Biomatemática **27 (2)** *2017, 85–96* ***ISSN 1679-365X***
Uma Publicação do Grupo de Biomatemática IMECC – UNICAMP

Measure of covariance on fuzzy interactive Malthusian model

Francielle Santo Pedro[1],
IMECC, UNICAMP, 13083-859, Campinas/SP, Brazil.

Laécio C. Barros[2], Estevão E. Laureano[3],
IMECC, UNICAMP, 13083-859, Campinas/SP, Brazil.

Abstract. In this manuscript, we study the measure of covariance in a linearly correlated fuzzy processes. The analysis was done using the Malthusian decay model, where we compute the covariance of the solution at instants $t + h$ and t, and observe its behavior when t tends to infinity.

Keyword: *Fuzzy Interactive Malthusian Model; Fuzzy Differential Equation; Measure of Covariance.*

1. Introduction

Generally, evolutionary phenomena in time as population dynamics and epidemiology, involve correlated data(Robinson, 2009; Kelley and Schmidt, 1995; Barlow, 1994). So, it is essential to study interactivity in these processes. In (Barros and Santo Pedro, 2017), we studied autocorrelated fuzzy processes, more particularly linearly correlated fuzzy processes, that is, processes which states are linearly locally correlated. These processes have ideas similar to processes with memories (time series). In this paper we study concepts of mean value, variance, measure of interactivity and measure of covariance in the linearly correlated fuzzy processes, which play a fundamental role in both possibility and probability theory.

[1]fran.stopedro@gmail.com
[2]laeciocb@ime.unicamp.br
[3]eelaureano@ime.unicamp.br

1.1. Preliminary

In this section we will denote the space of the real numbers by $\mathbb{R}$, the space of strictly positive real numbers by $\mathbb{R}^+$ and the space of fuzzy number by $\mathbb{R}_\mathcal{F}$.

A fuzzy number A is a fuzzy subset in $\mathbb{R}$ with normal, fuzzy convex and continuous membership functions of bounded support. The membership function of the fuzzy number A is denoted by $A(x)$ and fuzzy numbers can be considered as possibility distributions (Fullér and Majlender, 2004). The well-known α-levels sets of the fuzzy number A are given by

$$[A]_\alpha = \{x \in \mathbb{R} : A(x) \geq \alpha\}, \text{ for } \alpha > 0$$

and

$$[A]_0 = \text{cl}\{x \in \mathbb{R} : A(x) > 0\}, \text{ for } \alpha = 0.$$

We will denote by

$$[A]_\alpha = [a^-(\alpha), a^+(\alpha)],$$

where $a^-(\alpha), a^+(\alpha)$ are the left and right end points (Barros et al., 2017).

A n-dimensional joint possibility distribution J is a fuzzy subset of $\mathbb{R}^n$ with a normal membership function and a compact support. We denote by $\mathcal{F}(\mathbb{R}^n)$ the family of joint possibility distribution of $\mathbb{R}^n$.

Let $A_1, A_2, ..., A_n$ be fuzzy numbers and $J \in \mathcal{F}(\mathbb{R}^n)$, then (Carlsson et al., 2004) defines that J is a joint possibility distribution of $A_1, A_2, ..., A_n$ if

$$\max_{x_j \in \mathbb{R}, j \neq i} J(x_1, ..., x_n) = A_i(x_i). \tag{1.1}$$

Moreover, A_i is called the i-th marginal distribution marginal of J and we denote $A_i = \pi_i(J)$, where π_i is the projection operator in $\mathbb{R}^n$ onto ith axis, $i = 1, \dots n$ (Fullér and Majlender, 2004).

The interactivity between fuzzy numbers is determined from a possibility distribution (Carlsson et al., 2004).

If J is a possibility distribution of fuzzy numbers $A_1, A_2, ..., A_n$, then the following relationship holds

$$J(x_1, ..., x_n) \leq \min\{A_1(x_1), ..., A_n(x_n)\}.$$

In addition,

$$[J]_\alpha \subseteq [A_1]_\alpha \times ... \times [A_n]_\alpha, \ \forall \alpha \in [0, 1].$$

We say that the fuzzy numbers $A_1, A_2, ..., A_n$ are non interactive when

$$J(x_1, ..., x_n) = \min\{A_1(x_1), ..., A_n(x_n)\},$$

or equivalently,

$$[J]_\alpha = [A_1]_\alpha \times ... \times [A_n]_\alpha, \ \forall \alpha \in [0,1].$$

Otherwise they are interactive.

