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The double integral method applied to heat conduction problems

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Abstract

This paper presents the application of the double integral method (DIM) to solve problems of transient one-dimensional conduction heat transfer. This method is a mathematical technique that transforms the non-linear boundary value problem into an initial value problem, whose solution can often be expressed in a closed analytical form. The partial differential equations are integrated twice, the first integration being performed within the domain and the second along the phenomenological distance. This double integration allows the gradient vector at the surface to be approximated using the simple integral method (SIM). Thus improvements can be attained by changing the derivative at the boundary by an integral relation, since the process of differentiation amplifies any difference between the assumed temperature profile and the exact solution. In this the work results obtained were are compared with exact and approximate analytical solutions found in the literature.

Palavras-chave: *Integral Methods; Heat Conduction.*

1. Introduction

The proliferation of numerical and computational techniques and the availability of software packages have neglected analytical methods for solving heat transfer problems. There is no doubt that the computational programs represent a breakthrough especially in problems of irregular geometries. However it is important to study and develop exact and approximate analytical

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methods, so that the computational programs can be optimized demanding less processing time. In this sense, this work has the objective to develop approximate analytical solutions for transient one-dimensional heat conduction problems in semi-infinite body. For this purpose, this study uses the double integral method developed by Volkov (1965) that consists of a refinement of the widely utilized simple integral method, known as the Karman-Pohlhausen method for the boundary layer or Goodman method for phase change.

The first application of double integral method in the literature is due to Volkov (1965), conducting a thorough study on the method of Karman-Pohlhausen used for solving boundary layer equations. In this article,Volkov suggests that this method could be improved if the calculation of the gradient vector at the boundary were expressed by an integral relation, since improvements would result due to the elimination of the differentiation that will not appear directly, but only as part of an integrand.

It is interesting to notice that Bromley (1952) used a similar strategy to that used by Volkov (1965) applied to a specific problem to determine the heat transfer coefficient for the case of high heat convection sensitivity of a laminar film. However he did not present significant considerations of the characteristics of the strategy utilized, as well as the increased accuracy respective to the simple integral method. Tse-Fou Z(1971,1979) published a series of three articles between 1971 and 1979 where he proposed to solve the momentum equation including the effect of large variations of the Prandtl number, the calculation of skin friction on porous plate, and the calculation of transient heat conduction. In this group of works he utilized the same strategy of Volkov (1965) to calculate the gradient vector at the surface.

Sucec (1977) applied the simple integral method in transient heat conduction problems for step temperature and heat flux variations. This work is mentioned here, since Sucec (1979)references it extensively when solving the energy equation for laminar flow over a flat plate, with constant properties, with specified heat flux at the surface. Sucec (1995) also applied the double integral method for boundary layer equations with the objective to obtain solutions to locate the separation point. El-Genk e Cronenberg. (1979) applied this method to verify the accuracy of the solution in phase change problems.El-Genk e Cronenberg (1979) also produced another paper to obtain an approximate solution for the growth of a frozen crust in forced flow.

The problems presented here were taken from the chapter 9 Application of Integral Methods to Transient Nonlinear Heat Transfer written by Goodman (1964). This choice is justified, since the application of the double integral method deals with the boundary conditions using the simple integral method. Thus, the work of Goodman (1964) serves as a reference for the treatment of the boundary conditions, as well as in the evaluation of the results obtained with the double integral method.

This work is intended to be an analysis of the behavior of the double integral method to solve several physical situations as well as an assessment of its performance when applied to these cases.

2. Integral Methods

The simple integral methods due to Karman-Pohlhausen, as well as the double integral method of Volkov (1965), are mathematical techniques to reduce boundary value problems into an ordinary initial value problem, whose solution can be frequently expressed in an analytical closed form. As observed by M. N (1980), the solution of a boundary value problem and initial condition defined by a partial differential equation by means of an exact analytical method, leads to a solution that is valid at all points of the domain considered. However, when this problem is solved by an integral method, the solution is satisfied only in the average for the considered domain.

The solutions obtained by integral methods are associated with the choice of the velocity profiles for the case of boundary layer or temperature in the case of heat transfer problems. However, the choice of these profiles is not arbitrary, but must meet the boundary conditions of the problem to be solved, since both integral methods require in their development that such conditions are satisfied.

In the development of this work, the double integral method will be applied utilizing only polynomial profiles. This choice is justified because these profiles are easy to obtain and to manipulate in the integration and differentiation operations of the integral methods. Moreover, this choice allows the validation of the results obtained here with the work of Goodman (1964). It should be noted that the utilization of polynomial profiles does not minimizes the generality of the application of the method, since more sophisticated functions as exponentials, trigonometric or hyperbolic trigonometric can be expressed by means of a polynomial by a Taylor series expansion. In order to demonstrate the influence of increasing the degree of the polynomial to obtain better approximations, the problems presented here will be solved with polynomials of degree two, three and, when convenient, with the profile recommended by the literature. According to Langford (1973), the use of additional boundary conditions to obtain high-order de polynomial approximations can inhibit the precision of the integral methods.

