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Montroll's model applied to a population growth data set using type-1 and type-2 fuzzy parameters

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Abstract. A comparative study is established through modeling Peruvian population data from 1961 to 2013 by means of Montroll's classic model. To model the population dynamics it is considered three kinds of growth rates: the first is a real number, the second is a triangular type-1 fuzzy parameter and the third is an interval type-2 fuzzy set. For the second kind the Zadeh's Extension Principle is used in order to obtain a type-1 fuzzification in function of time and for the third kind it was applied Zadeh's Extension Principle for both the upper and lower membership functions at each time-instant. Defuzzification methods are employed for both types of fuzzy sets. The comparison study is made through the approximations of the deterministic solution, the type-1 and type-2 defuzzification, with the current population data, using the maximum relative error. The best result is obtained by the defuzzification through the type-2 fuzzy parameter.

Key words: *Interval type-2 fuzzy set; Zadeh's extension principle; Montroll's model; population dynamics; Peruvian population.*

1. Introduction

The population of Peru comprises in 2013 nearly 30.475 million inhabitants, presenting an increase of 1.13 percent in relation to 2012. Nowadays,

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this population was quadruplicated in relation to 1950, making the country the fourth most populated in South America after Brazil, Colombia and Argentine. When Peru celebrates its Independence Bicentennial on July 28, 2021, it is estimated that three million people will be added to its population. According to the National Institute of Statistics and Information (INEI), the expectation for the Peruvian population for the year 2050 is around 40 million people.

Mathematical models utilized to describe population growth evolved along history, undergoing several modifications after Malthus' model (Malthus, 2008). One of the most important and well known models was proposed by Belgium sociologist P. F. Verhulst (1838) (Verhulst, 2013). In this model it is assumed that all populations are prone to suffer natural inhibitions in their growths, with a tendency to reach a steady limit value as time increases. Another important proposal was developed by Montroll (Goel et al., 1971) which consists of a general model, that translates the asymptotic growth of a variable taking into account the position of the inflexion point of the curve.

When the fuzzy set theory was introduced by Zadeh (1965), a paradigm breakage took place, opening new trends to the study of population dynamics as in the case of HIV (Jafelice et al., 2005). A new development in the application of this theory, the type-2 set theory, deals with uncertain or gradual models, as well as with uncertain or gradual information provided by experts. Examples from several areas, such as engineer (Ibarra et al., 2015), medicine (Undre et al., 2015), economy (Kiliça and Kayab, 2015), mathematics (Takata et al., 2015), computational science (Perez-Ornelas et al., 2015), among others, can be found in literature.

Our objective is to establish a comparative case study of a Peruvian population data set from 1961 to 2003 (provided by INEI) using Montroll's model. The approach compares the classic Montroll's Model with the model produced by replacement of the real number that represents the population growth rate by type-1 and type-2 sets. The latter models are based on fuzzification through Zadeh Extension Principle of type-1 and type-2 sets, followed by defuzzification via the centroid (type-1) and the generalized proximity (type-2), although such methodologies are quite different from each other.

Six sections comprise this text: in section 2 a description of Montroll's classic model and its application to a Peruvian population data set is performed; in section 3 it is introduced the main concepts of the fuzzy set theory and Zadeh's Extension Principle for type-1 fuzzy set and it is presented the fuzzification and defuzzification of Montroll's model with a triangular type-1 fuzzy parameter applied to the data; in section 4 Zadeh's Extension Principle and the deffuzification for interval type-2 fuzzy set is defined and the fuzzification and defuzzification of Montroll's model with an interval type-2 fuzzy parameter is applied to the data; in section 5 it is discussed the comparison of the approximations of the deterministic solution, type-1 and type-2 defuzzifications, with the actual population data using the maximum relative errors; in section 6 it is presented the main conclusions.

