Dip correction for coherence-based time migration velocity analysis

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ABSTRACT

Migration velocity analysis (MVA) is a seismic processing step that aims to translate residual moveout in an image gather after migration with an erroneous velocity model into velocity updates. An analysis of the position of a reflection event in an image gather after migration with an incorrect velocity allows us to extend the original coherence-based MVA approach to dipping reflectors. The extended MVA technique includes the reflector dip which is treated as an additional search parameter that is to be detected together with the velocity updating factor. Both parameters are searched for simultaneously by the application of two-parameter search techniques. The search consists of determining trial curves as a function of the search parameters and stacking the migrated data along these curves. The highest coherence determines the best-fitting curve and thus the optimal parameter pair. A numerical example demonstrates that the additional search parameter improves the quality of the velocity updates, thus requiring less iterations in the MVA.

INTRODUCTION

Migration velocity analysis (MVA) is a seismic processing step that exploits the redundancy of seismic data to improve an a priori velocity model. The basic idea is to use the velocity information that is contained in the residual moveout in an image gather, i.e., a CMP gather of the prestack migrated image cube, after migration with an erroneous velocity model (Yilmaz and Chambers, 1984). MVA aims at translating this residual moveout into velocity updates (Fowler, 1985; Al-Yahya, 1989). The process can be applied iteratively, thus generating a loop between migration, MVA, and velocity updates, which terminates when the residual moveout is sufficiently flattened.

The initial concepts soon led to a wide range of more sophisticated techniques. One of the first developments was the picking-free differential semblance optimization (DSO) of Symes and Carazzone (1991). Lafond and Levander (1993) generalized Al-Yahya’s method to arbitrary heterogeneous media, based on ray tracing rather than an analytical formula for the residual moveout. Bradford and Sawyer (2002) suggested three implementations of MVA methods that progress from relatively simple to relatively complex and computationally intensive. Further developments include a residual moveout analysis on a sparse grid of common midpoints (CMPs) (Audebert et al., 1998; Woodward et al., 1998) or, more recently, along a fine grid following horizons (see, e.g., Billette et al., 2002). In parallel, other algorithms, based on picking of either continuous (Liu, 1997) or locally coherent (Chauris et al., 2002a,b) events, were suggested. Earlier, Biondi and Sava (1999) proposed a residual-migration wave-equation tomographic technique. The efficiency and accuracy of these various migration-based methods strongly depend on the density of points where the analysis is carried out (Billette et al., 2003).

In this paper, we follow the lines of the coherence-based approach of Al-Yahya (1989). In other words, we propose a semblance analysis along certain stacking curves within the data. This approach has the advantage that no picking of migrated events in the image gather is needed. The method of Al-Yahya is restricted to horizontal reflectors. Therefore, it requires an iterative approach even for a dipping reflector below a homogeneous overburden. To overcome this disadvantage, we extend Al-Yahya’s method to dipping reflectors. A first attempt in this respect was undertaken by Lee and Zhang (1992) using near-offset and small-dip approximations. Here, we keep the assumption of small dips (≤ 25°) but drop the restriction of small offsets.

Our extended MVA technique treats the reflector dip as an additional search parameter that is to be detected together with the velocity updating factor. We searched for both parameters simultaneously by the application of techniques that have been developed in connection with the common-reflection-surface (CRS) stack (Biloti et al., 2002). Like that method, the search consists of determining trial curves as a function of the search parameters and stacking the migrated data along these curves. The highest coherence determines the best-fitting curve and thus the optimal, i.e., best-possible, parameter pair.

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As in Al-Yahya (1989), the development of the present technique relies on the assumption of constant velocity and should be applied iteratively in inhomogeneous media to estimate root-mean-square (rms) velocities. The main objective of including the reflector dip as an additional parameter is to achieve higher accuracy in the updated velocity, thus reducing the number of necessary iterations.