2.1. Algorithm of the double integral method

The application of the double integral method can be systematized through a simple five steps algorithm which is described below. One should note that when treating with problems involving temperature dependent properties, minor changes should be made.

Step 1) A first integration of the heat conduction differential equation is performed with respect to x inside the domain, that is, the extremes of the integration are the values at $x = 0$ and $x = x$.This first integration is intended to make the gradient vector explicit at $x = 0$.

Step 2) The boundary condition at x=0 must be analyzed and, in case it is of the second kind, it need not to be approximated implicitly via the simple integral method. Otherwise, a sub-routine to treat the boundary condition must be used and it is described as follow:

Step 2.1) Sub-Routine to Treat the Boundary Condition

Again the heat conduction differential equation is considered and the simple integral method algorithm is applied, as can be seen in Milanez e Ismail (1984). Thus, one can determine implicitly the gradient vector in the boundary.

Step 3) The temperature profiles that satisfy the boundary conditions of the problem to be solved are replaced in the equation obtained in the previous step, and a second integration is performed along the phenomenological distance $\delta(t)$ thus removing from the conduction equation the partial derivative respective to the space.

Step 4) An ordinary differential equation for the thermal thickness $\delta(t)$ is then obtained, which is solved with the initial condition for $\delta(t) = 0$.

Step 5) Once the function $\delta(t)$ is known, the temperature distribution is obtained in function of the time and of the position.

3. Applications of the double integral method

This section deals with the application of double integral method, with the problems taken from the work of Goodman (1964).Six cases were selected, which are; semi-infinity body with boundary condition of first, second and third kind, and the body of finite size, body semi-infinity with generating internal energy, and finally a study when the thermal properties are not constant with temperature.

Problem 1) Non homogeneous Dirichlet boundary condition

Consider initially a semi-infinite body which extends along the axis $x > 0$ with initial condition $T(x, 0) = 0^0 C$. Furthermore, assume that at its surface, the boundary condition is $T(0, t) = T_s$. For this physical system the mathematical model is given by:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.1}$$

The natural boundary conditions of the problem are:

$$T(0, t) = T_s \tag{3.2}$$

$$T(\delta, t) = 0 \tag{3.3}$$

$$\frac{\partial T}{\partial x}(\delta, t) = 0 \tag{3.4}$$

To obtain solutions with cubic temperature profile, a fourth boundary condition is required. This condition is obtained by deriving partially Eq.(3.3) with respect to t and evaluating Eq.(3.1) at point (δ, t), so that:

$$\frac{\partial^2 T}{\partial x^2}(\delta, t) = 0 \tag{3.5}$$

This equation is known as the smoothing condition, because it makes the temperature profile to be equal to the initial temperature:

$$T(x,0) = 0, x > 0 \tag{3.6}$$

By applying the double integral method in Eq.(3.1):

$$\begin{aligned}\alpha \int_0^\delta \frac{\partial T}{\partial x}(x,t)dx \quad &- \quad \alpha \int_0^\delta \frac{\partial T}{\partial x}(0,t)dx \\ &= \quad \int_0^\delta \frac{\partial}{\partial t}\int_0^x T(x,t)dx \end{aligned} \tag{3.7}$$

Calculating the gradient vector in the boundary using the simple integral method results:

$$\alpha \frac{\partial T}{\partial x}(0,t) = \int_0^\delta \frac{\partial T}{\partial t} dx \tag{3.8}$$

Substituting Eq.(3.8) into Eq.(3.7):

$$\int_0^\delta \alpha \frac{\partial T}{\partial x} dx + \int_0^\delta \int_0^\delta \frac{\partial T}{\partial t} dx dx = \int_0^\delta \int_0^x \frac{\partial T}{\partial t} dx dx \tag{3.9}$$

Assuming quadratic and cubic polynomial profiles for the temperature distribution:

$$T(x,t) = T_s \left(1 - \frac{x}{\delta(t)}\right)^2 \tag{3.10}$$

$$T(x,t) = T_s \left(1 - \frac{x}{\delta(t)}\right)^3 \tag{3.11}$$

Substituting Eq.(3.10) and Eq.(3.11) in Eq.(3.9), the following ordinary differential equations are obtained:

$$6\alpha = \delta(t)\frac{d\delta}{dt} \tag{3.12}$$

$$10\alpha = \delta(t)\frac{d\delta}{dt} \tag{3.13}$$

Solving both ordinary differential equations with the initial condition, and then substituting the obtained solutions in the respective profiles, they are completely determined. Table 1 shows the temperature profiles and the calculation of the heat flow at the surface for the integral and similarity methods.

Tabela 1: Heat Flux

Method	**Heat Flux**	**Error**
Similarity	$\frac{\sqrt{\frac{1}{\pi}}T_s k}{\sqrt{\alpha t}}$	0%
Double Integral(Quadratic Profile)	$\frac{\frac{3}{\sqrt{20}}T_s k}{\sqrt{\alpha t}}$	2.3%
Double Integral (Cubic Profile)	$\frac{\frac{3}{\sqrt{20}}T_s k}{\sqrt{\alpha t}}$	18.8%
Simple Integral(Cubic Profile)	$\frac{\sqrt{\frac{3}{8}}T_s k}{\sqrt{\alpha t}}$	8.5 %

It may be observed that the expression obtained for the heat flow at the boundary differs of a single constant when the expressions obtained by the integral methods are compared with the exact analytical solution.