Let J be a joint possibility distribution of $A_1, ..., A_n \in \mathbb{R}_\mathcal{F}$ and $f : \mathbb{R}^n \to \mathbb{R}$ a continuous function. The function f_J is said to be the extension principle of f via J (Carlsson et al., 2004) and will be defined by

$$f_J(A_1, ..., A_n)(y) = \begin{cases} \sup_{y=f(x_1,...,x_n)} J(x_1, ..., x_n) & \text{if} \quad y \in \mathrm{Im}(f) \\ 0 & \text{if} \quad y \notin \mathrm{Im}(f) \end{cases}. \tag{1.2}$$

Theorem 1.1 *(Carlsson et al., 2004) Let $A_1, ..., A_n$ be completely correlated fuzzy numbers, J their joint possibility distribution and $f : \mathbb{R}^n \to \mathbb{R}$ a continuous function. Then,*

$$[f_J(A_1, ..., A_n)]_\alpha = f([J]_\alpha),$$

for all $\alpha \in [0,1]$.

From (Fullér and Majlender, 2003), the weighting function $w : [0,1] \longrightarrow \mathbb{R}$, it is a non-negative, monotone increasing and normalized over the unit interval function, that is,

$$\int_0^1 w(s)ds = 1.$$

Let J be a joint possibility distribution in $\mathbb{R}^n$, let $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ be an integrable function, and let $\alpha \in [0,1]$. The central value of p on $[J]_\alpha$ is given by

$$\mathcal{C}_{[J]_\alpha}(p) = \frac{1}{\int_{[J]_\alpha} dx} \int_{[J]_\alpha} p(x)dx. \tag{1.3}$$

Additionally, if J is a degenerated set then we compute $\mathcal{C}_{[J]_\alpha}(p)$ as the limit case of a uniform approximation of $[J]_\alpha$ with non-degenerated sets (Fullér and Majlender, 2004).

Let J be a joint possibility distribution in $\mathbb{R}^n$, let $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ be an integrable function, and let w be a weighting function. Then the expected value of p on J with respect to w is given by (Fullér and Majlender, 2004)

$$E_w(p; J) = \int_0^1 \mathcal{C}_{[J]_\alpha}(p) w(\alpha) d\alpha. \tag{1.4}$$

Notice that $E_w(p;J)$ computes the w-weighted average of the central values of function p on the level sets of J and, for any possibility distribution J, we have that $E_w(\cdot;J)$ is a linear operator.

Let J be a joint possibility distribution in $\mathbb{R}^2$ with marginal possibility distributions $A = \pi_x(J)$ and $B = \pi_y(J)$, and let $\alpha \in [0,1]$. Then the measure of interactivity between the α-level sets of A and B (with respect to $[J]_\alpha$) is given by (Fullér and Majlender, 2004)

$$I_\alpha(\pi_x,\pi_y) = \mathcal{C}_{[J]_\alpha}(\pi_x\pi_y) - \mathcal{C}_{[J]_\alpha}(\pi_x)\mathcal{C}_{[J]_\alpha}(\pi_y) =$$
$$\frac{1}{\int_{[J]_\alpha} dxdy}\int_{[J]_\alpha} xy\,dxdy - \left(\frac{1}{\int_{[J]_\alpha} dxdy}\int_{[J]_\alpha} x\,dxdy\right)\left(\frac{1}{\int_{[J]_\alpha} dxdy}\int_{[J]_\alpha} y\,dxdy\right),$$

for all $\alpha \in [0,1]$.

The measure of covariance between A and B (with respect their joint possibility distribution J and weighting function w) is given by

$$\mathrm{Cov}_w(A,B) = E_w(p;J) = \int_0^1 \mathrm{I}_\alpha(A,B)w(\alpha)d\alpha, \tag{1.5}$$

where p is the interactivity function associated with $[J]_\alpha$, $\alpha \in [0,1]$. In other words, the covariance between A and B is computed as the expected value of the interactivity function on the joint distribution J. That is, the equation (1.5) is the average interactivity between the levels of A and B (by w).

Carlsson, Fúller and Majlender (Carlsson et al., 2004) introduced the concept of completely correlated fuzzy numbers using the concept of possibility distribution.