MIGRATION WITH INCORRECT VELOCITY

Dipping reflector

As a basis for the stacking technique, we need a theoretical expression for the position of the migrated image of a dipping reflector as a function of the (incorrect) migration velocity. For this purpose, we consider a dipping reflector with dip \( \mathbf{m} = \tan \theta \), where \( \theta \) is the dip angle. The depth \( z \) of this reflector at a horizontal position \( x \) is thus described by the formula \( z = mx + z_0 \), where \( z_0 \) is the depth of the reflector vertically below the coordinate origin (see Figure 1). We consider this reflector \( z = mx + z_0 \) to be buried in a homogeneous medium with true velocity \( v \).

Reflection traveltimes

Next, we consider a 2D seismic experiment being carried out over this medium along the \( x \)-axis. Sources and receivers are positioned at points \( x_l = y - h \) and \( x_r = y + h \), where \( y \) and \( h \) are the (varying) midpoint and half-offset coordinates (see again Figure 1). As shown in the first section of the Appendix, the reflection traveltimes can then be conveniently expressed as

\[
t_{\text{rel}} = \frac{2r}{v} \cos \theta,
\]

where \( r \) is the distance between the reflector point vertically below the midpoint \( y \) and source or receiver (see Figure 1).

Reflector image

The reflector event located in the data at the traveltimes expressed by equation 1 is now to be migrated using the (incorrect) migration velocity \( v_m \). In the second section of the Appendix, we show that an approximate analytic expression for the position of the time-migrated reflection event in an image gather can be constructed as the envelope of isochrons for all points \((y, t_{\text{rel}})\) on the reflection traveltime curve. This approximation is given by

\[
t_{\text{ig}}(h) = \tau + (1 - \gamma^2) \left(4\gamma^2h^2 + v_m^2z_0^2\right)(4h^2 - v_m^2z_0^2)\frac{m^2}{2v_m^4r^3},
\]

(2)

where

\[
\tau = \sqrt{t_0^2 + (\gamma^2 - 1)4h^2/v_m^2}.
\]

(3)

Here, \( t_0 \) stands for vertical time and \( \gamma \) denotes the ratio between the migration and true medium velocities as defined by Al-Yahya (1989), i.e.,

\[
\gamma = v_m/v.
\]

(4)

We immediately observe that equation 3 is exactly Al-Yahya’s expression for the image position of a horizontal reflector. The additional factor 4 in the last term under the square root is due to the fact that Al-Yahya considers a migration slowness that corresponds to twice the medium slowness, i.e., his \( w_m \) relates to \( v_m \) as \( w_m = 2/v_m \). Because Al-Yahya’s expression is the first term of formula 2, we can interpret its second term as a dip correction to Al-Yahya’s formula. Note that equation 2 is a third-order approximation in \( m \), but with the first- and third-order terms in \( m \) equal to zero. Therefore, the image gather contains no information about the sign of \( m \), i.e., of the direction of the dip. Moreover, the fact that there is no first-order term in equation 2 shows that Al-Yahya’s formula 3 is already a first-order approximation.

There are two important conclusions to be drawn from the second term in expression 2. First, the proportionality factor \( 1 - \gamma^2 \) confirms Al-Yahya’s observation that the dip dependence of the reflector image position decreases as \( \gamma \) approaches 1, i.e., as the migration velocity \( v_m \) approaches the true medium velocity \( v \). Second, the equality at zero offset between \( t_{\text{ig}} \) and \( t_0 \), which Al-Yahya observed for horizontal reflectors, is no longer true for dipping reflectors. For zero offset, equation 2 reduces to

\[
t_{\text{ig}}(0) \approx t_0 + (\gamma^2 - 1)t_0m^2/2.
\]

(5)

This equation describes the well-known fact that a nonzero reflector dip causes the zero-offset image of the reflector to shift downward if the migration velocity is too high, and upward if the migration velocity is too low.