Figure 1 shows the temperature distribution obtained with similarity, simple and double integral methods with quadratic and cubic profiles. It may be observed that the double integral method with quadratic profile exhibits better precision in the temperature profile when compared with the method of Goodman with cubic profile. In this same figure it may be observed that the double integral method with cubic profile is almost indistinguishable when compared with analytical solution.

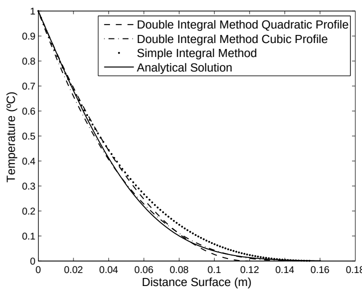

Figura 1: Surface Temperature.

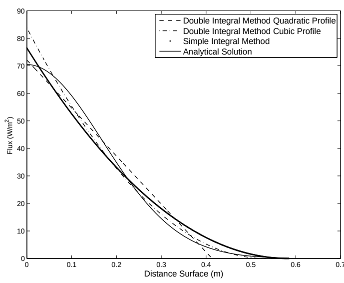

Figura 2: Flux Profile.

The analysis of Figure 2 illustrates the superiority of the double integral method to describe the heat flow through the body when compared with the method of Goodman. This best approximation is a consequence of the better description of the temperature profile obtained by this method as shown in Figure 1.

Problem 2) Non homogeneous Neumann boundary condition

Consider a semi-infinite body which extends along the axis $x > 0$, with initial condition $T(0,t) = 0^0C$. Assume that at its surface there is a constant heat flux of $f(t) = 1000\frac{W}{m^2}$. For this physical system the mathematical model is given by:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.14}$$

The natural boundary conditions of the problem are:

$$-k\frac{\partial T}{\partial x} = f(t) \tag{3.15}$$

$$\frac{\partial T}{\partial x}(\delta, t) = 0 \tag{3.16}$$

$$T(\delta, t) = 0, x > 0 \tag{3.17}$$

And the profile smoothing condition is given by:

$$\frac{\partial^2 T}{\partial x^2} = 0 \tag{3.18}$$

Assuming quadratic Eq.(3.19) and cubic Eq.(3.20) polynomial profiles for the temperature distribution:

$$T(x,t) = \frac{f}{2k\delta}(\delta - x)^2 \tag{3.19}$$

$$T(x,t) = \frac{f}{3k\delta^2}(\delta - x)^3 \tag{3.20}$$

By applying the double integral method in Eq.(3.14) the follow expression results:

$$\int_0^\delta \alpha \frac{\partial T}{\partial x}(x,t)dx - \int_0^\delta \frac{f(t)}{k}dx = \int_0^\delta \int_0^\delta \frac{\partial T}{\partial t}dxdx \tag{3.21}$$

Substituting the temperature profiles Eqs.(3.19) and (3.20) in Eq. (3.21), the following ordinary differential equations result:

$$\frac{5\delta}{12}\frac{d\delta}{dt} = \alpha \tag{3.22}$$

$$\frac{7\delta}{40}\frac{d\delta}{dt} = \alpha \tag{3.23}$$

The resolution of differential equations (3.22) and (3.23), with the initial condition $\delta(0) = 0$ leads to the temperature profiles represented in Table 2 that also shows the profiles obtained by Goodman (1964) using the simple integral method and Carslaw e Jaeger (1959) using the similarity method. Both integral methods differ only of a single numerical constant from the exact analytical solution in the calculation of the surface temperature.

Tabela 2: Temperature in T(0,t)

Method	**T(0,t)**	**Error**
Similarity	$\sqrt{\frac{4}{\pi}}\frac{f\sqrt{\alpha t}}{k}$	0%
Double Integral(Quadratic Profile)	$\sqrt{1.2}\frac{f\sqrt{\alpha t}}{k}$	3.53 %
Double Integral(Cubic Profile)	$\sqrt{1.26}\frac{f\sqrt{\alpha t}}{k}$	0.66 %
Simple Integral(Cubic Profile)	$\sqrt{\frac{4}{3}}\frac{f\sqrt{\alpha t}}{k}$	2.18 %
Simple Integral(Quadratic Profile)	$\sqrt{1.5}\frac{f\sqrt{\alpha t}}{k}$	9 %

