\} \). Let us consider the equation

\[
TS(x) = NN(x), \forall x \in \mathbb{R}.
\]

We obtain

\[
\sum_{i=0}^{n} A_i(x) \cdot y_i = w_0 + \sum_{i=1}^{n} w_i \varphi(a_i(x - b_i)).
\]
We can illustrate the TS fuzzy system as below

\[ TS(x) = \sum_{i=0}^{n} A_i(x) \cdot y_i. \]
TS Fuzzy System as an Interpretable Neural Network
TS Fuzzy System as an Interpretable Neural Network
Open problems

- Backpropagation for TS fuzzy systems
- Multiple hidden layers
- New proof of Cybenko’s theorem with error estimates
- Connecting with Łukasiewicz fuzzy systems
- Real-time applications such as simulations and videogames
- Data Analysis applications
- Combine interpretability with adaptivity and learning