SEARCH TECHNIQUE

As suggested by Al-Yahya (1989) for its first term, we use equation 2 to carry out coherence analysis along trial lines defined by this equation. However, instead of searching for a single parameter \( \gamma \) for each \( t_0 \), we search for two parameters: \( \gamma \) and \( m \). Although \( \gamma \) as before, is meant to determine an estimate for the true medium velocity from the migration velocity as

\[
v_{\text{est}} = v_m/\gamma,
\]

(6)

the main role of the second parameter \( m \) is to stabilize the search and achieve better estimates for \( \gamma \) in situations with dipping interfaces and/or where the first guess for \( v_m \) was quite bad. The actual values of \( m \) can be useful at the interpretation stage.

To determine the optimal values for \( \gamma \) and \( m \), a biparametrical search has to be carried out. To make this bidimensional search computationally feasible, we need starting values for both parameters as
close as possible to their optimal values. To determine these starting values, we first search one-dimensionally for each of them independently with the following strategy.

Note that Al-Yahya’s formula, which is accurate up to first order in \( m \), depends only on \( \gamma \). Only the second-order term of the proposed formula depends on both parameters, \( \gamma \) and \( m \). This means that the velocity parameter \( \gamma \) plays a more important role than the dip parameter \( m \). This observation suggests that the 1D search for \( \gamma \) should precede the search for \( m \).

Because our dip correction was derived from a Taylor expansion for small \( m \), the first stage of the search is to look for a value of \( \gamma \) that maximizes the coherence measure with \( m \) set to zero. In other words, the starting value for \( \gamma \) is obtained by a search using Al-Yahya’s formula only. When an initial estimate for \( \gamma \) is found, a second 1D search is performed with fixed \( \gamma \) to obtain an approximation for \( m \).

This procedure is only intended to obtain rough estimates for these two parameters, which serve as starting values for the subsequent bidimensional search. Finally, a derivative-free optimization method (Kolda et al., 2003) is employed to obtain the best-possible values for \( \gamma \) and \( m \) simultaneously.

Because the purpose of our implementation is to demonstrate the technique, we employ a simple optimization method. It is a realization of a standard simplex search method (Nelder and Mead, 1965). In two dimensions, its principle is to evaluate the target function at the vertices of a triangle and, depending on the resulting values, to reflect, invert, stretch, and/or shrink the triangle for the next search step until a given precision or a maximum step number is reached. [For details about the implementation, see Galassi et al. (2005)].

The described procedure follows the same search technique successfully employed to simultaneously obtain the three traveltime attributes of the CRS stack (Birgin et al., 1999). In the present application, the bidimensional search technique is even more advantageous than in the case of the CRS stack, where the search for the second and third parameters was quite time consuming. Also, the search for the additional parameter \( m \) (instead of looking only for \( \gamma \)) has a small impact on computation time.

**Figure 2.** Model for the synthetic experiment. Three interfaces with dips of 10\(^\circ\), 15\(^\circ\), and 20\(^\circ\), respectively, are embedded in a homogeneous background medium with a velocity of 2 km/s. The bold line on top of the model indicates the extension of midpoints along the seismic line and the two vertical dashed lines indicate the region to be time migrated.

**Figure 3.** Image gather of the data migrated with an incorrect velocity at 3000 m. The theoretical predictions of the image positions in this gather are indicated with (short-dashed white curves) and without (long-dashed black curves) dip correction as described by equation 2 using the true values for \( t_{00} \), \( \gamma \), and \( m \).

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**NUMERICAL EXAMPLE**

In this section, we demonstrate the application of the dip-corrected MVA to a synthetic data set. The depth model is depicted in Figure 2. It consists of three dipping planar interfaces embedded in a homogeneous background medium with a velocity of 2 km/s. The three interfaces have dips of 10\(^\circ\), 15\(^\circ\), and 20\(^\circ\), respectively.