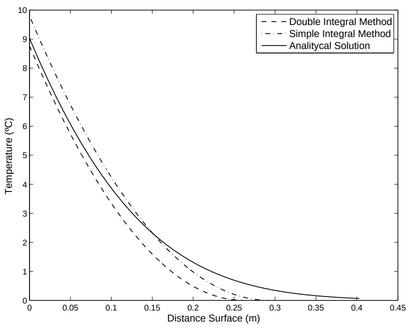

Figura 3: DIM quadratic profile

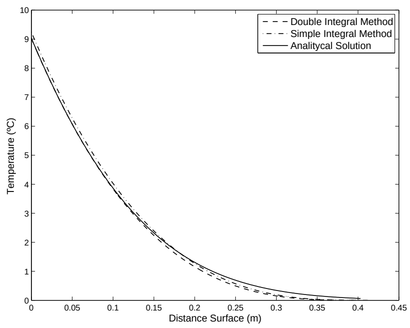

Figura 4: DIM cubic profile

Figures 3 and 4 exhibit, the temperature distributions for each of the methods mentioned previously. Figure 3 depicts both integral methods, with quadratic profiles and in Figure 4 the integral methods utilize cubic profiles in the description of the temperature. As shown in the figures, the double integral method has greater accuracy in the description of the temperature profile in regions near the boundary; furthermore the solution obtained by this method do not crosses the analytical solution, which does not happen with the method of Goodman.

Problem 3) Robin boundary condition

Consider a semi-infinite body which extends along the axis $x > 0$, with initial condition $T(x,0) = 0^0C$ and assume that at its surface a boundary condition of the Robin type be specified. For this physical system the mathematical formulation is given by:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.24}$$

$$T(\delta, t) = 0 \tag{3.25}$$

$$\frac{\partial T}{\partial t}(\delta, t) = 0 \tag{3.26}$$

$$\frac{\partial T}{\partial t}(0, t) = -f(z, t) \tag{3.27}$$

$$\frac{\partial^2 T}{\partial x^2}(\delta, t) = 0 \tag{3.28}$$

Assuming quadratic Eq.(3.29) and cubic Eq.(3.30) polynomial profiles for the temperature distribution:

$$T(x,t) = \frac{f(z,t)}{2\delta}(\delta - x)^2 \tag{3.29}$$

$$T(x,t) = \frac{f(z,t)}{3\delta^2}(\delta - x)^3 \tag{3.30}$$

Performing a change of variables given by Eqs.(3.31) to (3.33), the temperature profiles can be rewritten as function of the surface temperature, as expressed in Eq.(3.34) and Eq.(3.35).

$$z(t) = T(0, t) \tag{3.31}$$

$$\delta(t) = \frac{2z}{f(z,t)} \tag{3.32}$$

$$\delta(t) = \frac{3z}{f(z,t)} \tag{3.33}$$

After some algebraic manipulations, the temperature profiles can be expressed as

$$T(x,t) = \frac{z(t)}{\delta(t)^2}(\delta - x)^2 = T(0,t)(1 - \frac{x}{\delta})^2 \tag{3.34}$$

$$T(x,t) = \frac{z(t)}{\delta(t)^3}(\delta - x)^2 = T(0,t)(1 - \frac{x}{\delta})^3 \tag{3.35}$$

Applying the double integral method in the conduction equation, Eq.(3.24), results:

$$\int_0^\delta \int_0^x \alpha \frac{\partial^2 T}{\partial x^2} dx dx = \int_0^\delta \int_0^x \frac{\partial T}{\partial t} dx dx \tag{3.36}$$

With some algebraic manipulations

$$\int_0^\delta \alpha \frac{\partial T}{\partial x} dx + \int_0^\delta \int_0^\delta \frac{\partial T}{\partial t} dx dx = \int_0^\delta \int_0^x \frac{\partial T}{\partial t} dx dx \tag{3.37}$$

Substituting the quadratic profile of Eq.(3.34), and the cubic profile of Eq. (3.35) into Eq.(3.37), the respective ordinary differential equations are obtained:

$$\frac{d}{dt}\frac{z^3}{f^2(z,t)} = 3\alpha z \tag{3.38}$$

$$\frac{d}{dt}\frac{9z^3}{f^2(z,t)} = 20\alpha z \tag{3.39}$$

Assuming that the function $f(z,t)$ given by the boundary condition of Eq.(3.28) is defined as illustrated in Eq. (3.40) and performing the necessary substitutions, the ordinary differential equations written in the form of Eqs. (3.38) and (3.39) can be expressed as Eqs. (3.41) and (3.42).

$$f(z_1) = \left(\frac{h}{k}\right)(z_0 - z_1) \tag{3.40}$$

$$\frac{3z_1 f^2(z_1)\frac{dz_1}{dt} - z_1^3 f(z_1)\frac{df}{dz_1}\frac{dz_1}{dt}}{f^2(z_1)} = 3\alpha z_1 \tag{3.41}$$

$$\frac{27z_1^2 f^2(z_1)\frac{dz_1}{dt} - 18z_1^3\frac{df}{dz_1}\frac{dz_1}{dt}}{f^4(z_1)} = 20\alpha z_1 \tag{3.42}$$

