A model for optimal chemical control of leaf area damaged by fungi population - parameters dependence

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Abstract

We present a model to study a fungi population submitted to chemical control, incorporating the fungicide application directly in the model. From that, we obtain an optimal control strategy that minimizes both the fungicide application (cost) and leaf area damaged by fungi population during the interval between the moment when the disease is detected (t = 0) and the time of harvest (t = t_f). Initially, the parameters of the model are considered constant. Later, we consider the apparent infection rate depending on the time (and the temperature) and do some simulations to illustrate the case to compare with the constant case.

Keywords: optimal control, bang-bang control, fungi population, commutation point.

1 Introduction

Although biological control would be better from an environmental perspective, a combination of chemical and biological control is often necessary. Since 1882, a large application of fungicides has been used to avoid agricultural damages caused by pathogens. During this period until the end of the 70’s the commonly applied fungicides had unspecific action, that is, they were not highly selective against pathogens to be controlled [3]. It is worthy to state that the resistance to the unspecific fungicides was rarely observed.

However, the application of fungicides acting selectively to specific pathogenic agents, that occurred since the beginning of the 80’s, led to an increasing number of observations related to fungal resistance phenomenon. Since then biochemical researches have been carried out in order to establish the possible mechanisms between the selective action of the fungicide and the resistance phenomenon [5]. Thus, the dynamics studies regarding resistant fungi became very important to obtain control strategies taking into account the resistance to specific fungicide. Many studies have been developed to better understanding the behavior of the resistant population when submitted to the application of one fungicide, alternating fungicides or a mixture of fungicides ([1], [4], [5], [11], [19], [13]).
In this study our objective is to analyze not only the resistant fungi population but also the
cost of the fungicide applied. We are seeking an optimal strategy of fungicide application
within a pre-determined period (from the instant the disease is detected to the time of
harvest) that minimizes the total fungi population at the end of the period as well as the
cost of the fungicide applied.

According to Kimati (in [4]), no matter how technically efficient a fungicide may be its use
is dependent on economic factors. Thus, the inclusion of the minimization of the fungicide
applied is based on a concern not only with financial aspects, but also with the environmental
aspects.

We analyze this problem using the Optimal Control Theory and, consequently, the Pon-
tryagin Maximum Principle.

The application of Optimal Control Theory in Ecology appeared in the end of the 60's.
Shoemaker ([8], [9], [10]) investigated the application of optimization techniques, especially
the Dynamic Programing, to decision making in agricultural pest management. Leitmann [7]
illustrated the application of optimal control theory to obtain optimal strategies considering
a simple prey-predator system. The time of completion of the control program and the final
state of the system and the commutation points were obtained numerically (time instants
at which the control changes).

In Section 2 we describe all the parameters of the model and the formulation of the optimal
control problem associated to it. Section 3 analyzes the optimal control problem applying
the Pontryagin Maximum Principle. This Principle is considered when all the parameters are
constant and, further, some simulations considering the infection apparent rate depending
on the time (and temperature) are accomplished to exemplify and to be compared with the
constant case. In Section 4, we present some final remarks.

Even with several limitations of the model, it will be important to remark the difficulties of
obtaining analytic results related to the optimal control problem. For this reason, numerical
simulations are of fundamental importance to determine not only the feasible control but
also to provide scenarios resulting from different kinds of control strategy.

2 Model

The first models that appeared in the study of fungi population dynamics without fungicide
application assumed exponential growth in susceptible and resistant populations relative to
a specific fungicide. Besides that, the parameters involved in the models are considered
constant, which is also true in a great deal of mathematical models in Ecology.

The growth of a fungus or the production of a fungal metabolite is the result of interactions
of numerous enzymatically controlled processes; each one may have different optimal
temperature coefficients. Fungal activity will start at a minimum temperature, increase to
an optimum, and then decline and stop at a maximum temperature. These are known as the
cardinal temperatures for that activity, but they are dependent upon other factors, which
include time of exposure to any temperature [3].
As an attempt of incorporating the intraspecific and interspecific competition, Barrett [1] presented a general treatment of a pathogen population that includes two interacting subpopulations. Since host tissue is limited, the Lotka-Volterra competition model is suitable to describe the changes in the numbers of each subpopulation. With epidemic foliar diseases, using the proportion of leaf area affected by disease as an index of disease severity is much more convenient.