2. Montroll's Deterministic Model Applied to the Data

Let $P(t)$ be a population value at an instant t; P_∞, the finite limit value of $P = P(t)$ when t tends to infinity; λ, the growth rate of the population and the parameter $\alpha > 0$ which is the indicator of the position of the curve inflection point $P(t)$. Montroll's model is given by the non-linear differential equation:

$$\frac{dP}{dt} = \lambda P \left[1 - \left(\frac{P}{P_\infty}\right)^\alpha\right], \qquad \lambda > 0, \qquad \alpha > 0. \tag{2.1}$$

The solution of (2.1) is given by:

$$P(t) = P_\infty \left[\frac{e^{\alpha\lambda t}}{\left(\frac{P_\infty}{P_0}\right)^\alpha + e^{\alpha\lambda t} - 1}\right]^{1/\alpha}. \tag{2.2}$$

We apply equation (2.1) to the Peruvian population data (Table 1). In the second column of Table 1 it is presented the values of t_i that are differences between two consecutive years. In the third column it is shown the actual population data and in the fourth column it is depicted $\frac{\Delta P}{\Delta t}\frac{1}{P}$ which is an approximation of $\frac{dP}{dt}\frac{1}{P}$,

$$\frac{\Delta P_i}{\Delta t_i}\frac{1}{Pi} = \frac{P_{i+1} - Pi}{t_{i+1} - t_i}\frac{1}{Pi} \text{ for } i = 0, 1, \cdots, 5, \text{ and } \frac{\Delta P_5}{\Delta t_6}\frac{1}{P_6} = \frac{P_6 - P_5}{t_6 - t_5}\frac{1}{P_6}.$$

Table 1: Population data obtained from Peru censuses in the period 1961 to 2013. Source: (INEI, 2015).

Year	t_i	Population ($P(t_i)$)	$\dfrac{\Delta P}{\Delta t}\dfrac{1}{P}$
1961	0	10,420,357	0.032
1972	11	14,121,564	0.028
1981	20	17,762,231	0.022
1993	32	22,639,443	0.017
2007	46	28,220,764	0.0133
2013	52	30,475,000	0.0123

We utilize the least squares method to adjust $\dfrac{\Delta P}{\Delta t}\dfrac{1}{P}$ to the function

$$f(P) \;=\; \lambda + \; aP^{\alpha},$$

where $a = -\dfrac{\lambda}{P_\infty^\alpha}$, for $0.1 \leq \alpha \leq 1.5$. We obtained the highest determination coefficient ($r^2 = 0.9919$) for $\alpha = 0.1$. For $\alpha = 0.1$ we obtain $\lambda = 0.214$, $a = -0.0361$ and $P_\infty \backsimeq 53,587,774$.

In figure 1 it is shown the graph of the solution (2.2) for the calculated values of α, λ and P_∞, and the line given by $P = P_\infty$ that is the finite limit value of $P(t)$ when t tends to infinity.

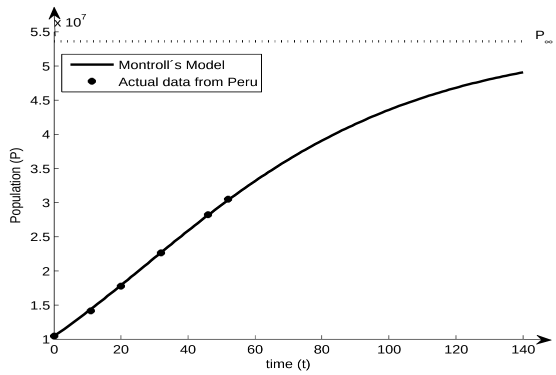

Figure 1: Montroll's Deterministic Model for $\alpha = 0.1, \lambda = 0.214$ and $P_\infty \backsimeq 53,587,774$.

3. Montroll's Model with Type-1 Fuzzy Parameter

A fuzzy subset A of the universe X is a membership function $\mu_A(x) \in [0,1]$, called the membership degree x. The α-cuts of a fuzzy set D, denoted by $[D]^\alpha$ is defined as $[D]^\alpha = \{x \in X, \mu_D(x) \geq \alpha\}$, $0 < \alpha \leq 1$; $[D]^0 = \overline{\text{supp}(D)}$, where $\text{supp}(D) = \{x \in X, \mu_D(x) > 0\}$ is the support of D (Pedrycz and Gomide, 1998, 2007). A fuzzy set D is called a *fuzzy number* when $X = \mathbb{R}$, there is $x \in X$ such that $\mu_D(x) = 1$, all α-cuts of D are nonempty closed intervals and the support of D is bounded.