The previous differential equations are of the separable type, the resolution of each one leads respectively to the following relations

$$\begin{aligned} \frac{4}{3}\left(\frac{h}{k}\right)^2 \alpha t &= \frac{4}{9}\left(1 - \frac{1}{1-\frac{z}{z_0}}\right) + \frac{4}{9}\left(\frac{1}{1-\frac{z}{z_0}} - 1\right) \\ &+ \frac{4}{9}\ln\left(1 - \frac{z}{z_0}\right) \end{aligned} \tag{3.43}$$

$$\begin{aligned} \frac{4}{3}\left(\frac{h}{k}\right)^2 \alpha t &= 0.6(\frac{1}{\left(1-\frac{z}{z_0}\right)^2} - 1) + 0.6\ln\left(1 - \frac{z}{z_0}\right) \\ &+ 0.6\left(1 - \frac{1}{1-\frac{z}{z_0}}\right) \end{aligned} \tag{3.44}$$

The exact analytical solution presented by Carslaw e Jaeger (1959) is :

$$\begin{aligned} \frac{z}{z_0} = erfc(\frac{x}{2\sqrt{\alpha t}}) - \exp\left(\frac{h}{k} + h^2\alpha t\right) \\ erfc(\frac{x}{2\sqrt{\alpha t}} + \frac{h}{k}\sqrt{\alpha t}) \end{aligned} \tag{3.45}$$

Eqs.(3.43) and Eq.(3.44) must be rewritten in function of two of the three dimensionless terms below

$$\frac{x}{2\sqrt{\alpha t}}, \frac{h}{k}\sqrt{\alpha t}, \frac{h}{k}x \tag{3.46}$$

so that they can be compared with the solution presented by Carslaw e Jaeger (1959).

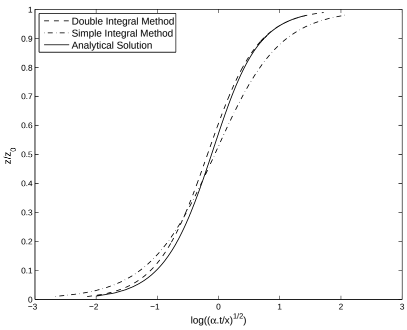

Figura 5: DIM quadratic profile

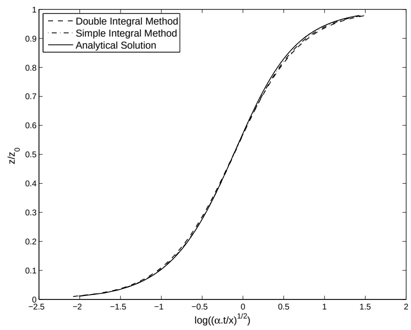

Figura 6: DIM cubic profile

Figures 5 and 6 show the graph of the exact analytical solution, as well as the solutions obtained by both integral methods. In the following graphs the solutions were plotted as function of the variables $\frac{z}{z_0}$ versus $\log_{10}(\frac{h}{k}\sqrt{\alpha t})$ considering $\frac{x}{2\sqrt{\alpha t}} = 0$. In Figure 5 the integral methods are utilizing quadratic approximations. As it may be observed, the double integral method exhibits little sensitivity respective to the choice of the profile utilized. In Figure 6 the integral methods utilize cubic profiles in the approximations. There is a significant improvement in the approximation of the simple integral method. Furthermore, it is impossible to distinguish differences in the solutions obtained by the methods.

Problem 4)Body of finite dimension

In the problems previously considered, the double integral method was applied to semi-infinite bodies. The importance of the study of heat transfer in such geometries is due to the fact that they are able to give the temperature profile in the initial stages of the transient regime. However, when the function penetration depth $\delta(t)$ reaches the total length of the body being analyzed, it does not behave as a semi-infinite solid any more, and begins to behave as a finite body.

The problem considered next, consists of a solid of finite length with boundary conditions of first and second kinds where the initial stage described before was reached. Thus it is intended to evaluate the usefulness of the double integral method in this second stage. The mathematical model of the previous problem is:

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{3.47}$$

$$\frac{\partial T}{\partial x}(0,t) = -\frac{f(t)}{k} \tag{3.48}$$

$$T(l,t) = 0 \tag{3.49}$$

$$\frac{\partial^2 T}{\partial x^2}(l,t) = 0 \tag{3.50}$$

Having the mathematical model to be solved, the temperature profiles must be assumed for the application of the integral methods. For this second stage, the penetration depth is not needed and therefore only three de boundary conditions are necessary to obtain the temperature profiles. Goodman (1964) recommends that the cubic profile to be adopted to satisfy the boundary conditions should be equal to

$$T(x,t) = \left(\frac{3z}{2l} - \frac{f}{2k}\right)(l-x) + \frac{1}{2l^2}\left(\frac{f}{k} - \frac{z}{l}\right)(l-x)^3 \tag{3.51}$$