Our model for the fungi population control is based on Barret’s model [1] incorporating the fungicide application and the growth rate as a function of \( t \). The assumptions of the model are the following:

a) The fungi population (represented by the total occupied area \( N \)) analyzed in a given crop is subdivided into susceptible and resistant. We assign \( S(t) \) and \( R(t) \) to represent, at each time \( t \), the proportion of damaged leaf area for susceptible and resistant, respectively;

b) the intrinsic infection rates for susceptibles and resistsants are \( r_s(t) \) and \( r_r(t) \), respectively;

c) the change rate of susceptible to resistant is given by \( \alpha \);

d) the efficacy of fungicide is given by \( \beta \), which incorporates the dependency on the fungicide concentration;

e) the rate of fungicide application is given by \( u = u(t) \), a piecewise function. Moreover, we consider a limitation in fungicide application of the form \( 0 \leq u(t) \leq 1 \), that is, we always consider \( u \) relative to the value \( u_{\text{max}} \);

f) once a leaf is attacked by fungi population, the damaged area cannot be regenerated. Therefore we have \( dS/dt > 0 \) and \( dR/dt > 0 \).

Based on the above considerations, the fungi population can be described by the following system of differential equations:

\[
\begin{align*}
\dot{S}(t) &= (1 - \alpha) r_s(t) S(1 - S - R)(1 - \beta u) \\
\dot{R}(t) &= r_r(t) R(1 - S - R) + \alpha r_s(t) S(1 - S - R)(1 - \beta u),
\end{align*}
\]

where \( 0 \leq u(t) \leq 1 \) and the initial conditions are given by \( S(0) = S_0, R(0) = R_0 \). Although \( S, R, N \) and \( u \) are functions of \( t \), we will omit the variable \( t \) for simplification from this point on.

Letting \( r_s(t) = r_r(t) = r(t) \) and using the relation \( N = S + R \), the system of differential equations (1) and the initial conditions can be rewritten as:
\[
\begin{align*}
\dot{N} &= r(t)N(1-N)(1-\beta u) + r(t)R\beta u(1-N) \\
\dot{R} &= r(t)R(1-N) + \alpha r(t)(N-R)(1-N)(1-\beta u), \\
N_0 &= S_0 + R_0, \quad 0 \leq u \leq 1.
\end{align*}
\] (2)

The **optimal control problem** associated with the dynamics system (2) consists of finding \( u \) that minimizes the functional

\[
J(u) = N(t_f) + c_1 \int_0^{t_f} u(t)dt.
\] (3)

The function \( u^* \) is the optimal control which satisfies

\[
J(u^*) = \min_u \left( N(t_f) + c_1 \int_0^{t_f} u(t)dt \right).
\]

Note that we are considering a fixed final time \( t_f \), which corresponds to the harvest day or to the allowed time of fungicide application.

### 3 Model Analysis

The Hamiltonian related to the problem described in the previous section is given by:

\[
H(t, u, \lambda_1, \lambda_2, S, R) = [c_1 - r(t)(1-N)(N-R)\beta(\lambda_1 + \alpha\lambda_2)]u + \lambda_1 r(t)N(1-N) + \lambda_2 r(t)R(1-N) + \alpha r(t)\lambda_3(N-R)(1-N).
\] (4)

where \( \lambda_1 = \lambda_1(t) \) and \( \lambda_2 = \lambda_2(t) \) are the co-state variables.