Next, we enunciate the Zadeh's Extension Principle for the type-1 fuzzy sets. Let X and Y be universes and $f : X \longrightarrow Y$ a function. Given a fuzzy set D, the fuzzy set $\widehat{f}(D)$, with membership function given by

$$\mu_{\widehat{f}(D)}(y) = \begin{cases} \sup\limits_{\{x:f(x)=y\}} \mu_D(x) & \text{if } \{x : f(x) = y\} \neq \phi, \\ 0, & \text{otherwise,} \end{cases}$$

is called Zadeh's extension of D by f. $\widehat{f}(D) = f(D)$ if D is a classical set of X. We utilize proposition 1 to obtain the fuzzification through the $\alpha-$cuts of a fuzzy set.

Proposition 1 *(Barros and Bassanezi, 2010) Let X and Y be non-empty metric spaces, D be a fuzzy set of X and $f : X \longrightarrow Y$ be a continuous function. For each $0 \leq \alpha \leq 1$, $\left[\widehat{f}(D)\right]^\alpha = f([D]^\alpha)$.*

Proposition 1 claims that the $\alpha-$cuts of a fuzzy set obtained by Zadeh's Extension Principle are the images of the $\alpha-$cuts obtained through the function f.

The Center of Gravity (COG) defuzzification method for type-1 fuzzy sets is defined for a discrete domain:

$$\text{COG} = \frac{\sum\limits_{i=0}^{n} \mu(z_i) z_i}{\sum\limits_{i=0}^{n} \mu(z_i)}, \tag{3.3}$$

where n is the number of discretization points of the domain.

In order to obtain the fuzzification of equation (2.2), we consider $P_\lambda(t)$ as the deterministic solution given by (2.2), the triangular fuzzy number A with zero-cut $[A]^0$ and the function $S_t : [A]^0 \longrightarrow \mathbb{R}$ such that $S_t(\lambda) = P_\lambda(t)$ (Bertone et al., 2013). We build this fuzzification through Zadeh's extension of S_t at each fixed point t. The continuity of $S_t(\cdot)$ allows the utilize of proposition 1 and of the $\alpha-$cuts of A.

In figure 2 it is presented the fuzzy number A where the parameter $\overline{\lambda}$ is the modal value and δ_i $(i = 1, 2)$ are dispersions of the fuzzy set A. The analytical form is given by:

$$\mu_A(\lambda) = \begin{cases} \dfrac{1}{\delta_1}(\lambda - \overline{\lambda} + \delta_1) & if \quad \overline{\lambda} - \delta_1 < \lambda \leq \overline{\lambda}, \\ -\dfrac{1}{\delta_2}(\lambda - \overline{\lambda} - \delta_2) & if \quad \overline{\lambda} < \lambda \leq \overline{\lambda} + \delta_2, \\ 0 & otherwise. \end{cases} \tag{3.4}$$

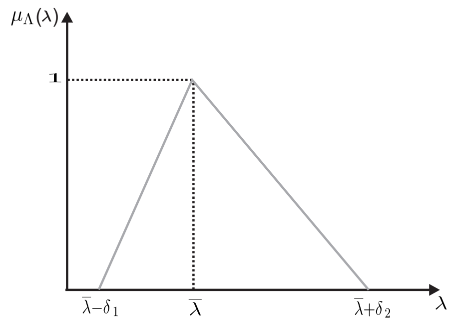

Figure 2: Fuzzy number A.

The simulation of this fuzzification is performed using numerical methods developed by (Bertone et al., 2013). We utilize the continuity of $S_t(\lambda)$ regarding λ (Cabrera, 2014), which is necessary for applying proposition 1.

In figure 3 it is presented the fuzzification of the deterministic solution of Montroll's model with triangular fuzzy parameter A for the Peruvian population, considering $\alpha = 0.1$, $P_\infty \backsimeq 53,587,774$, $\overline{\lambda} = 0.214$, $\delta_1 = 0.003$ and $\delta_2 = 0.001$. In the graph, the lightest gray region represents the highest membership degree (equal to 1) and the darkest represents the lowest membership degree (equal to 0).

