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A delay in the vaccine term exacerbates the backward bifurcation phenomenon

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Abstract. In this work, an epidemiological differential model is constructed using a discrete time delay in the vaccine term of the model. The model without delay shows a backward bifurcation phenomenon when $R_0 = 1$. In this case, a bistability phenomenon occurs between the disease-free equilibrium and an endemic equilibrium point, for $R_0 < 1$. However, when a discrete delay in the vaccine term is considered, a different type of bistability phenomenon occurs. In this scenario, the bistability phenomenon can occur between the disease-free equilibrium and either an endemic equilibrium point or an periodic orbit. We show that the model with delay can present a Hopf bifurcation for $R_0 < 1$, which can be catastrophic for the susceptible population. Finally, numerical simulations are presented to illustrate the results.

Key-words*: Multiple endemic states; backward bifurcation; Hopf bifurcation; delay model.*

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1. Introduction

In theoretical epidemiology, there has been a big effort to understand and to know the consequences of a public health strategy when it is applied in a population. In Kribs-Zaleta and Velasco-Hernández (2000) it was analyzed a model without delay where a strategy of vaccination is applied over the susceptible population. They show that multiple endemic equilibrium points can appear for $R_0 < 1$ (see Figure 1). In this case, a bistability phenomenon occurs and depending on the number of infectious individuals at the beginning of the disease, there will be an epidemic outbreak or the disease will be eradicated. A backward bifurcation is considered an undesirable phenomenon because the classical policy of bringing R_0 below 1 is not a sufficient condition to control the disease (Sharomi and Gumel, 2009; Hadeler and Castillo-Chávez, 1995; Garba et al., 2008; Dushoff et al., 1998; Zhang and Liu, 2009; Lou et al., 2013).

It is known that, the application of a vaccine has been an important public health strategy because the vaccine application has not only reduced considerably the possible number of infectious individuals of some infectious diseases such as rubella, measles and mumps, but also, in some cases, the eradication of the disease has been possible. This is the case for the smallpox, which has been eradicated (see Bazin (2000)), and the rubella and the measles will be eradicated from Europa in the next years. In the mathematical epidemiology literature there are models analyzing how the application of a vaccine affects the evolution of a disease; for example, the effect of different awareness of vaccination programs has been studied through epidemiological models (see Misra et al. (2011); Zuo and Liu (2014); Zuo et al. (2015); Shim and Galvani (2012)), additionally, the efficiency and efficacy of a vaccine have been analyzed in other works (see Shim and Galvani, 2012; Kribs-Zaleta and Velasco-Hernández, 2000).

Through the years there has been a lot of discussion about differential equation models without delay; however, the use of a time delay in demographic or epidemiological variables has not been profusely analyzed. Some works have modeled the delay in media coverage of an epidemic outbreak; (Zhao et al. (2014)). There are time delay models considering the level of disease awareness (Misra et al. (2011); Zhao et al. (2014); Zuo and Liu (2014); Zuo et al. (2015)), and pulse vaccination models with a delayed term representing disease incubation time have been reviewed, too (Meng et al., 2010; Schenzle, 1984; Agaba et al., 2017).

With this in mind, we propose an epidemiological model with delay in the vaccine term. We analyze the effect of the delay over the bistability phenomenon. To do this, in section 2 we present a time-delayed epidemiological model with vaccination and the basic reproduction number is calculated. Section 3 contains analytical results on feasibility and stability of the equilibrium points of the model. Additionally, conditions for the existence of a Hopf bifurcation in one endemic steady state are given. In Section 4, numerical simulations are carried out to illustrate the behavior of the solutions of the model for different scenarios. Finally, in Section 5 the discussion of the results are given.

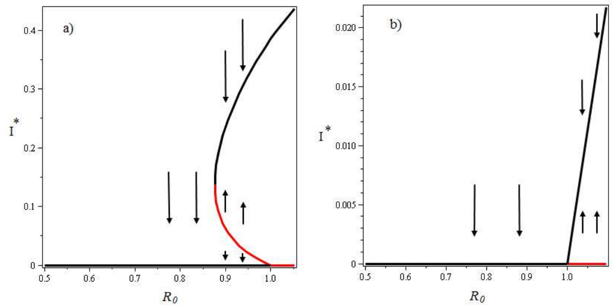


Figure 1: Figure a) shows a backward bifurcation in $R_0 = 1$. Multiple endemic states are showed when $\mathrm{R}_0 < 1$. In this case, a phenomenon of bistability appears, meanwhile, b) shows a forward bifurcation in $R_0 = 1$. Equilibria endemic do not exist for values of R_0 below one, and an unique endemic equilibrium exists for $R_0 > 1$. Black lines are locally stable equilibria points and the red lines are unstable equilibria.