Let A_1 and A_2 be fuzzy numbers. Then, (Carlsson et al., 2004) defines that A_1 and A_2 are completely correlated fuzzy numbers if exist $q \neq 0$ and r real numbers, such that their joint possibility distribution is given by

$$\begin{aligned} C(x_1,x_2) &= A_1(x_1)\mathcal{X}_{\{qx_1+r=x_2\}}(x_1,x_2) \\ &= A_2(x_2)\mathcal{X}_{\{qx_1+r=x_2\}}(x_1,x_2), \end{aligned} \tag{1.6}$$

where $\mathcal{X}_{\{qx_1+r=x_2\}}(x_1,x_2)$ is the characteristic function of the line $\{(x_1,x_2) \in \mathbb{R}^2 : qx_1 + r = x_2\}$.

Barros and Santo Pedro (2017), introduced the concept of linearly correlated fuzzy numbers, in order to avoid the knowledge of the joint distribution. Two fuzzy numbers A and B are called linearly correlated (Simões, 2017) if there exist $q, r \in \mathbb{R}$ such that

$$[B]_\alpha = q[A]_\alpha + r,$$

for all $\alpha \in [0,1]$.

When $q > 0$ ($q < 0$), we say that A and B are linearly positively (negatively) correlated.

Now let us assume that $A, B \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$ are linearly correlated and the joint possibility distribution is given by C in (1.6). The covariance between A and B, with respect to their joint possibility distribution C, is (Fullér and Majlender, 2004)

$$\begin{aligned}\mathrm{Cov}_w(A,B) &= \int_0^1 I_\alpha(A,B)w(\alpha)d\alpha \\ &= \pm\frac{1}{12}\int_0^1 [a^+(\alpha)-a^-(\alpha)][b^+(\alpha)-b^-(\alpha)]w(\alpha)d\alpha,\end{aligned} \tag{1.7}$$

where the sign is positive if A and B are linearly positive and negative if A and B are linearly negative.

From the fact that A and B are fuzzy numbers linearly correlated, we have that there exists $q > 0$ (or $q < 0$), such that $B = qA + r$ and

$$b^+(\alpha) - b^-(\alpha) = q(a^+(\alpha) - a^-(\alpha)), \;\; \forall\alpha \in [0,1].$$

Thereby,

$$\mathrm{Cov}_w(A,B) = \frac{q}{12}\int_0^1 [a^+(\alpha)-a^-(\alpha)]^2 w(\alpha)d\alpha. \tag{1.8}$$

1.2. Linearly correlated fuzzy process and interactive derivative.

A fuzzy process $F : [a,b] \longrightarrow \mathbb{R}_{\mathcal{F}}$ is a fuzzy-valued function which, for each t, associates a fuzzy number. The level sets of $F(t)$ are non-empty, compact and convex subsets of $\mathbb{R}$.

The process F is called autocorrelated fuzzy process when, for h with absolute value sufficiently small, there is a joint possibility distribution that relates $F(t+h)$ with $F(t)$ for all $t, t+h \in [a,b]$.

In particular, when there exist $q(h)$ and $r(h)$, for h with absolute value sufficiently small, such that, in levels, we have

$$[F(t+h)]_\alpha = q(h)[F(t)]_\alpha + r(h), \tag{1.9}$$

then we say that F is a locally linearly correlated fuzzy process.

The equation (1.9) means that the future value $F(t+h)$ is linearly correlated with the present value $F(t)$, for each h with absolute value sufficiently small.

The metric used is the Pompieu-Hausdorff distance $d_\infty : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ \cup \{0\}$, and it is defined (Barros et al., 2017) by

$$d_\infty(A, B) = \sup_{0 \le \alpha \le 1} \max\{|a^-(\alpha) - b^-(\alpha)|, |a^+(\alpha) - b^+(\alpha)|\},$$

where $A, B \in \mathbb{R}_\mathcal{F}$, $[A]_\alpha = [a^-(\alpha), a^+(\alpha)]$ and $[B]_\alpha = [b^-(\alpha), b^+(\alpha)]$.

Let $F : [a, b] \to \mathbb{R}_\mathcal{F}$ be an linearly correlated fuzzy process. According to (Barros and Santo Pedro, 2017), F is called L-differentiable at t_0, if there exists a fuzzy number $F_L^{'}(t_0)$ such that the limit

$$\lim_{h \to 0} \frac{F(t_0 + h) -_L F(t_0)}{h}$$

exists and is equal to $F_L^{'}(t_0)$ using the metric d_∞. In addition, $F_L^{'}(t_0)$ is called linearly correlated fuzzy derivative of F at t_0. At the endpoints of $[a, b]$, we consider only one-sided derivative.