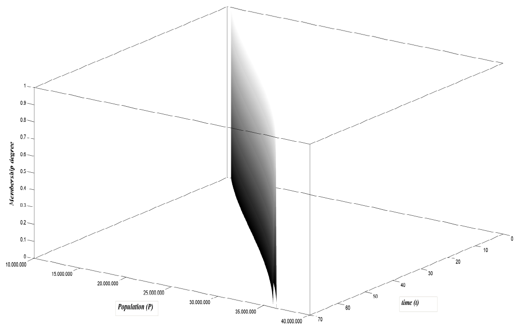

Figure 3: Montroll's model with triangular fuzzy parameter A.

4. Montroll's model with Interval Type-2 Fuzzy Parameter

A type-2 fuzzy set $\widetilde{A}$ in X is the graphic of a function

$$\begin{array}{rcccc} \mu_{\tilde{A}} & : & X \times [0,1] & \rightarrow & [0,1] \\ & & (x,u) & \rightarrow & \mu_{\tilde{A}}(x,u), \end{array}$$

where $\mu_{\tilde{A}}$ is called the membership function of $\tilde{A}$. In other words, the set $\tilde{A}$ is given by:

$$\tilde{A} = \left\{((x,u), \mu_{\tilde{A}}(x,u)) | (x,u) \in X \times [0,1], \mu_{\tilde{A}}(x,u) \in [0,1]\right\}.$$

In particular, when $\mu_{\widetilde{A}}(x,u) = 1$, $(x,u) \in X \times [0,1]$, it is called interval type-2 fuzzy set. The upper and lower membership function are denoted by $\overline{\mu}_{\widetilde{A}}(x)$, $\underline{\mu}_{\widetilde{A}}(x)$, $x \in X$, respectively, and defined by:

$$\overline{\mu}_{\widetilde{A}}(x) = \sup\{u | u \in [0,1], \mu_{\widetilde{A}}(x,u) > 0\},$$

$$\underline{\mu}_{\widetilde{A}}(x) = \inf\{u | u \in [0,1], \mu_{\widetilde{A}}(x,u) > 0\}.$$

We enunciate Zadeh's Extension Principle for interval type-2 fuzzy sets, adapted from (Mendel et al., 2009).

Let X and Y be universes and $f : X \longrightarrow Y$ a function. Given the interval type-2 fuzzy set $\widetilde{A}$ defined in X, the fuzzy set $\widetilde{B} = \widehat{f}(\widetilde{A})$ with membership function given by

$$\mu_{\widehat{f}(\widetilde{A})}(y) = \begin{cases} \sup\limits_{\{x:f(x)=y\}} \mu_{\widetilde{A}}(x) & \text{if } \{x : f(x) = y\} \neq \phi, \\ 0, & \text{otherwise,} \end{cases}$$

is called the type-2 Zadeh's extension of $\widetilde{A}$ by f.

Denote by A_l and A_u the to particularly embedded type-1 fuzzy sets corresponding to the upper $\overline{\mu}_{\widetilde{A}}$ and lower $\underline{\mu}_{\widetilde{A}}$ membership functions of $\widetilde{A}$. In (Mendel et al., 2009) it was demonstrated that if f is monotonic then the extension of $\widetilde{A}$ by f is given by

$$\mu_{\widetilde{B}}(y) = \left[\widehat{f}(A_l)(y), \widehat{f}(A_u)(y)\right]. \tag{4.5}$$

We utilize proposition 1 to obtain the extension of $\widetilde{A}$ for a continuous monotonic mapping f using the $\alpha-$cuts of A_l and A_u.

The Karnik-Mendel algorithm (Karnik and Mendel, 2001) was proposed for computing a defuzzification method for interval type-2 fuzzy sets. This method includes the well-known type-1 reduction of interval type-2 fuzzy set method. Following, we described this method for a discrete universe $X = \{x_1 < x_2 < \cdots < x_N\} \subset \mathbb{R}$.