We used a seismic ray-tracing algorithm to generate synthetic data, with a sampling rate of 2 ms. For the numerical experiment, we simulated 241 midpoints at every 25 m along the seismic profile between CMP coordinates 2200 and 8200 m (indicated at the top of Figure 2) with 81 source-receiver offsets at every 50 m from 0 up to 4000 m. These data have been Kirchhoff time migrated with the incorrect (constant) migration velocity of 3.5 km/s to grid points at every 10 m between 2000 and 4000 m (indicated by dashed lines in Figure 2), with a sampling rate of 4 ms up to a maximum time of 3 s. White noise with a signal-to-noise ratio of 3 was added to the migrated data.

A coherence-based MVA has been applied to the resulting image gathers, using Al-Yahya’s original time-image formula 3 and our dip-corrected formula 2. Figure 3 shows a typical image gather at a migrated CMP at the horizontal position \( x = 3000 \) m. Note the strong moveout resulting from the extremely high migration velocity. Figure 3 also shows the predicted positions of the migrated reflectors as determined by equations 2 and 3, using the correct values for \( t_{00} \), \( \gamma \), and \( m \). We observe a much better prediction of the actual mi-
grated positions by the dip-corrected formula than by Al-Yahya’s original formula.

As the next step, the search procedure described above was applied to all image gathers. Figure 4 shows the maximum-coherence values for the optimal values of m and γ at each midpoint and each time value in the migrated domain as obtained with Al-Yahya’s and our formulas. In both sections, we observe rather high coherence values, i.e., close to 1.0. This means that both formulas can be adjusted rather well to the data, however for a different set of parameters. The difference between the formulas lies in the obtained parameter values.

Along each migrated reflection event, we detect the position of maximum coherence. To discard outliers, a maximum was accepted rather well to the data, however for a different set of parameters. The difference between the formulas lies in the obtained parameter values.

The difference between the formulas lies in the obtained parameter values. In both sections, we observe rather high coherence values, i.e., close to 1.0. This means that both formulas can be adjusted rather well to the data, however for a different set of parameters. These provisional reflector images, which we refer to as paths of maximum coherence, will be extracted for the updating of the velocity model. Note the difference in position between the maximum-coherence paths in the two panels. The mispositioning in Figure 4a is because of the bias in the detected t0 caused by the dipping reflector.

Figure 5 shows the comparison of the optimal values of the velocity correction factor γ, as obtained with formulas 2 and 3. The desired value of γ is 1.75, which is represented by white color. Note that the values of γ are only meaningful at the positions of the migrated reflector images (i.e., the maximum-coherence paths indicated by the solid lines). The values off these positions are random values without any physical meaning. Upon inspection of these two sections at the reflector images, we observe that Al-Yahya’s formula tends to underestimate the parameter γ, while the dip-corrected formula determines more reliable values. Because Al-Yahya’s formula does not consider dips, the resulting values of γ are biased.

This observation is confirmed in Figure 6, where the values of γ extracted from the sections in Figure 5 along these maximum-coherence paths are compared. The values of γ obtained with Al-Yahya’s formula 3 (circles) show a significant deviation from the true value of 1.75. The values obtained with the dip-corrected formula 2 (crosses) are much closer to the true gamma value of 1.75. The quality of the obtained values degrades slightly from the first reflector (top) to the third one (bottom), i.e., for increasing dip. This behavior was to be expected because the derivation of formula 2 relies on an approximation for small dips.

The dip-corrected formula 2 provides, as an additional parameter, the optimal dip. As explained before, the attribute optimal is the parameter value that achieves best fit between the traveltime curve (equation 2) and the positions of the migrated reflector images. Figure 7 shows the resulting dip angles along the maximum-coherence curves: the dip angle of 10° of the first reflector is almost perfectly recovered (circles), the dip of the second reflector (15°) is slightly underestimated (crosses), and that of the third reflector (20°) is more significantly underestimated (squares). These results indicate that the recovered dip angle is more sensitive to the small-dip approximations than the accuracy of the velocity correction factor γ.