2. Interactive Malthusian Model

2.1. Malthusian decay

Consider the fuzzy initial value problem given by the Malthusian decay

$$\begin{cases} X^{'}(t) & = & -\lambda X \\ X(t_0) & = & X_0 \end{cases}, \tag{2.10}$$

where $\lambda > 0$ and X_0 is a triangular fuzzy number.

In Barros and Santo Pedro (2017) we can see that FIVP (2.10) can be written in levels by:

$$\begin{cases} [(x_t^+)^{'}(\alpha), (x_t^-)^{'}(\alpha)] = [-\lambda x_t^+(\alpha), x_t^-(\alpha)] & \text{if} \quad 0 < q(h) < 1 \\ [(x_t^-)^{'}(\alpha), (x_t^+)^{'}(\alpha)] = [-\lambda x_t^+(\alpha), x_t^-(\alpha)] & \text{if} \quad q(h) > 1 \end{cases}, \tag{2.11}$$

and the solution is given, in levels, by:

$$[X(t)]_\alpha = [x_0^-(\alpha), x_0^-(\alpha)]e^{-\lambda\, t} \ \text{ if } \ 0 < q(h) < 1. \tag{2.12}$$

$$\begin{cases} x_\alpha^-(t) & = & c^-(\alpha)e^{\lambda t} + c^+(\alpha)e^{-\lambda t} \\ x_\alpha^+(t) & = & -c^-(\alpha)e^{\lambda t} + c^+(\alpha)e^{-\lambda t} \end{cases} \ \text{if } \ q(h) > 1 \tag{2.13}$$

where

$$c^-(\alpha) = \frac{x_0^-(\alpha) - x_0^+(\alpha)}{2} \text{ and } c^+(\alpha) = \frac{x_0^-(\alpha) + x_0^+(\alpha)}{2}.$$

Therefore, by equation (1.8), we have:

– for $0 < q(h) < 1$

$$\mathrm{Cov}_w(X(t), X(t+h)) = \frac{e^{-\lambda(2t+h)}}{12} \int_0^1 [x_0^+(\alpha) - x_0^-(\alpha)]^2 w(\alpha) d\alpha; \quad (2.14)$$

– for $q(h) > 1$

$$\mathrm{Cov}_w(X(t), X(t+h)) = \frac{e^{\lambda(2t+h)}}{12} \int_0^1 [x_0^+(\alpha) - x_0^-(\alpha)]^2 w(\alpha) d\alpha. \quad (2.15)$$

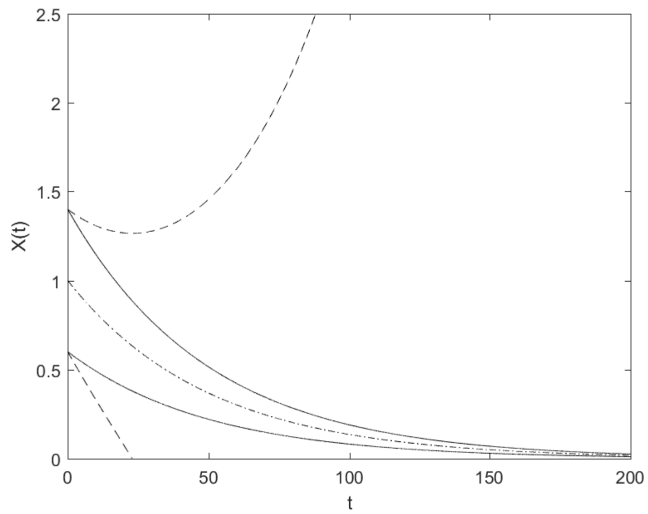

Figure 1: Solutions of the Malthusian model (2.10). The 0-level (continuous curve) of the solution with $0 < q_2 < 1$, the 0-level (dashed curve) of the solution with $q_2 > 1$ and the 1-level (dash-point curve) of both. We consider $x_0 = (0.6; 1; 1.4)$ and $\lambda = 0.02$.

Consider $X_0 = (0.6; 1; 1.4)$ the triangular fuzzy number such that $[X_0]_\alpha = [0.6 + 0.4\alpha, 1.4 - 0.4\alpha]$ and $\lambda = 0.02$. Next we compute $\mathrm{Cov}_w(X(t+h), X(t))$ for some weighting functions w.