A type-1 fuzzy set A_e is said to be embedded in the interval type-2 fuzzy set $\widetilde{A}$ if and only if

$$\underline{\mu}_{\widetilde{A}}(x_i) \leq \mu_{A_e}(x_i) \leq \overline{\mu}_{\widetilde{A}}(x_i), \quad \text{for all} \quad x_i \in X.$$

In figure 4 it is shown type-1 fuzzy sets embedded in the interval type-2 fuzzy set $\widetilde{A}$.

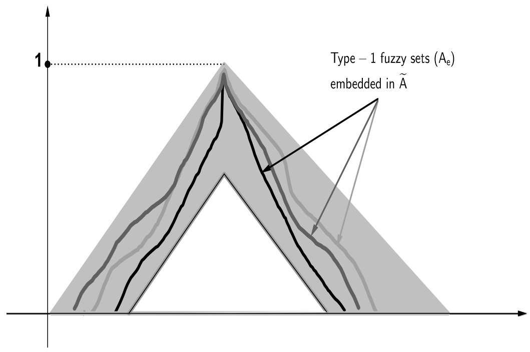

Figure 4: Type-1 fuzzy sets embedded in the interval type-2 fuzzy set $\widetilde{A}$ (Karnik and Mendel, 2001).

Given an interval type-2 fuzzy set $\widetilde{A}$, defined in X with membership function $\mu_{\widetilde{A}}(x)$, its centroid, $C(\widetilde{A})$, is defined as the collection of the centroids of all of its embedded type-1 fuzzy sets. In order to find the centroid of $\widetilde{A}$ consider an embedded type-1 fuzzy set $A_e(l)$ defined as

$$\mu_{A_e}(l) = \begin{cases} \overline{\mu}(x_i) \text{ if } x_i \leq x_l \\ \underline{\mu}(x_i) \text{ if } x_i > x_l, \end{cases}$$

where l is the so-called *switch point* of $A_e(l)$. Then, it has been demonstrated (Mendel and Wu, 2007) that $c_L(\widetilde{A}) = \min\limits_{l \in \mathbb{N}} centroid(A_e(l))$ exists and the switch point of the corresponding embedded set is $L \in \mathbb{N}$.

Likewise the fuzzy set $A_e(r)$ given by

$$\mu_{A_e}(r) = \begin{cases} \underline{\mu}(x_i) \text{ if } x_i \leq x_r \\ \overline{\mu}(x_i) \text{ if } x_i > x_r, \end{cases}$$

where r is the switch point of $A_e(r)$, $c_R(\widetilde{A}) = \max\limits_{r \in \mathbb{N}} centroid(A_e(r))$ exists with $R \in \mathbb{N}$ as a switch point of the corresponding embedded set. It has also been shown (Mendel and Wu, 2007) that

$$c_L(\widetilde{A}) = \frac{\sum\limits_{i=1}^{L} x_i \overline{\mu}_{\widetilde{A}}(x_i) + \sum\limits_{i=L+1}^{N} x_i \underline{\mu}_{\widetilde{A}}(x_i)}{\sum\limits_{i=1}^{L} \overline{\mu}_{\widetilde{A}}(x_i) + \sum\limits_{i=L+1}^{N} \underline{\mu}_{\widetilde{A}}(x_i)}.$$

$$c_R(\widetilde{A}) = \frac{\sum_{i=1}^{R} x_i \underline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=R+1}^{N} x_i \overline{\mu}_{\widetilde{A}}(x_i)}{\sum_{i=1}^{R} \underline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=R+1}^{N} \overline{\mu}_{\widetilde{A}}(x_i)}, .$$

The Karnik-Mendel algorithms (Mendel and Wu, 2007) locate the switch points L and R and as a consequence the centroid of $\widetilde{A}$. Then, the defuzzified value of the centroid is given by:

$$C = \frac{c_L(\widetilde{A}) + c_R(\widetilde{A})}{2}.$$

In figure 5 it is depicted an interval type-2 fuzzy set with the parameter $\widetilde{A}$ considered in equation (2.2). In this figure the $\overline{\lambda}$ parameter is a modal value; δ_i and θ_i $(i = 1, 2)$, are the dispersions of the upper and lower membership functions of $\widetilde{A}$.