2. The model

At any given time t, a size N population is considered which is divided into three classes; susceptible individuals, infectious individuals and vaccinated individuals. All the model parameters are considered non-negative constants. The death rate is denoted by μ and the number of deaths in each class is considered proportional to the size of each class. Λ is the recruitment rate of susceptible individuals and all new born are assumed to be susceptible individuals. β is the infectious rate and new infectious individuals are modelled using the mass-action-law. Therefore, βSI represents the number of new infectious individuals per time unit. c is the rate at which individuals recover and return to susceptible class from infectious class and the number of recovered individuals is assumed to be proportional to the size of the class. The susceptible population is vaccinated at a rate ϕ. Vaccinated individuals experience loss of immunity after an average time $\frac{1}{\theta}$. The parameter $\sigma \in [0,1]$ is used to describe the vaccine efficiency. So, $\sigma = 0$ means a totally effective vaccine whereas $\sigma = 1$ describes a totally useless vaccine. Finally, τ represents the time the vaccine needs for a vaccinated individual to be protected. So, the term $\phi S(t-\tau)$ describes the susceptible individuals that will be in the vaccinated class τ time after being vaccinated.

Then, the delay epidemiological model is given by:

$$\begin{aligned} \dot{S} &= \Lambda - \beta SI - (\mu + \phi)S + cI + \theta V, \\ \dot{I} &= \beta SI + \sigma\beta VI - (\mu + c)I, \\ \dot{V} &= \phi S(t-\tau) - \sigma\beta VI - (\mu + \theta)V. \end{aligned} \tag{2.1}$$

The initial conditions for the model (2.1) are $S(\theta) = S_0$, $I(0) = 0$ and $V(\theta) = V_0$, $\theta \in [-\tau, 0)$. Also, the positive octant is invariant under the system 2.1.

For the epidemiological model (2.1), the basic reproduction number is given by

$$R_0 = \frac{\beta\Lambda(\mu + \theta + \sigma\phi)}{\mu(\mu + c)(\mu + \theta + \phi)}. \tag{2.2}$$

Observe that the delay τ does not appear in the basic reproduction number.

3. Existence of equilibria points

For system (2.1), taking the system of non linear equations that gives the fixed points and solving for the first and the third equation, for S^* and V^* respectively and substituting those values in the second equation, the following equilibrium equation is obtained.

$$I^* f(I^*) = 0. \tag{3.3}$$

Where

$$f(I^*) = A(I^*)^2 + BI^* + C, \tag{3.4}$$

and the coefficients of the quadratic function (3.4) are given by

$$\begin{aligned} A &= -\sigma\beta\mu, \\ B &= \Lambda\sigma\beta - \mu(\sigma(\mu + c + \phi) + \mu + \theta), \\ C &= \Lambda(\mu + \theta + \phi)(1 - \frac{1}{R_0}). \end{aligned} \tag{3.5}$$

From equation (3.3) the disease-free equilibrium point is obtained, which has the coordinates $I^* = 0$, $V^* = \frac{\Lambda\phi}{\mu(\mu+\theta+\phi)}$ and $S^* = \frac{\Lambda(\mu+\theta)}{\mu(\mu+\theta+\phi)}$. The disease-free equilibrium is denoted by E_0.

The endemic equilibrium points are given by the positive solutions of the equation (3.3). Then, analyzing the coefficients given in (3.5), it can be observed that

- A is always negative for all the values of the models parameters.
- $C > 0$ if and only if $R_0 > 1$.
- $f(0) = C$.
- $f(\frac{\Lambda}{\mu}) < 0$.

Therefore, if $R_0 > 1$ then $C > 0$, and a unique positive zero of $f(I^*)$ exists. If $R_0 < 1$, there can be either two positive roots of $f(I^*)$ or one double positive root of $f(I^*)$ or no positive roots, depending on whether some conditions over the coefficients (3.5) are satisfied. These observations can be summarized in the following result:

Theorem 1 *For model (2.1), the number of endemic states with positive entries behaves according to the following options:*

a) If $R_0 > 1$, the model have exactly one positive endemic state.

b) If $R_0 < 1$ and $B < 0$ the model have no positive endemic states.

c) If $R_0 < 1$, $B > 0$ and $B^2 - 4AC = 0$ the model has a unique positive endemic state, which is a double root of $f(I^*)$.

d) If $R_0 < 1$, $B > 0$ and $B^2 - 4AC > 0$ the models have two positive endemic states.