We note from Figures 6 and 7 that the estimated velocity and dip values present certain fluctuations. These instabilities result from noisy data and local maxima of the coherence measure. Also, small variations on the velocity correction factor can be compensated by deviation of the estimated dip because both parameters are not decoupled completely. These effects are much stronger for the dip estimate because formula 2 is more sensitive to γ than to m.

To eliminate this high-frequency error of both parameters, the extracted values have been smoothed with optimal splines. This technique is an optimized cubic spline approximation that has been developed by Biloti (2002). It differs from conventional spline interpolation in the fact that it detects, for a given number of control points, their best possible positions in a least-squares sense. The smoothing technique preserves lateral trends as well as abrupt changes while eliminating outliers and short-range fluctuations. For this purpose,
approximately 20 neighboring image gathers are required. In this way, the final estimate becomes more reliable than each single estimate.

Finally, the actual delivery of an MVA is the updated migration velocity field. To construct this field, the velocities are updated along the maximum-coherence paths by multiplication with the corresponding gamma values. For this purpose, not only the gammas but also the maximum-coherence paths are smoothed by optimal splines (Biloti, 2002; Biloti et al., 2003). Where no velocity is known, the field is filled in by linear interpolation of the slowness. Above and below the first and last paths, the corresponding velocity values are repeated. The so-constructed velocity fields based on the competing formulas are compared in Figure 8, which shows that, the dip correction has indeed succeeded in providing a better updated velocity field than Al-Yahya’s original formula. Not only are the updated velocities closer to the true value of 2 km/s, but also the velocity field is less heterogeneous, reflecting the homogeneity of the original model.

A more quantitative analysis is carried out in Figure 9. It shows the relative velocity error after velocity updating. Note the diminished error of the dip-corrected results. Although the error after Al-Yahya’s correction is in the order of 2%, the dip correction reduces this to less than 0.5%.

As a final test, the original data have been Kirchhoff time migrated with the updated rms velocity fields. The results are compared in Figure 10. The improvement of the resulting image using the dip correction is clearly visible. Although the quality of the image resulting from using the dip correction is satisfactory, the velocity update using Al-Yahya’s formula would require another iteration. It should be kept in mind that one more iteration not only means to repeat the search process, but also to perform another migration with the improved velocity model. Because the effort for any of the one- or two-parametric searches is negligible compared to a migration, eliminating just a single iteration can significantly reduce the overall cost of the MVA.

![Figure 6. Values of γ as extracted from the γ sections (Figure 5) along the maximum-coherence curves of Figure 4 (Al-Yahya: circles; dip-corrected: crosses), from top to bottom, for the first, second, and third reflectors, respectively.](image)

![Figure 7. Dip angles along the maximum-coherence curves (first reflector: circles; second reflector: crosses; third reflector: squares) as obtained from MVA using the dip-corrected formula 2. Also shown are the results of a smoothing of these data (solid lines).](image)

### DISCUSSION

We have presented a dip correction to the coherence-based approach to MVA of Al-Yahya (1989). This approach is based on a coherence analysis along possible moveout curves in a migrated image gather for different values for the ratio (γ) between the migration and true medium velocities. The highest coherence determines the optimal values for this ratio, which can then be used to update the migration velocity. This approach has the advantage that no event picking in the migrated section is needed.

We generalized the method of Al-Yahya, which was restricted to horizontal reflectors, to dipping reflectors. In our extended MVA technique, the reflector dip is treated as an additional search parameter, which is to be detected together with the velocity updating factor γ. We searched for both parameters simultaneously by the application of techniques that have been developed in connection with the CRS stack (Biloti et al., 2002). Like with that method, the search is carried out by determining trial curves as a function of the search parameters and stacking the migrated data along these curves. The highest coherence determines the best-fitting curve and thus the optimal, i.e., best-possible, parameter pair. In this way, estimates can be found not only for the velocity ratio γ, but also for the reflector dip.