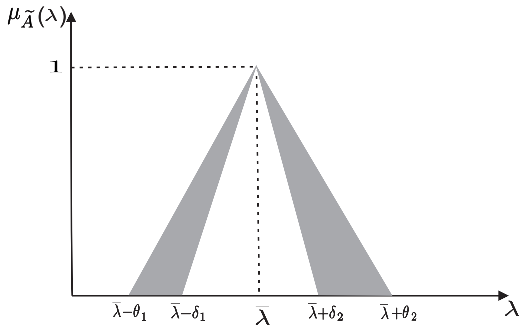

Figure 5: An interval type-2 fuzzy set $\widetilde{A}$.

The analytic forms of the upper and lower membership functions $\widetilde{A}$ are:

$$\overline{\mu}_{\widetilde{A}}(\lambda) = \begin{cases} \frac{1}{\theta_1}(\lambda - \overline{\lambda} + \theta_1), & if \quad \overline{\lambda} - \theta_1 < \lambda \leq \overline{\lambda}, \\ -\frac{1}{\theta_2}(\lambda - \overline{\lambda} - \theta_2), & if \quad \overline{\lambda} < \lambda \leq \overline{\lambda} + \theta_2, \\ 0, & otherwise, \end{cases} \tag{4.6}$$

and

$$\underline{\mu}_{\widetilde{A}}(\lambda) = \begin{cases} \dfrac{1}{\delta_1}(\lambda - \overline{\lambda} + \delta_1), & if \qquad \overline{\lambda} - \delta_1 < \lambda \leq \overline{\lambda}, \\ -\dfrac{1}{\delta_2}(\lambda - \overline{\lambda} - \delta_2), & if \qquad \overline{\lambda} < \lambda \leq \overline{\lambda} + \delta_2, \\ 0, & otherwise. \end{cases} \tag{4.7}$$

Both membership functions are triangular.

The extension of $\widetilde{A}$ is built through the continuous monotonic mapping S_t of the A_l and A_u. We utilize proposition 1 as a result of the assertion (4.5). The method is shown in figure 6.

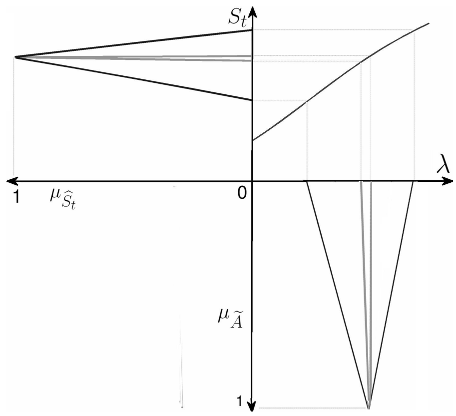

Figure 6: Zadeh's Extension Principle for the upper and lower membership functions of an interval type-2 fuzzy set.

In figure 7 it is presented the evolution of the upper and lower membership function of the Peruvian population at instant t with fuzzy parameters $\widetilde{A}$, considering $\alpha = 0.1$, $P_\infty = 53,587,774$, $\overline{\lambda} = 0.214$, $\theta_1 = 0.035$, $\delta_1 = 0.003$, $\theta_2 = 0.025$ and $\delta_2 = 0.001$. The lighter gray region of in this graph represents the highest membership degree (equal to 1) and the black region represents the smallest membership degree (equal to zero).

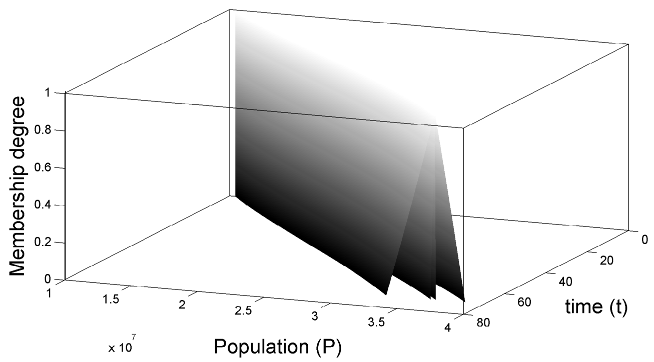

Figure 7: Evolution of the upper and lower membership function for the Peruvian population at instant t.