Notice that, $R_0^* = \frac{4\Lambda A(\mu+\theta+\theta\sigma\phi)}{B^2+4\Lambda A(\mu+\theta+\theta\sigma\phi)}$ is a critical value of R_0 below one, where a pair of endemic equilibrium states are suddenly created. Then, R_0^* is a bifurcation point, which is found solving the equation $B^2 - 4AC = 0$. Therefore, combining the results of the theorem (1), it can be concluded that multiple non-negative endemic states can occur for values of $R_0^* < R_0 < 1$ when some conditions over the parameters are satisfied.

The bifurcation diagram for system (2.1) when $\tau = 0$ may be one of two possible options, which are showed in Figure 1 (see Kribs-Zaleta and Velasco-Hernández (2000)). Which one actually occurs depends only on the parameters values of the models. However, the bifurcation diagrams may change because a periodic solution could exist for $\tau > 0$ when $R_0 < 1$.

In the following section, the stability of the equilibrium points for $\tau > 0$ will be analyzed.

3.1 Stability of the disease-free equilibrium point

Knowing the stability of the equilibrium states of the model is fundamental to understand the dynamics of the model solutions. For this, analysis of stability of the equilibrium states will be done as follows.

The linearized system associated to the delay model (2.1) is given by;

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = A_1 \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} + A_2 \begin{pmatrix} x(t-\tau) \\ y(t-\tau) \\ z(t-\tau) \end{pmatrix}. \tag{3.6}$$

Where matrices A_1 and A_2 depend on the parameters model.

In the following, the linearized system associated to the system (2.1) in the disease-free equilibrium point will be analyzed.

Theorem 2 *The disease-free equilibrium E_0 for system (2.1) is locally asymptotically stable for all $\tau \geq 0$ if and only if $R_0 < 1$.*

Proof.

The matrices A_1 and A_2, for the linearized model (2.1), evaluated in the disease free equilibrium, are given by;

$$A_1 = \begin{pmatrix} -(\mu+\phi) & -\frac{\beta\Lambda(\mu+\theta)}{\mu(\mu+\theta+\phi)} + c & \theta \\ 0 & \frac{\beta\Lambda(\mu+\theta+\sigma\phi)}{\mu(\mu+\theta+\phi)} - (\mu+c) & 0 \\ 0 & -\frac{\sigma\beta\Lambda\phi}{\mu(\mu+\theta+\phi)} & -(\mu+\theta) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \phi & 0 & 0 \end{pmatrix}.$$

The characteristic equation associated to the system (3.6), for the model (2.1), is given by

$$\Delta(\lambda) = |\lambda I - A_1 - e^{-\lambda\tau} A_2| = 0. \tag{3.7}$$

Where,
$\Delta(\lambda) = (\frac{\mu\,(\mu+\theta+\phi)\lambda - \Lambda\,(\mu+\theta+\sigma\,\phi)\beta + \mu\,(\mu+\theta+\phi)(\mu+c)}{\mu(\mu+\theta+\phi)})(\lambda^2 + a_1\lambda + a_2 + a_3 e^{-\lambda\tau})$,

and

$$\begin{aligned} a_1 &= 2\mu + \theta + \phi, \\ a_2 &= (\mu+\theta)(\mu+\phi), \\ a_3 &= -\theta\phi. \end{aligned} \tag{3.8}$$

$\lambda_1 = \Lambda(\mu+\theta+\phi)(1 - \frac{1}{R_0})$ is an eigenvalue of the linearization of the model (2.1) in the disease-free equilibrium. The other eigenvalues are given by the solutions of the next transcendental equation

$$(\lambda^2 + a_1\lambda + a_2 + a_3 e^{-\lambda\tau}) = 0. \tag{3.9}$$

The analysis of the existence of roots of the equilibrium equation (3.9) is too difficult because the equilibrium equation is a transcendental equation with infinite eigenvalues; however, stability analysis can still be done (see Culshaw and Ruan (2000)). It is known that the equilibrium solution is locally asymptotically stable if all the roots of its characteristic equation have negative real parts; however, the Routh-Hurwitz criterion cannot be applied to the transcendental equation, but a first result can be obtained for the equilibrium equation (3.9). For $\tau = 0$ the expression (3.9) is a polynomial and, in this case, the Routh-Hurwitz criterion can be used to analyze the stability of the equilibrium point, which will be done as follows.

Lemma 1 *Let $\tau = 0$. The disease-free equilibrium point is locally asymptotically stable if and only if $R_0 < 1$.*

Proof.
Let $\tau = 0$. Then, the characteristic equation (3.9) is

$$\lambda^2 + a_1\lambda + a_2 + a_3 = 0. \tag{3.10}$$

Observe that a_1 and $a_2 + a_3$ are positive constants. Therefore, the Routh-Hurwitz criterion is satisfied. Then, all the roots of the characteristic equation (3.9) have negative real parts. Furthermore, when $\tau = 0$, the disease-free equilibrium is locally asymptotically stable when $R_0 < 1$ and unstable when $R_0 > 1$ ■.