Example 2.1 *Let us consider*

$$w(\alpha)=\begin{cases} 2\alpha, & \alpha\in[0,1] \\ 0, & other\ wise \end{cases}.$$

So,

– *for* (2.14)

$$\begin{aligned} Cov_w(X(t),X(t+h)) &= \frac{e^{-0.02(2t+h)}}{12}\int_0^1 [0.8-0.8\alpha]^2 2\alpha\, d\alpha \\ &= \frac{0.1067e^{-0.02(2t+h)}}{12} = 0.0089e^{-0.02(2t+h)}; \end{aligned} \tag{2.16}$$

– *for* (2.15)

$$\begin{aligned} Cov_w(X(t),X(t+h)) &= \frac{e^{0.02(2t+h)}}{12}\int_0^1 [0.8-0.8\alpha]^2 2\alpha\, d\alpha \\ &= \frac{0.1067e^{0.02(2t+h)}}{12} = 0.0089e^{0.02(2t+h)}. \end{aligned} \tag{2.17}$$

Example 2.2 *Consider the weighting function given by*

$$w(\alpha)=\begin{cases} \frac{2\alpha}{k}, & \alpha\in[0,k] \\ \frac{2(\alpha-1)}{k-1}, & \alpha\in[k,1] \\ 0, & other\ wise \end{cases},$$

where $k\in(0,1)$, *so we have*

– *for* (2.14)

$$\begin{aligned} Cov_w(X(t),X(t+h)) = \frac{e^{-0.02(2t+h)}}{12}\Bigg(\int_0^k [0.8-0.8\alpha]^2\frac{2\alpha}{k}\, d\alpha \\ + \int_k^1 [0.8-0.8\alpha]^2\frac{2(\alpha-1)}{k-1}\, d\alpha\Bigg) \\ =(0.32k^3-0.8533k^2+0.64k+0.32(k-1)^3)\frac{e^{-0.02(2t+h)}}{12}; \end{aligned} \tag{2.18}$$

– *for* (2.15)

$$Cov_w(X(t),X(t+h)) = \frac{e^{0.02(2t+h)}}{12}\left(\int_0^k [0.8-0.8\alpha]^2 \frac{2\alpha}{k}\,d\alpha + \int_k^1 [0.8-0.8\alpha]^2 \frac{2(\alpha-1)}{k-1}\,d\alpha\right) = (0.32k^3 - 0.8533k^2 + 0.64k + 0.32(k-1)^3)\frac{e^{0.02(2t+h)}}{12}. \tag{2.19}$$

Taking $k = 0.5$ *in* (2.18) *and* (2.19), *we have, respectively,* $Cov_w(X(t), X(t+h)) = 0.1067e^{-0.02(2t+h)}$ *and* $Cov_w(X(t), X(t+h)) = 0.1067e^{-0.02(2t+h)}$.

Example 2.3 *Consider the weighting function given by*

$$w(\alpha) = \beta(\alpha) = \frac{\alpha^{a-1}(1-\alpha)^{b-1}}{B(a,b)}, \ \ \alpha \in [0,1]$$

where $B(a,b) = \int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}$ *is the beta function. The beta function can we written by* $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, *where* $\Gamma(z) = \int_0^\infty \alpha^{z-1}e^{-\alpha}d\alpha$ *(Barros, 2015). According to (Gupta and Nadarajah, 2004), this weighting function is suitable for variables measure on interval* $(0,1)$, *for example, rates and proportions. So we have*

– *for* (2.14)

$$Cov_w(X(t),X(t+h)) = \frac{e^{-0.02(2t+h)}}{12}\int_0^1 [0.8-0.8\alpha]^2 \frac{\alpha^{a-1}(1-\alpha)^{b-1}}{B(a,b)}\,d\alpha = \left(\frac{16\Gamma(a)\Gamma(b+2)}{25B(a,b)\Gamma(a+b+2)}\right)\frac{e^{-0.02(2t+h)}}{12}, \tag{2.20}$$

where $\Gamma(\alpha)$ *is the gamma function;*

– *for* (2.15)

$$Cov_w(X(t),X(t+h)) = \frac{e^{0.02(2t+h)}}{12}\int_0^1 [0.8-0.8\alpha]^2 \frac{\alpha^{a-1}(1-\alpha)^{b-1}}{B(a,b)}\,d\alpha = \left(\frac{16\Gamma(a)\Gamma(b+2)}{25B(a,b)\Gamma(a+b+2)}\right)\frac{e^{+0.02(2t+h)}}{12}, \tag{2.21}$$

where $\Gamma(\alpha)$ *is the gamma function.*

In (2.20) *and* (2.21)*, taking* $a = 3$ *and* $b = 1$*, we have, respectively,* $Cov_w(X(t), X(t+h)) = 0.005e^{-0.02(2t+h)}$ *and* $Cov_w(X(t), X(t+h)) = 0.005e^{-0.02(2t+h)}$*, and taking* $a = 4$ *and* $b = 1$*, we have, respectively,* $Cov_w(X(t), X(t+h)) = 0.003e^{-0.02(2t+h)}$ *and* $Cov_w(X(t), X(t+h)) = 0.003e^{-0.02(2t+h)}$.