The optimal control \( u^*(t) \) must satisfy the system (2) and also the co-state equations obtained from Hamiltonian [6], that are given by

\[
\dot{\lambda}_1^* = -\partial H/\partial N \quad \text{and} \quad \dot{\lambda}_2^* = -\partial H/\partial R
\]

and, in its simplified form, they are described as:
\[
\begin{aligned}
\lambda_1^* &= [r(t)(1 - \beta u^*)(1 - 2N^*) - r(t)3R^*u^*] \lambda_1^* = \\
&\lambda_2^* [-r(t)R^* + \alpha r(t)(1 - \beta u^*)(1 - 2N^* + R^*)]
\end{aligned}
\]
\[
\begin{aligned}
\dot{\lambda}_2^* + r(t)(1 - N^*)[1 - \alpha(1 - \beta u^*)]\lambda_2^* &= \lambda_1^* r(t)3u^*(1 - N^*)
\end{aligned}
\]
with \( \lambda_1^*(t_f) = 1 \) and \( \lambda_2^*(t_f) = 0 \).

The final conditions for \( \lambda_1^* \) and \( \lambda_2^* \) are obtained when \( t_f \) is fixed and, \( N^*(t_f) \) and \( R^*(t_f) \) are free.

Thus, using \( g(t) \) to denote the function that decides the control, it becomes

\[
g(t) = c_1 - r(t)(1 - N)(N - R)3(\lambda_1 + \alpha \lambda_2),
\]

and therefore, applying the Pontryagin's Maximum Principle to Hamiltonian, the optimal control results in:

\[
u^*(t) = \begin{cases} 
0, & \text{if } g(t) > 0 \\
1, & \text{if } g(t) < 0 \\
\text{indet}, & \text{if } g(t) = 0
\end{cases}
\]

3.1 Constant growth rate

The nonlinearity of the model complicates the analysis of the differential equations for \( \lambda_1 \) and \( \lambda_2 \) making it practically impossible to obtain analytic expressions for \( \lambda_1 \) and \( \lambda_2 \). We initially consider the growth rate being constant, that is, \( r(t) = r, \forall t \).

3.1.1 Analytic results

In the constant case \( r(t) = r \), we analyze which types of control are feasible in the following lemmas.

Lemma 1: If the change rate \( \alpha \) is null and the optimal control is zero \( (u^*(t) = 0) \) during some time interval \( (\bar{t}, t_f) \) then \( u^*(t) = 0 \) \( \forall t \in [0, t_f] \).

Proof:
Suppose that the control occurs in this way:

\[
u^*(t) = \begin{cases} 1, & 0 \leq t < \bar{t} \\
0, & \bar{t} < t \leq t_f
\end{cases}
\]
that is to say, the control ends with zero and it has only one change in \(\bar{t}\) \((g(\bar{t}) = 0)\).

As \(u^*(t) = 0\) for \(t \in (\bar{t}, t_f)\) we have \(g(t) > 0\) in this interval, and the differential equations \((5)\) for \(\lambda_1^*\) and \(\lambda_2^*\) take the following form:

\[
\dot{\lambda}_2^* + r(1 - N^*)(1 - \alpha)\lambda_2^* = 0
\]

\[
\dot{\lambda}_1^* + r(1 - 2N^*)\lambda_1^* = -\lambda_2^* [\lambda R^* + \alpha r(1 - 2N^* + R^*)]
\]

The unknown \(\lambda_2^*(t)\) is obtained using \(\lambda_2^*(t_f)\), resulting in

\[
\lambda_2^*(t) = \lambda_2^*(t_f) e^{\int_{t_f}^t r(1 - \alpha)(1 - N)ds} = 0,
\]

\(\forall \ t \in (\bar{t}, t_f)\), because \(\lambda_2^*(t_f) = 0\).

Consequently, we have the expression for \(\lambda_1^*(t)\), using \(\lambda_1^*(t_f) = 1\)

\[
\lambda_1^*(t) = e^{\int_{t_f}^t r(1 - 2N^*) ds} > 0, \quad \forall \ t \in (\bar{t}, t_f).
\]

Analyzing the function behavior \(g(t)\), defined in \((6)\), we have:

\[
g'(t) = r^2 \beta (1 - N^*) (N^* - R^*) [\alpha \lambda_1^*(1 - N^*) + \lambda_2^* [(\alpha - 1) R^* + \alpha (1 - N^*)]]
\]

Taking \(\alpha \approx 0\) (in practice it is between \(10^{-9}\) and \(10^{-5}\)) and \(\lambda_2^* = 0\), we have \(g'(t) = 0 \ \forall \ t \in (\bar{t}, t_f)\). Thus, \(g(t)\) is constant, \(\forall \ t \in (\bar{t}, t_f)\). By continuity, it is not possible that \(g(\bar{t}) = 0\), that is to say, \(\bar{t}\) doesn’t exist. Thus, \(u^*(t) = 0\) for every \(t\) in \([0, t_f]\).