Above, the stability of the disease-free equilibrium was proved for $\tau = 0$. In the following, distribution of the roots of the transcendental equation (3.9) will be analytically studied. Also, conditions over the parameters of the model (2.1) are derived to ensure that the steady state of the delay model is still stable for all delay $\tau > 0$.

By Rouche's theorem and continuity arguments, the characteristic equation (3.7) has roots with positive real part if and only if the characteristic equation has purely imaginary roots. In this sense, in the paragraphs below, it will be proved that this is not the case for the disease-free equilibrium. That is, the characteristic equation does not have purely imaginary roots for all delay, Therefore, the steady state is always stable because all roots of the characteristic equation have negative real part.

Notice that, $\lambda = i\omega$ ($\omega > 0$) is a root of the characteristic equation (3.9) if and only if

$$-\omega^2 + a_1\omega i + a_2 + a_3\cos(\omega\tau) - a_3 i \sin(\omega\tau) = 0. \tag{3.11}$$

Separating the real and imaginary parts of the equation (3.11), the following expressions are obtained

$$a_1\omega = a_3\sin(\omega\tau),$$
$$-\omega^2 + a_2 = -a_3\cos(\omega\tau).$$

Adding up the squares of both the equations, the following expression is obtained;

$$\omega^4 + (a_1^2 - 2a_2)\omega + a_2^2 - a_3^2 = 0. \tag{3.12}$$

Let $z = \omega^2, \alpha = a_1^2 - 2a_2$ and $\delta = a_2^2 - a_3^2$. be. Then, left side of equation (3.12) can be reduced to the following quadratic function

$$h(z) = z^2 + \alpha z + \delta. \tag{3.13}$$

Where

$$\alpha = a_1^2 - 2a_2 = (\mu + \theta)^2 + (\mu + \phi)^2,$$
$$\delta = a_2^2 - a_3^2 = \mu(\mu + \theta + \phi)(\mu\phi + 2\theta\phi + \mu^2 + \mu\theta).$$

Notice that, both coefficients are positive.

Then, analyzing the quadratic function (3.13), the following result is achieved.

Lemma 2 *The quadratic function $h(z)$ has no positive real roots.*

Then, Lemma 2 implies that $i\omega$ is not an eigenvalue of the characteristic equation (3.9).

Therefore, the disease-free equilibrium E_0 is locally asymptotically stable for all delay $\tau \geq 0$ because the eigenvalues do not cross the imaginary axis.

Then, the proof of the Theorem 2 is concluded ■.

The stability analysis of the endemic equilibrium points, which are in the branches of the bifurcation diagram, will be done as follows.

To study the stability of the endemic states for the model (2.1), the following changes of variable are proposed.

$$x(t) = S(t) - S^*, y(t) = I(t) - I^*, z(t) = V(t) - V^*. \tag{3.14}$$

Matrices A_1 and A_2 associated to the linearized system (3.6) of the model (2.1), evaluated in the endemic states, is showed below.

For model (2.1) matrices A_1 and A_2 are given by;

$$A_1 = \begin{pmatrix} -(\mu + \phi) - \beta I^* & -\beta S^* + c & \theta \\ \beta I^* & \beta(S^* + \sigma V^*) - (\mu + c) & \sigma\beta I^* \\ 0 & -\sigma\beta V^* & -\sigma\beta I^* - (\mu + \theta) \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \phi & 0 & 0 \end{pmatrix}.$$

The characteristic equations $\Delta(\lambda) = \mid \lambda I - A_1 - e^{-\lambda\tau} A_2 \mid = 0$ associated to the linearized system (3.6) evaluated in the non trivial equilibria of the system (2.1) is given by

$$\Delta_P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 e^{-\lambda\tau} + a_4\lambda e^{-\lambda\tau} + a_5 = 0 \tag{3.15}$$