In this profile l is the total thickness of the body, $f(t)$ is the heat flux assumed constant and $z(t)$ a compact notation for the surface temperature, that is $z(t) = T(0,t)$. Applying the double integral method in the conduction equation, the resulting ordinary differential equation is

$$\int_0^\delta \int_0^x \alpha \frac{\partial^2 T}{\partial x^2} dx dx = \int_0^\delta \int_0^x \frac{\partial T}{\partial t} dx dx \tag{3.52}$$

$$\frac{dz}{dt} + \frac{5\alpha}{2l} z = \frac{5\alpha}{2l} \frac{f(t)}{k} \tag{3.53}$$

the solution of the differential equation involves the knowledge of the surface temperature at the instant of time when the body ceases to behave as a semi-infinite solid. It is considered that this instant of time is t_0 and the temperature at this instant is $z(t_0)$. Given the above considerations, the resolution of Eq.(3.54) leads to the following expressions

$$\int_{t_0}^{t} \frac{d}{dt_1}\left(z e^{-\frac{5\alpha}{2l^2}t}\right) dt_1 = \frac{5\alpha}{2l} \int_{t_0}^{t} \frac{f(t)}{k} e^{-\frac{5\alpha}{2l^2}t} dt_1$$
$$t >= t_0 \tag{3.54}$$

$$z(t) = z(t_0)e^{(}\frac{5\alpha}{2l^2}(t_0 - t)) + \frac{lf(t)}{k}\left[1 - e^{\frac{5\alpha}{2l^2}}(t_0 - t)\right] \quad t >= t_0 \tag{3.55}$$

The determination of the variable $z(t_0)$ of Eq.(3.55) is attained by means of an expression recommended by Goodman (1964), where it is evaluated at the instant of time $t = t_0$ and $\delta(t_0) = l$.

$$\delta(t) = \frac{3z(t)}{\frac{f(t)}{k}} \tag{3.56}$$

With some algebraic manipulations it is possible to show that $z(t_0)$ can be written as

$$z(t_0) = \frac{lf(t_0)}{3k} \tag{3.57}$$

Having the above expression, it is possible to calculate the instant of time t_0 with the aid of Table 2 that gives for the double integral method with cubic profile the following expression

$$t_0 = \frac{l^2}{11.421\alpha} \tag{3.58}$$

With the determination of variables t_0 and $z(t_0)$, Eq.(3.55) can be re-written as follows

$$z(t) = \frac{lf(t)}{k}\left[1 - 0.829e^{-\frac{5\alpha t}{2l^2}}\right] \tag{3.59}$$

After $z(t)$ is determined it may be compared with the exact analytical solution due to Carslaw e Jaeger (1959) as well as with the approximated analytical solution due to Goodman (1964). Table 3 gives the expressions obtained for each one and Figure 7 shows profile of the solutions obtained.

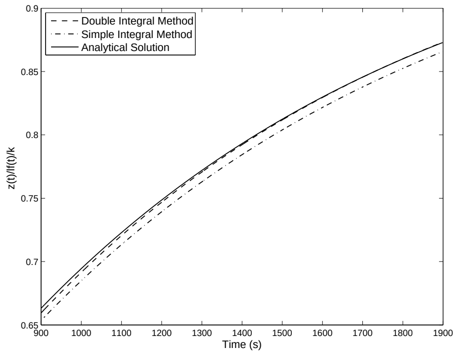

Figura 7: Body of finite dimension

Tabela 3: Plate with Finite Thickness

Method	**Profile Solution** $\frac{z(t)}{\frac{lf(t)}{k}}$	**Eigenvalue**	**Error**
Separation of Variables	$\left[1-\frac{8}{\pi^2}e^{-\frac{\pi^2\alpha}{4l^2}t}\right]$	2.467	0%
Simple Integral	$\left[1-0.814e^{-\frac{12\alpha}{2l^2}t}\right]$	2.4	2.71 %
Double Integral	$\left[1-0.829e^{-\frac{15\alpha}{2l^2}t}\right]$	2.5	1.33%

Problem 5) Conduction with internal heat generation

Consider a semi-infinite body which extends along the axis $x > 0$ with initial condition $T(0,t) = 0^0C$ and boundary condition given by $T(0,t) = 0^0C$. Furthermore it is assumed that an internal source of heat generation per unit of time and volume $q(t)$ is specified in the domain. For this physical system, the associated mathematical model is described by:

$$\frac{\partial t}{\partial t} - \alpha\frac{\partial^2 T}{\partial x^2} = \frac{q(t)}{\rho c} \tag{3.60}$$

The boundary conditions for the problem are:

$$\frac{\partial T}{\partial x}(\delta, t) = 0 \tag{3.61}$$

$$T(0,t) = 0 \tag{3.62}$$

$$\frac{\partial^2 T}{\partial x^2}(\delta, t) = 0 \tag{3.63}$$

In order to obtain approximations with the cubic profile, four boundary conditions are required. To obtain this fourth condition, the conduction equation is applied at a point far from the boundary where the temperature gradient is zero, resulting after some algebraic manipulations the following expression

$$T(\delta, t) = \frac{Q(t)}{\rho c} = \frac{1}{\rho c}\int_0^t q(t)dt \tag{3.64}$$

With the necessary boundary conditions, the quadratic and cubic profiles are:

$$T(x,t) = \frac{Q(t)}{\rho c}\left[1 - \left(1 - \frac{x}{\delta(t)}\right)^2\right] \tag{3.65}$$

$$T(x,t) = \frac{Q(t)}{\rho c}\left[1 - \left(1 - \frac{x}{\delta(t)}\right)^3\right] \tag{3.66}$$