From lemma 1, we observe that the optimal control, if it is not always null, should end with \(u^*(t) = 1\).

**Lemma 2:** If the optimal control ends with \(u^*(t) = 1\) and \(\alpha \approx 0\), then it will have at most one commutation point.

**Proof:**

Suppose that there exists \(\bar{t}_1\) and \(\bar{t}_2\), such that \(0 \leq \bar{t}_1 < \bar{t}_2 < t_f\), and the control is

\[
u^*(t) = \begin{cases} 
1 & \text{for } 0 \leq t < \bar{t}_1 \\
0 & \text{for } \bar{t}_1 < t < \bar{t}_2 \\
1 & \text{for } \bar{t}_2 < t \leq t_f
\end{cases}
\]
Consequently, we have

\[
g(t) = \begin{cases} 
< 0 ; & 0 \leq t < \bar{t}_1 \\
> 0 ; & \bar{t}_1 < t < \bar{t}_2 \\
< 0 ; & \bar{t}_2 < t \leq t_f
\end{cases}
\]

Considering \( \alpha \approx 0 \), we will analyze the function behavior \( g(t) \) in each subinterval:

- For \( 0 < t < \bar{t}_1 \) we have \( g(t) < 0 \) and therefore \( g(t) \) should be increasing so that a commutation point can exist in \( t = \bar{t}_1 \). Thus, \( g'(t) > 0 \Rightarrow \lambda_2^*(t) < 0 \).
- In \( \bar{t}_1 < t < \bar{t}_2 \) the differential equations for \( \lambda_1^* \) and \( \lambda_2^* \) are:

\[
\begin{align*}
\dot{\lambda}_1^* + r(1 - 2N^*)\lambda_1^* &= rR\lambda_2^* \\
\dot{\lambda}_2^* + r(1 - N^*)\lambda_2^* &= 0
\end{align*}
\]

for which the solution for \( \lambda_2^* \) is

\[
\lambda_2^*(t) = \lambda_2^*(\bar{t}_1) \exp \int_{\bar{t}_1}^{t} -r(1 - N^*)ds . \tag{10}
\]

As \( g(t) > 0 \) for \( \bar{t}_1 < t < \bar{t}_2 \) and \( g(t) < 0 \) in \( \bar{t}_2 < t \leq t_f \), \( g(t) \) it should continue increasing for \( t > \bar{t}_1 \), to reach a maximum in \( t = \bar{t} \) (\( \bar{t} \in (\bar{t}_1, \bar{t}_2) \)) so that later on \( g(\bar{t}_2) = 0 \).

Thus, \( \lambda_2^*(t) < 0 \) in \( \bar{t}_1 < t < \bar{t} \), \( \lambda_2^*(\bar{t}) = 0 \) and \( \lambda_2^*(t) > 0 \) for \( (\bar{t}, t_f) \).

Returning to (10) we conclude that \( \lambda_2^*(t) < 0 \) in the whole interval \( (\bar{t}_1, \bar{t}_2) \), what brings us to a contradiction.

**Theorem 1.** For the control problem (2)-(3), with \( \alpha \approx 0 \), the optimal control \( u^*(t) \) is of the type

\[
u^*(t) = \begin{cases} 
0 , & 0 \leq t < \bar{t} \\
1 , & \bar{t} < t \leq t_f
\end{cases}
\]

with \( t \in [0, t_f] \).

**Proof:** See Lemmas 1 and 2.

\( \dagger \)From the previous theorem we may identify the possible control types which naturally depend on the parameters involved in the model: \( S_0, R_0, \alpha, \beta, c_1, \) and \( r \). However, for the model studied we did not obtain an explicit formula among the parameters to identify when \( u^*(t) = 0, \forall t \in [0, t_f] \).
3.1.2 Numerical results

In the work of Basseu et al.\cite{basseu} some monocyclic parameters of rust (Uromyces appendiculatus) and of angular leaf spot (Phaeoisariopsis griseola) in two bean (Phaseolus vulgaris) cultivars (Rosinha G-2 e Carioca) were determined, as for example, leaf area, lesion density (number of lesions per \(\text{cm}^2\)), average lesion size and growth rate of the lesions.