Where

$$a_1 = \beta(\sigma+1)I^* - (\sigma V^* + S^*)\beta + 3\mu + c + \theta + \phi,$$

$$\begin{aligned} a_2 = & -\beta\,(\sigma\,\beta\,I^* + 2\,\mu + \phi + \theta)\,S^* - \sigma\,\beta\,(\beta\,I^* + 2\,\mu + \theta + \phi)\,V^* + \beta^2 {I^*}^2 \sigma + \\ & + \beta\,(\phi\,\sigma + 2\,\mu + \theta + c\sigma + 2\,\mu\,\sigma)\,I^* + \\ & + c\theta + 3\,\mu^2 + \phi\,c + 2\,c\mu + 2\,\phi\,\mu + 2\,\mu\,\theta + \phi\,\theta, \end{aligned}$$

$$a_3 = \phi(\theta\beta(\sigma V^* + S^*) - \theta(\mu + c) + \sigma\beta I^*(\beta S^* - c)),$$

$$a_4 = -\theta\phi,$$

$$\begin{aligned} a_5 = & -\beta\,(\mu + \phi)\,(\mu + \sigma\,\beta\,I^* + \theta)\,S^* - \sigma\,\beta\,\left(\beta\,I^*\mu + \mu^2 + \phi\,\theta + \phi\,\mu + \mu\,\theta\right)V^* + \\ & + \beta^2 {I^*}^2 \mu\,\sigma + \beta\,\left(\mu^2 + \phi\,c\sigma + c\mu\,\sigma + \mu\,\theta + \sigma\,\mu^2 + \phi\,\sigma\,\mu\right)I^* + \\ & (\mu + \theta)\,(\mu + \phi)\,(\mu + c)\,. \end{aligned}$$

First, the endemic equilibrium points in the lower branch of the bifurcation diagram will be analyzed.

Theorem 3 *For the system (2.1). Let I^* be an equilibrium point such that $0 < I^* < -\frac{B}{2A}$. If $R_0 < 1$, then the branch of equilibrium points is given by unstable equilibrium points for all $\tau \geq 0$.*

Proof.

Notice that for the system (2.1),

$$\lim_{\lambda \mapsto \infty} \Delta_P(\lambda) = \infty \tag{3.16}$$

and

$$\Delta_P(0) = a_3 + a_5. \tag{3.17}$$

Expanding the expression (3.17),

$$\begin{aligned} \Delta_P(0) = a_3 + a_5 &= \sigma\beta\mu(I^*)^2 + \mu(\mu + \theta + \sigma\phi + \sigma(\mu + c) - \sigma\beta(S^* + V^*))I^* + \\ &+ \mu(\mu + \theta + \phi)(\mu + c) - \beta\mu(\mu + \theta + \phi)(S^* + \sigma V^*) = \\ &= A_1^P(I^*)^2 + B_1^P I^* + C_1^P. \end{aligned} \tag{3.18}$$

Where

$$\begin{aligned} A_1^P &= \sigma\beta\mu, \\ B_1^P &= \mu(\mu + \theta + \sigma\phi + \sigma(\mu + c) - \sigma\beta(S^* + V^*)), \\ C_1^P &= \mu(\mu + \theta + \phi)(\mu + c) - \beta\mu(\mu + \theta + \phi)(S^* + \sigma V^*). \end{aligned} \tag{3.19}$$

Let $F_P(I^*) \equiv A_1^P(I^*)^2 + B_1^P I^* + C_1^P$, be. Then the following limits are calculated.

$$\lim_{R_0 \mapsto 1} F_P(I^*) = 0 \quad \text{and} \quad \lim_{R_0 \mapsto R_0{}^*} F_P(-\frac{B}{2A}) = 0. \tag{3.20}$$

Observe that, $\Delta_P(0) = F_P(I^*)$ is a convex quadratic function because $A_1^P > 0$. Also $I^* = 0$ and $I^* = -\frac{B}{2A}$ are zeros of the function $\Delta_P(0)$. Then, $\Delta_P(0) < 0$ for all the values of I^* on $0 < I^* < -\frac{B}{2A}$. Therefore, the characteristic equations $\Delta_P(\lambda)$ has at least one positive real eigenvalue. That is, the endemic equilibrium points that belong to the interval $0 < I^* < -\frac{B}{2A}$ are asymptotically unstable ■.

Second, stability of the equilibrium points of the upper branch of the bifurcation diagram for the model (2.1) will be discussed below.

Kribs-Zaleta and Velasco-Hernández ((see Kribs-Zaleta and Velasco-Hernández (2000))) proved that the equilibrium points in the upper branch of the bifurcation diagram are locally asymptotically stables when $\tau = 0$. That is, the coefficients of the characteristic equation (3.15) with $\tau = 0$ satisfy the Routh-Hurwitz criterion for some conditions over the parameters of the model.

It is known that equation (3.15) has roots with positive real part if and only if it admits purely imaginary roots.

Let $\lambda(\tau) = \eta(\tau) + \mathrm{i}\omega(\tau)$ ($\omega > 0$) an eigenvalue of (2.1). In the following we will determine if $\lambda = \mathrm{i}\omega(\tau)$ is a root of (3.15).