5. Comparisons of the Models with actual data from Peru

The real data (Table 1) was compared to the three models: the classic model solution, type-1 and type-2 defuzzified solutions of Montroll's model. We perform some empirical tests in order to determine which of the models fit best the actual data, which is the one that yields the lowest of the maximum of the relative errors. For the parameter of the type-1 A_i $(i = 1, 2, 3)$ model we utilize the following features:

- A_1: $\overline{\lambda} = 0.214$, $\delta_1 = 0.001$ and $\delta_2 = 0.003$;
- A_2: $\overline{\lambda} = 0.214$, $\delta_1 = 0.001$ and $\delta_2 = 0.001$;
- A_3: $\overline{\lambda} = 0.214$, $\delta_1 = 0.003$ and $\delta_2 = 0.001$.

For the type-2 parameter we utilize $\widetilde{A}_i$ $(i = 1, 2, 3)$ with the following features:

- $\widetilde{A}_1$: $\overline{\lambda} = 0.214$, $\delta_1 = 0.001$, $\theta_1 = 0.035$, $\delta_2 = 0.003$ and $\theta_2 = 0.025$;
- $\widetilde{A}_2$: $\overline{\lambda} = 0.214$, $\delta_1 = 0.001$, $\theta_1 = 0.035$, $\delta_2 = 0.001$ and $\theta_2 = 0.025$;

- $\widetilde{A}_3$: $\overline{\lambda} = 0.214$, $\delta_1 = 0.003$, $\theta_1 = 0.035$, $\delta_2 = 0.001$ and $\theta_2 = 0.025$.

The comparison is performed utilizing the maximum of the relative errors given by:

$$S_i = \max\left(\frac{|d_j - n_{ij}|}{|d_j|}\right), \quad i = 1, 2, 3,$$

$\forall\, d_j \in D$ and $n_{ij} \in N_i$, with $j = 1, \cdots, 6$, where,

- D is the data set of populations of Peru.
- N_1 is the set of population values for Montroll's classic model.
- N_2 is the set of defuzzified values obtained with A_i $i = 1, 2, 3$.
- N_3 is the set of defuzzified values obtained with $\widetilde{A}_i$ $i = 1, 2, 3$.

Then we determine:

$$S_1 = 0.018170862,$$

and

- for A_1, $S_2 = 0.01863121$;
- for A_2, $S_2 = 0.01817091$;
- for A_3, $S_2 = 0.01771091$.

We observe that the lowest S_2 occurs in the case that $\delta_1 > \delta_2$.

For the case of type-2 fuzzy set, we obtain the following results:

- for $\widetilde{A}_1$, $S_3 = 0.01465$;
- for $\widetilde{A}_2$, $S_3 = 0.01482$;
- for $\widetilde{A}_3$, $S_3 = 0.01409$.

Similarly to the type-1 parameter, the smallest S_3 occurs for $\delta_1 > \delta_2$ and $\theta_1 > \theta_2$, both fixed.

It is interesting to highlight that the Montroll's Model interval type-2 fuzzy parameters $\widetilde{A}_i$ $(i = 1, 2, 3)$, and their corresponding defuzzication through the centroid method (C) is the closest model to the real population data of Peru. These conclusions confirm the reported results of authors such as Mendel et al. (2006), who state that type-2 fuzzy logic systems may have higher performance than type-1 fuzzy systems.

6. Conclusion

We conclude that the closest approximation to the original Peruvian population data result from the defuzzified number for Montroll's model where the growth rate is taken as an interval type-2 fuzzy parameter.

Another important finding is that the Montroll's classic model presents reasonable predictions for population of Peru for the period 2021-2050, that are 33,126,956 and 41,258,926 inhabitants. Predictions from the INEI for the same years are 33,149,000 and 40,111,000 inhabitants.

The results emphasize the joint use of ordinary differential equations and fuzzy set theory, as paramount mathematical tools.

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