Each fungal disease was analyzed considering the damaged leaf area and it was observed that the optimal temperature for the development of the rust is 17°C and for angular leaf spot is 24°C.

We used some data from \cite{basseu} in the numerical simulations and we observed the control changes when we altered each of the parameters, since we did not obtain an explicit formula among \(\bar{t}\) and the other parameters of the model.

In the numerical simulations the parameters were used with the following units: \(\alpha\) (dimensionless), \(t\), \(t_f\) e \(\bar{t}\) (days), \(r\) e \(\beta\) (days\(^{-1}\)), \(c_1\) (damaged area\(\text{days}\)). We also used a software developed in Fortran language to solve Optimal Control Problems with an algorithm of nonlinear programming known as BOX. The objective of this algorithm is to solve problems such as

\[
\begin{align*}
\min f(x) \\
x_{\min} \leq x \leq x_{\max}
\end{align*}
\]

where \(f: \mathbb{R}^n \rightarrow \mathbb{R}\) is differentiable and \(x_{\min}\) and \(x_{\max} \in \mathbb{R}^n\). Excel software was used to generate the tables and figures. For simplification, the units will be omitted in the description of the simulations as well as in most of the graphs.

We consider that in the crop, the total fraction of infected area is 10\% (\(N(0) = 0.1\)), where \(S(0) = 0.08\) e \(R(0) = 0.02\) (\(t = 0\) is the instant in which the disease is observed) and that the disease was detected with 80 days remaining before harvest; that is to say, \(t_f = 80\) days. Let us see some analyzed situations:

a) the influence of in the determination of \(\bar{t}\) (commutation point)

Considering the parameters \(\beta, c_1, \alpha\) fixed and the initial conditions \(S_0, R_0\), then we used some values of \(r\) obtained in \cite{basseu} for the angular leaf spot in the cultivar Rosinha G-2. The chosen values of \(r\) are: \(r_1 = 0.026\ A/day\) where \(A = 10\ mm^2\), \(r_2 = 0.31\ A/day\) and \(r_3 = 0.0336\ A/day\). Although the maximum growth rate obtained is \(r_3\), we considered several values of \(r\) (larger than \(r_3\) and also smaller than \(r_1\)) with the purpose of observing the variation of \(\bar{t}\) as a function of \(r\).

In the following figures, we may observe the \(\bar{t}\) variation as a function of \(r\), as well as the total damaged leaf area \((S^*(t_f), R^*(t_f), N^*(t_f))\) after the respective control. From theorem 1, if there exists control change, it happens from \(u = 0\) to \(u = 1\). For \(\bar{t} = 80\) we have \(u^*(t) = 0\), \(\forall t \in [0, 80]\) and for \(\bar{t} = 0\) we have \(u^*(t) = 1\).

In Table 1, we take \(S_0 = 0.08\), \(R_0 = 0.02\), \(c_1 = 0.002\), \(\beta = 0.6\) and \(\alpha = 10^{-5}\). We can observe that, with \(\beta\) and \(c_1\) fixed, as \(r\) increases the control change exists so that the
fungicide application period increases ($\bar{t}$ decreases) until for $r \in [0.031, 0.045]$ the application becomes necessary during the whole interval $[0, 0.80]$. For values of $r$ larger than 0.045, the values of $\bar{t}$ increase and the fungicide application becomes unfeasible for large values of $r$ (see Figure 1). In Figure 2 we observe the behavior of damaged leaf area in final time $t_f$ for each type of control obtained (as a function of $r$).

![Graph](image1.png)

Figure 1: Commutation point $\bar{t}$ as a function of the growth rater (data of Table 1).

![Graph](image2.png)

Figure 2: $S^*(t_f)$, $R^*(t_f)$ and $N^*(t_f)$ as a function of $r$ for the data in Table 1 $N^*(t_f)$ is the percentage of damaged leaf area at the final time $t_f$ for each value of fixed $r$.