Notice that $\mathrm{i}\omega$ is an eigenvalue of (2.1) if and only if $\Delta(i\omega) = 0$. Then

$$-\mathrm{i}\omega^3 - a_1\omega^2 + \mathrm{i}a_2\omega + (a_3 + a_4\omega\mathrm{i})cos(\omega\tau) + (-\mathrm{i}a_3 + a_4\omega)\sin(\omega\tau) + a_5 = 0. \tag{3.21}$$

Then, separating the real and imaginary parts of the complex equation (3.21), and adding up their squares, the following expression is obtained

$$\omega^6 + (a_1^2 - 2a_2)\omega^4 + (a_2^2 - 2a_1a_5 - a_4^2)\omega^2 + (a_5^2 - a_3^2) = 0. \tag{3.22}$$

Using the following variable changes $y = \omega^2$, $\alpha = (a_1^2 - 2a_2)$, $\kappa = a_2^2 - 2a_1a_5 - a_4^2$ and $\zeta = a_5^2 - a_3^2$, the equation (3.22) can be reduced to

$$O(y) = y^3 + \alpha y^2 + \kappa y + \zeta = 0, \tag{3.23}$$

Then, equation (3.15) has purely imaginary roots if $O(y)$ admits a positive root. Analyzing the coefficient of $O(y)$, in the former case, we can see that if $\zeta < 0$, $O(y)$ has at least one positive root since $O(0) < 0$ and $\lim_{y\to\infty} O(y) = \infty$. In the second case, if $\kappa < 0$, then (3.23) has a critical point in $y_1 = \frac{1}{3}\left(-\alpha + \sqrt{\alpha^2 - 3\kappa}\right) > 0$. Notice that $O(y_1) < 0$ if $\zeta < -\frac{2}{27}\alpha^3 - \frac{1}{27}\sqrt{(\alpha^2 - 3\,\beta)^3} + \frac{1}{3}\,\beta\,\alpha - \frac{1}{3}\sqrt{\alpha^2 - 3\,\beta}\left(\beta - 1/3\,\alpha^2\right)$. Then. Therefore, $O(y)$ has two positive roots and consequently (3.22) admits a purely imaginary root ω_0. In conclusion the characteristic equation (3.15) has a pair of purely imaginary roots $\pm\mathrm{i}\omega_0$.

Let τ_0 the delay value such that $\eta(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$. From (3.21) we obtain

$$\tau_j = \frac{1}{\omega_0}\arccos\left(\frac{a_4\omega_0^4 + (a_1a_3 - a_2a_4)\omega_0^2 - a_3a_5}{a_3^2 + a_4^4\omega_0^2}\right) + \frac{2j\pi}{\omega_0} \tag{3.24}$$

for $j = 0, 1, 2, ...$ Also, the transversality condition

$$\frac{d}{d\tau}\eta(\tau_0) > 0 \tag{3.25}$$

can be proved.

Then, by continuity arguments, $\lambda(\tau)$ cross the imaginary axis with positive derivative when $\tau > \tau_0$. In summary, the above analysis can be written as follows.

Theorem 4 *Suppose that* $a_1 > 0, a_3 + a_5 > 0, a_1(a_2 + a_4) - (a_3 + a_5) > 0$. *If either*

- *i)* $\zeta < 0$ *or*
- *ii)* $\zeta \geq 0, \zeta < -\frac{2}{27}\alpha^3 - \frac{1}{27}\sqrt{(\alpha^2 - 3\beta)^3} + \frac{1}{3}\beta\alpha - \frac{1}{3}\sqrt{\alpha^2 - 3\beta}\,(\beta - 1/3\,\alpha^2)$ *and* $\kappa < 0$

is satisfied, then the endemic equilibrium point E^* *of the delay model (2.1) is asymptotically stable when* $\tau < \tau_0$ *and unstable when* $\tau > \tau_0$*, where*

$$\tau_0 = \frac{1}{\omega_0}\arccos\left(\frac{a_4\omega_0^4 + (a_1a_3 - a_2a_4)\omega_0^2 - a_3a_5}{a_3^2 + a_4^4\omega_0^2}\right) + \frac{2j\pi}{\omega_0} \text{ for } j = 0, 1, 2, ...$$

Therefore, a Hopf bifurcation occurs when $\tau = \tau_0$. *That is, there exists a periodic orbit of the model (2.1).*

In this moment, the principal results of the models are about existence of a backward bifurcation and a Hopf bifurcation. In the next section, numerical results about the solutions of the model (2.1) are showed.