The heat conduction equation can then be written in the form

$$\frac{\partial}{\partial t}\left[T(x,t) - \frac{Q(t)}{\rho c}\right] = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.67}$$

Applying the double integral method in Eq.(3.67) results:

$$\int_0^\delta \int_0^x \frac{\partial}{\partial t}\left[T(x,t) - \frac{Q(t)}{\rho c}\right] dx dx = \int_0^\delta \int_0^x \alpha \frac{\partial^2 T}{\partial x^2} dx dx \tag{3.68}$$

Developing this expression, the following differential equations can be obtained

$$\frac{1}{12}\frac{d}{dt}\left[\delta(t)^2 Q(t)\right] = \alpha Q(t) \tag{3.69}$$

$$\frac{1}{20}\frac{d}{dt}\left[\delta(t)^2 Q(t)\right] = \alpha Q(t) \tag{3.70}$$

The solution of these equations with the boundary condition $\delta(0) = 0$ gives the respective results

$$\delta(t) = \sqrt{\frac{12\alpha \int_0^t Q(t_1)dt_1}{Q(t)}} \tag{3.71}$$

$$\delta(t) = \sqrt{\frac{20\alpha \int_0^t Q(t_1)dt_1}{Q(t_1)}} \tag{3.72}$$

With Eq.(3.71) and Eq.(3.72) the quadratic and cubic profiles are completely determined. Figure 8 shows the results using integral methods with quadratic profiles compared with the numerical solution obtained with the control volume method. However, as it can be observed in Figure 9, the approximation is improved when the integral methods are applied using cubic temperature profiles.

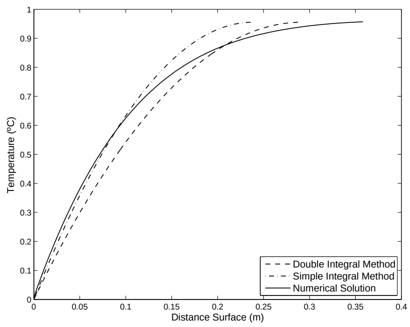

Figura 8: DIM quadratic profile

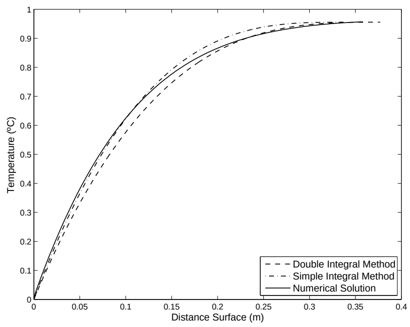

Figura 9: DIM cubic profile

Problem 6) Temperature dependent thermal properties

In many practical applications the properties of the body under study are temperature dependent which makes the mathematical model more complex. Due to the difficulties in obtaining an exact analytical solution, in general a numerical solution is used to solve such problems. However the integral methods allow that approximated analytical solutions can be obtained for this type of problem. Consider a semi-infinite body where its properties are temperature dependent, with initial condition $T(x, 0) = 0^0C$. Furthermore, assume that at the boundary a constant temperature equal to is prescribed $T(0, t) = T_s$. For this physical system, the associated mathematical model is described by:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) \tag{3.73}$$

$$T(0,t) = T_s \tag{3.74}$$

$$T(\delta,t) = 0^0 C \tag{3.75}$$

$$\frac{\partial T}{\partial x}(\delta,t) = 0 \tag{3.76}$$

$$\frac{\partial^2 T}{\partial x^2}(\delta,t) = 0 \tag{3.77}$$

By assuming that the properties k and ρc are temperature dependent, the solution of equation Eq.(74) must be carried out by a numerical method. Thus, in order to obtain an approximated analytical solution using an integral method, the change of variables proposed by Goodman Goodman (1964) will be adopted, given by Eq.(3.79).

$$\nu = \int_0^T \rho c dT \tag{3.78}$$

Considering Eq.(3.78) the heat conduction equation as well as the boundary conditions can be written as:

$$\frac{\partial \nu}{\partial t} = \frac{\partial}{\partial x}\left[\alpha(\nu) \frac{\partial \nu}{\partial x}\right] \tag{3.79}$$

$$\nu(0,t) = \nu_s \tag{3.80}$$

$$\nu(\delta,t) = 0 \tag{3.81}$$

$$\frac{\partial \nu}{\partial x}(\delta,t) = 0 \tag{3.82}$$

$$\frac{\partial^2 \nu}{\partial x^2}(\delta,t) = 0 \tag{3.83}$$

Having determined the boundary conditions, the resulting polynomial profiles are:

$$\nu(x,t) = \nu_s \left[1 - \frac{x}{\delta(t)}\right]^2 \tag{3.84}$$

$$\nu(x,t) = \nu_s \left[1 - \frac{x}{\delta(t)}\right]^3 \tag{3.85}$$

Applying the integral method with quadratic and cubic profiles, the following ordinary differential equations are obtained