Considering the same parameters of Table 1, except $c_1$ and $\beta$, that were considered as $c_1 = 0.004$ and $\beta = 0.8$, we obtained in Table 2 the $\bar{t}$ variation as a function of $r$ as well as the respective values for $S^*(t_f)$, $R^*(t_f)$ and $N^*(t_f)$.

In Table 2 we do not yet have the same type of behavior observed in Table 1. For this group of data we did not observe an interval of $r$ for which $u^*(t) = 1$ (as in the Table 1) but there exists an interval of $r$ where the commutation point practically doesn't change (Figure 3). In Figure 4 we also did not observe for this group of data the values of $S^*(t_f)$, $R^*(t_f)$ and $N^*(t_f)$ for each $r$ fixed.

If we take the same parameters for Table 1 but $c_1 = 0.02$, we found $u^*(t) = 0$ for any value of $r$ of the figure.
Figure 3: Commutation point $t$ as a function of the growth rate (data in Table 2).

Figure 4: $S^*(t_f)$, $R^*(t_f)$ and $N^*(t_f)$ as a function of $r$ for the data in Table 2. $N^*(t_f)$ is the percentage of damaged leaf area at the final of fixed $r$.

b) $\beta$ and $c_1$ influence in the determination of $\bar{t}$

Maintaining fixed $\alpha = 10^{-5}$, $S_0 = 0.08$, $R_0 = 0.02$, $r = 0.026$, $t_f = 80$ and varying the parameters $\beta$ and $c_1$, we observe the value of $\bar{t}$ in Table 3. For this group of data, with fixed $\beta$, there exists an interval $[a, b]$ of $c_1$ for which a commutation point exists and, as $c_1$ increases ($c_1 > b$) the fungicide application becomes unfeasible; if $c_1 < a$ then $u^*(t) = 1$ ($\bar{t} = 0$), $\forall t \in [0; 80]$.

We can visualize in the following illustrations, $\bar{t}$ as a function of $\beta$ (for each $c_1$ of Table 3) as well as $\bar{t}$ as a function of $c_1$.

Although we do not have an explicit formula among the parameters for $u^*(t)$, some results were only possible with the aid of numeric simulations.

3.2 Growth Rate as a function of $t$

We will consider $r$ as a function of $t$ (also of the temperature) to make a comparison with the constant case.
We use some data of [2] for angular leaf spot in the cultivar Rosinha G-2. For example: optimal temperature for the development of this fungal disease (24°C) and the values of $r$ at the temperatures 24°C ($r = 0.014\text{mm}^2/\text{h}$) and 29°C ($r = 0.002\text{mm}^2/\text{h}$). Those values of $r$ will be used with the same units of the previous simulations.

If we consider that the maximum temperature reached during the year is 39°C, and the minimum is 1°C, we can simulate the temperature as a function of $t$ (from January to December) as:

$$T(t) = 20 + 19\cos(\pi t/180) ,$$

and using the cardinal temperature concept, we assume the growth rate as a function of the temperature as given by

$$r(T) = a \exp[-b(T - T^*)^2] .$$

Since there was no development of angular leaf spot at 17°C and at 30°C, for simplification, to obtain $r$ in a continuous way we will use
\[ T = 24^\circ C \quad \Rightarrow r = 0.0336 \]
\[ T = 6^\circ C \quad \Rightarrow r = 0.0048 \]

Thus, we obtain the function \( r(t) = r(T(t)) \), given by

\[ r(t) = 0.0336 \exp[-0.006 \times (-1 + 19 \times \cos(\pi t/180))^2]. \]

Using \( S_0 = 0.08 \), \( R_0 = 0.02 \), \( \alpha = 10^{-5} \), \( \beta = 0.6 \), \( c_1 = 0.002 \) and the function \( r(t) \) above, we observe that the control obtained does not necessarily have the same type of behavior of the constant case \( r \).