Notice that β *distribution with* $a = 1 = b$ *is a uniform distribution on the interval* $[0, 1]$ *and we have, respectively,* $Cov_w(X(t), X(t+h)) = 0.0178e^{-0.02(2t+h)}$ *and* $Cov_w(X(t), X(t+h)) = 0.0178e^{-0.02(2t+h)}$.

Remark 2.1 (Malthusian growth) *If we consider the fuzzy initial value problem given by the Malthusian growth*

$$\begin{cases} X'(t) &= \lambda X \\ X(t_0) &= X_0 \end{cases}, \qquad (2.22)$$

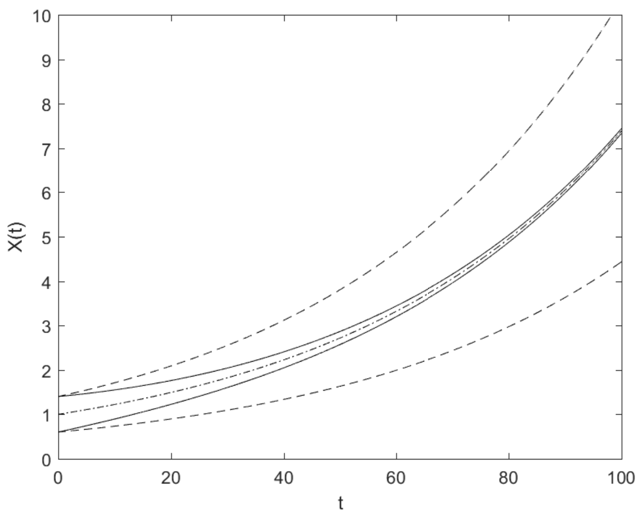

Figure 2: Solutions of the Malthusian model (2.22). The 0-level (continuous curve) of the solution with $0 < q_2 < 1$, the 0-level (dashed curve) of the solution with $q_2 > 1$ and the 1-level (dash-point curve) of both. We consider $x_0 = (0.6; 1; 1.4)$ and $\lambda = 0.02$.

where $\lambda > 0$ *and* X_0 *is a triangular fuzzy number. Then, by equation* (1.8)*, we have:*

– *for* $0 < q(h) < 1$,

$$Cov_w(X(t), X(t+h)) = \frac{e^{-\lambda(2t+h)}}{12}\int_0^1 [x_0^+(\alpha) - x_0^-(\alpha)]^2 w(\alpha)d\alpha; \quad (2.23)$$

– *for* $q(h) > 1$

$$Cov_w(X(t), X(t+h)) = \frac{e^{\lambda(2t+h)}}{12}\int_0^1 [x_0^+(\alpha) - x_0^-(\alpha)]^2 w(\alpha)d\alpha. \quad (2.24)$$

We obtain the same covariance for systems (2.10) *and* (2.22). *This result shows us that when the parameter q from L-derivative is between 0 and 1, the covariance decreases and when it is bigger than 1, the covariance increases over time no matter what is the signal of* λ. *Thus, what influences in the measure of covariance is the parameter q in the L-derivative and not the signal of parameter* λ. *Following, we have the final comments.*

3. Conclusion

In this paper, we compute the covariance between $X(t+h)$ and $X(t)$, where X is a linearly correlated fuzzy process and the solution of FIVP (2.10). By computing the covariance, we conclude, by (2.14) and (2.15), that for any weighting function w, when $t \to \infty$, we have

– for $0 < q(h) < 1$: $\text{Cov}_w(X(t), X(t+h)) \to 0$;

– for $q(h) > 1$: $\text{Cov}_w(X(t), X(t+h)) \to \infty$.

In this way, we also conclude that the parameter q in L - derivative is the one that determines if the fuzziness of the dynamic system increases or decreases over time (fuzziness in the sense of solution diameter) and not the signal of the parameter λ in (2.10). Thus, when modeling a phenomenon, we need to use a suitable derivative, i.e., a derivative that is capable of capturing the interactivity presented in the problem.

Acknowledgment

The authors would like to thank CNPq 305862/2013 – 8 and FAPESP 2016/26040 – 7.

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