$$\int_0^\delta \int_0^x \frac{\partial \nu}{\partial t} dx dx = \int_0^\delta \int_0^x \frac{\partial}{\partial x}\left(\alpha(v)\frac{\partial \nu}{\partial x}\right) dx dx \tag{3.86}$$

$$\frac{d}{dt}\delta^2 \nu_s = 12\alpha_s \nu_s \tag{3.87}$$

$$\frac{d}{dt}\delta^2 \nu_s = 20\alpha_s \nu_s \tag{3.88}$$

The solution of the previous equations with the condition $\delta(0) = 0$ leads to the following solutions

$$\delta(t) = \sqrt{\frac{12}{\nu_s}\int_0^t \nu_s \alpha_s dt} \tag{3.89}$$

$$\delta(t) = \sqrt{\frac{20}{\nu_s}\int_0^t \nu_s \alpha_s dt} \tag{3.90}$$

Goodman (1964) suggested as an application of the equations above assuming that the temperature T_s at the surface is prescribed so that ν_s is constant. Since α_s depends exclusively on ν_s , α_s is a constant. Therefore Eq.(3.89) and Eq.(3.90) are written as:

$$\delta(t) = \sqrt{12\alpha_s t} \tag{3.91}$$

$$\delta(t) = \sqrt{20\alpha_s t} \tag{3.92}$$

Substituting Eq.(3.91) and Eq.(3.92) in the temperature profiles:

$$\frac{\nu}{\nu_s} = \left(1 - \frac{x}{\sqrt{12\alpha_s t}}\right)^2 \tag{3.93}$$

$$\frac{\nu}{\nu_s} = \left(1 - \frac{x}{\sqrt{20\alpha_s t}}\right)^3 \tag{3.94}$$

In the present study, the double integral method is being applied to a solid geometry. Therefore, it may be assumed that the product ρc remains practically constant with temperature variation in this type of geometry. Thus, the variation of the thermal properties occurs due to the variation of the thermal conductivity. By virtue of the change of variables in Eq.3.78, ν is proportional to $T(x,t)$, and the thermal conductivity can be written as

$$k = k_0 \left(1 + \beta \frac{T}{T_s}\right) \tag{3.95}$$

Considering Eq.(3.96), the thermal diffusivity at the boundary can be expressed as:

$$\alpha_s = \frac{k_0(1+\beta)}{\rho c} \tag{3.96}$$

Tabela 4: Temperature Profiles

Method	**Temperature Profile**
Simple Integral (Cubic Profile)	$\frac{T}{T_s} = \left[1 - \frac{y}{\sqrt{6(1+\beta)}}\right]^3$
Double Integral (Quadratic Profile)	$\frac{T}{T_s} = \left[1 - \frac{y}{\sqrt{3(1+\beta)}}\right]^2$
Double Integral (Cubic Profile)	$\frac{T}{T_s} = \left[1 - \frac{y}{\sqrt{5(1+\beta)}}\right]^3$

Utilizing the dimensionless parameter of Eq.(3.96) defined by Goodman (1964), the temperature profiles are written as presented in Table 4 and and shown in Figure 10.

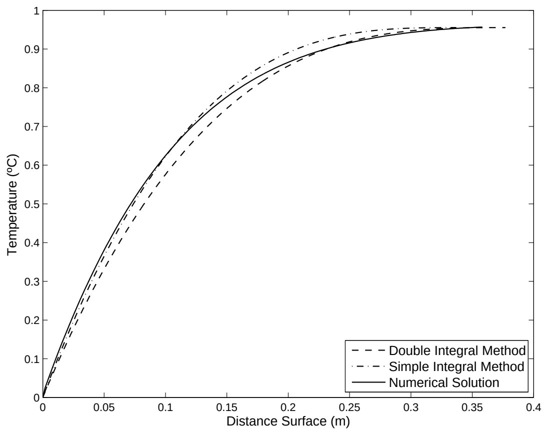

Figura 10: Temperature dependent properties.

4. Conclusions

Throughout this paper the double integral method was applied to six different types of transient heat transfer problems. The analysis of this method revealed that the double integral method exhibits higher sensitivity respective to the profile adopted, when applied to a semi-infinite plane geometry with boundary conditions of first or third kind.

The sensitivity regarding the choice of the profile is connected to the fact that these two problems are from the mathematics standpoint analogous to the problem originally solved by Volkov (1965) when developing the method. However, in applications involving semi-infinite body with internal heat generation or boundary condition of second kind, the double integral method presented a reduced sensitivity regarding the choice of the profile.

In either case, it was necessary to utilize cubic profiles to obtain good accuracy in the description of the temperature profile. The major difficulties in using the double integral method were found in the applications involving a wall of finite thickness, with boundary conditions of first and second kind. The adversity in this example is due to the fact that the approximation of the gradient vector at the surface by an integral relation is no longer necessary.

Besides, the derivation along the phenomenological distance makes no sense, because the body does not behave as a semi-infinite solid any more. However, even with all the adversities, the method is able to provide a good approximation for the eigenvalue, as well as to the surface temperature. Thus, this is an approximate analytical method relatively easy to use and capable of providing good accuracy results when compared to conventional methods.

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