Taking \( r \) for \( t \in [50, 130] \) a control change is obtained in \( \tilde{t}_1 = 4.22 \) and \( \tilde{t}_2 = 50.24 \) while, considering \( r \) for \( t \in [0, 80] \) we obtain \( \tilde{t} = 52.25 \). In the following illustrations we can observe these results and compare \( r(t) \) with the interval of time in that \( u^*(t) = 1 \): that is to say, \( u^*(t) = 1 \) for the interval in which \( r(t) \) is maximum.

![Graphs showing control and function behavior](image)

Figure 7: Interval of \( r(t) \) function and the respective control obtained - (a) \( r \) for \( t \in [50, 130] \), (b) \( r \) for \( t \in [0, 80] \).

It is worthwhile to stand out that the passage of \( u = 0 \) to \( u = 1 \), using numeric methods, it does not happen in an instantaneous way, what can be observed in the graphs for \( u^*(t) \).

Thus, the inclusion of the temperature effect in the lesions growth rate results in a different behavior in linear model [15] and in nonlinear model. In the first one the control change (when it exists) is unique, while in the second, the control change can happen in two different time instants.
Although these observation has only been made from numerical simulations, we verified that the nonlinearity of the model, with the inclusion of the competitions inter and intraspecific has a great influence for the determination of an optimal control, mainly when we consider \( r = r(t) \).

4 Final Considerations

Upon any economical circumstances the main goal is to achieve the maximum of productivity with small costs. In the specific case of this study we mathematically modelled an agricultural production situation subjected to the attacks of susceptible and resistant fungi population in order to maximize production (minimizing disease at the end of the harvest) controlling disease at low cost.

We considered the dynamics of the damaged leaf area with the intraspecific competition between susceptible and resistant fungi population for the occupation of leaf area. Such occupation is cumulative, that is, once the leaf area is damaged it does not recover, which causes a small production of the plant. The productivity is affected significantly if the damaged area exceeds some limits that varies for each cultivar. The control problem that we analyzed searched for the establishment of a a minimum value for the damaged leaf area in the final time (crop) with the minimum cost of fungicides.

In this model type, as well as in the linear model described in [15], we considered constant \( r \) and \( r = r(t) \) variable with the time. In the constant case, the control change (when it exists) is from \( u = 0 \) to \( u = u_{\text{max}} \), and for \( r(t) \) variable, the possibility of two control changes exists (although it was only observed in simulations) and \( u = u_{\text{max}} \) is obtained in the interval where the growth rate is maximum.

It is important we remark that the mutation of susceptible for resistant has little influence in the control measure when we want for identify how the fungicide must be applied as soon as the disease has been detected. In this case, we are not taking into account previous applications, but the level reached by resistant population just after the control, which can be analyzed for further applications.

We consider \( r(t) \) variable by incorporating only the effect of the temperature, because it is the variable that is more frequently related to biological answers and almost universally mentioned in the epidemic studies, besides affecting all the monocyclic components of the plants, as the host growth and diseases development.

Another fact to be considered is the plant growth (leaf area) was supposed constant. However, it is necessary being prudent when incorporating some more variables in the model because analytic results become very difficult and numerical methods are necessary.
## Appendix: Tables

<table>
<thead>
<tr>
<th>r</th>
<th>i</th>
<th>S(t_i)</th>
<th>R(t_i)</th>
<th>N(t_i)</th>
</tr>
</thead>
<tbody>
<tr>
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Table 1. $\bar{t}$ values and $(S^*(t_f), R^*(t_f), N^*(t_f))$ for each growth rate $r$, considering fixed the parameters and the initial conditions: $S_0 = 0.08$, $R_0 = 0.02$, $c_1 = 0.002$ and $\beta = 0.6$
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Table 2: \( \bar{t} \) values and \((S^* (t_f), R^* (t_f), N^* (t_f))\) for each growth rate \( r \), considering fixed the parameters and the initial conditions: \( S_0 = 0.08, \ R_0 = 0.02, \ c_1 = 0.004 \) and \( \beta = 0.8 \)

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Table 3: \( \bar{t} \) value for each \((\beta, c_1)\) \( c_1 = 0.003 \) and \( \beta = 0.7 \) we obtain \( \bar{t} = 47.43 \)

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References