4. Numerical simulations

Possible scenarios given by the solutions of model (2.1) will be showed. Particularly, a bistability phenomenon for model (2.1) is showed when both a backward bifurcation and a Hopf bifurcation exist simultaneously.

The values of the parameters for the simulations are given by; $\Lambda = 10, \beta = 0.0012, \mu = 0.02, c = 0.3, \theta = 0.001, \phi = 0.1, \sigma = 0.42$. For this values of the parameters, $R_0 = 0.9762396695$ and the equilibrium points of the model are given by: $E_0 = (86.78, 0, 413.22)$, $E_1 = (109.71, 16.59, 373.72,)$ and $E_2 = (164.14, 91.75, 244.10)$. The initial conditions are $S(0) = 200$, $I(0) = 100$, $V(0) = 50$ for all the scenarios shown in this section.

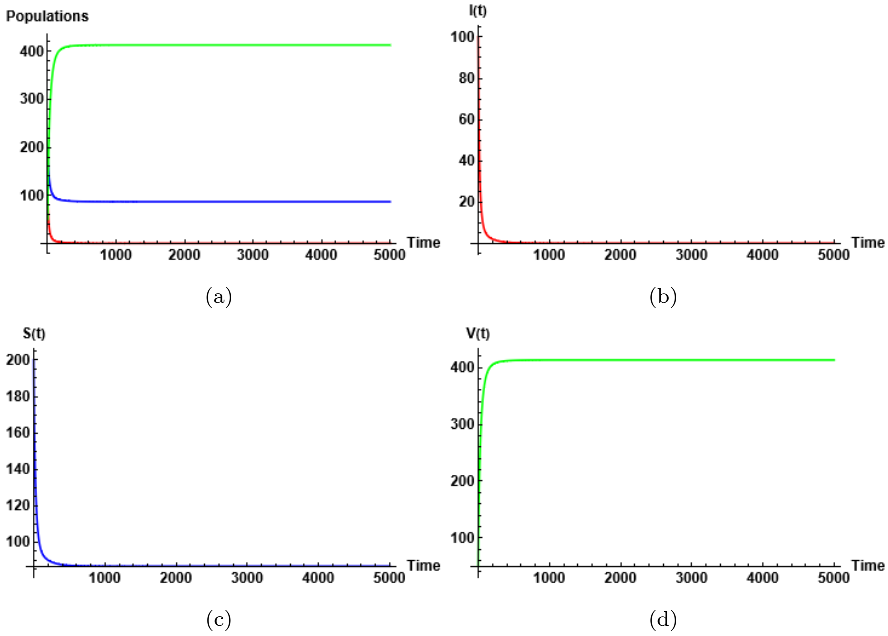
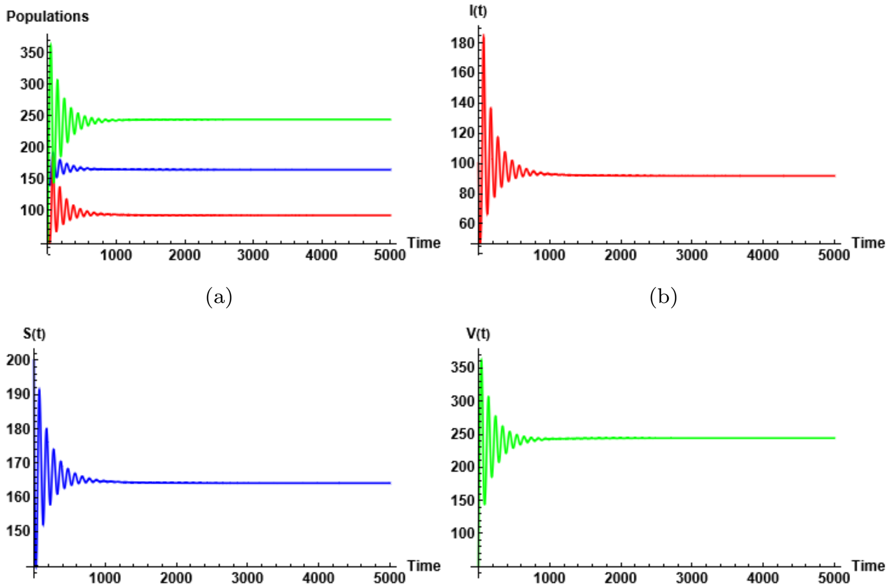


Figure 2: In this scenario $\tau = 0$. Populations go asymptotically towards a the disease-free equilibrium point E_0.



(a) (b) (c) (d)

Figure 3: In this scenario $\tau = 56$. Populations show forced oscillations and finally each solution goes to the endemic equilibrium point E_2. In this case, $a_1 = 0.2973500883$, $a_3 + a_5 = 0.0000834354196$, $a_1(a_2 + a_4) - (a_3 + a_5) = 0.002806442518$. Also $8.64656932 \times 10^{-8} < \zeta < 1.98587633 \times 10^{-7} \geq 0$ and $\kappa = -0.2365607137(10)^{-3} < 0$.

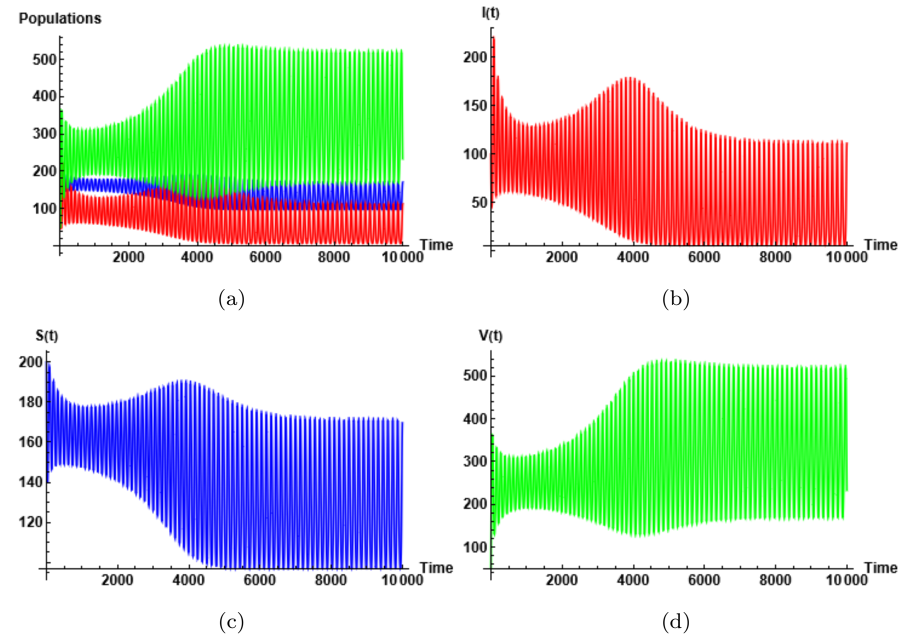


Figure 4: In this scenario $\tau = 76$. In this case, solutions go to a one periodic orbit. In this case, model (2.1) shows a Hopf bifurcation that emerges from the equilibrium point E_2.

5. Conclusions

Mathematical epidemiology literature is wide, particularly, when it is based on the construction of models with ordinary differential equations without delay. In contrast, there are still a lot of open questions concerning epidemiological differential models with a delay.

In this work, a delay differential equations model is analyzed. The results show that the solutions of the model without delay have a very different behavior compared with the model proposed (model with delay). Notice that, in the model without delay, when a backward bifurcation appears, a clear attraction basis for each stable equilibrium point exists (Kribs-Zaleta and Velasco-Hernández (2000)). However, in the model proposed a periodic orbit can exist when $R_0 < 1$. It is important to underline that simulations show different scenarios for the evolution of one solution of the model (2.1) with the same initial condition but different delays τ. Furthermore, the numerical simulations showed that the solutions will be going asymptotically to three different scenarios depending on the delay values and the initial condition. In the first scenario, the solution goes to the trivial equilibrium (see Figure 2). In the second one, the solution goes to a stable endemic equilibrium point (see Figure 3). In the third scenario, the solution goes to a periodic orbit (see Figure 4); therefore, a bistability phenomenon appears not only between an endemic equilibrium point and the disease-free equilibrium point but also between a periodic solution and the disease-free equilibrium point. The scenario showed in Figure 4 is different from the classical bistability phenomenon, which is related to the disease-free equilibrium and the biggest endemic equilibrium (see Figure 1). Observe that, the last scenario explained above is worse than the classical result associated to a backward bifurcation for models without delay since the amplitude of the oscillation can be big enough and this can be catastrophic for the susceptible population

The evidence shows that, an epidemic outbreak can be reach when there are few infectious individuals in the population. This behavior of the model solutions can be associated to the initial conditions, which are intervals, that influence the present time. That is, the present time is influenced for what happened during τ time ago. Notice that, this behavior is opposite to the case of the model without delay, where the initial condition influences only in an instant at time.

Besides, conditions for the stability of the equilibrium points, in terms of the parameters, are derived. Particularly, conditions for existence of a Hopf bifurcation are showed.

Ongoing work will analyze the effect of a delayed treatment over the bistability phenomenon associated to a backward bifurcation.

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