In accordance with Al-Yahya, we measured the coherence by means of semblance. Both methods benefit from more stable measures, whenever available. It should be mentioned that, being a one-parametric search, Al-Yahya’s method tends to run into less stability problems than our two-parametric approach. Because the parameters are not totally uncoupled, different combinations can result in a comparable fit, thus providing almost the same coherence.

Although the dip-corrected strategy tends to be the less stable method, it can be expected to always provide better velocity updates. Because for zero dip, the two-parametric curve reduces to Al-Yahya’s one, the search will result in zero dip unless another dip value improves the fit. Thus, the two-parametric fit must be equal to or better than that of Al-Yahya’s one-parameter curve. This observation is independent of whether the medium is homogeneous as in our example or inhomogeneous.

A numerical example has demonstrated that the additional search parameter can indeed be helpful for the analysis. Although curves
described by Al-Yahya’s original formula can be well fitted to the actual migration moveout, thus providing high coherence values, it turns out that the additional search parameter improves the estimates of the velocity ratio and its positioning in the migrated section. In our example, the velocity was recovered sufficiently in one step to produce a satisfactory final migrated image. With Al-Yahya’s formula, one more iteration would have been necessary. Because each iteration includes not only the search process, but also a subsequent migration, eliminating only one extra migration step significantly reduces the overall cost of the MVA. Also, the recovery of the dip was successful, providing reasonable estimates of the true reflector dip.

CONCLUSIONS

Adding a dip parameter to the description of a position of a reflection event in an image gather obtained from a migration with incorrect velocity has added accuracy to the velocity updating procedure with very little impact on the computational cost. Because of the additional degree of freedom, the resulting velocities show a stronger fluctuation around the improved mean value. This effect can be reduced by smoothing over a range of image gathers. Because of the improved velocity update, fewer iterations of the process, each step of which consists of one full migration and one velocity determination, are needed to achieve a reliable velocity model.

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APPENDIX A

MIGRATED REFLECTOR IMAGE

In this Appendix, we derive the theoretical expression 2 for the position in the image gather of the time-migrated image of a dipping reflector as a function of the (wrong) migration velocity.
Dipping reflector

We start with the derivation of a convenient expression (equation 1) of the reflection traveltime of a dipping reflector as parametrized in Figure 1. For this purpose, we consider a 2D seismic experiment being carried out over this medium along the x-axis. Sources and receivers are positioned at points \( x_1 = y - h \) and \( x_2 = y + h \), where \( y \) and \( h \) are the (varying) midpoint and half-offset coordinates (see again Figure 1). Then, the traveltime surface of the reflector in the data cube will be given by an expression of the form

\[
t_{\text{dif}} = \frac{1}{v} \left( \sqrt{(x-y+h)^2 + (mx+z_0)^2} + \sqrt{(x-y-h)^2 + (mx+z_0)^2} \right),
\]

(A-1)

where \( x \) denotes the horizontal coordinate of the reflector point. The actual traveltime can be determined by applying Fermat’s principle. It says that, of all the traveltimes described by the above equation, only those that are stationary will be observed. Thus, by setting to zero the derivative of the diffraction traveltime in equation A-1 with respect to \( x \), the reflection points can be determined as a function of \( y \) and \( h \):

\[
\frac{d t_{\text{dif}}}{d x} = \frac{1}{v} \left( \frac{\sqrt{(x-y+h)^2 + (mx+z_0)^2}}{\sqrt{(x-y-h)^2 + (mx+z_0)^2}} \right) = 0,
\]

(A-2)

which can be solved for \( x \) to yield

\[
x = \frac{(my+z_0)(y - mz_0) - mh^2}{(1+m^2)(my+z_0)}.
\]

(A-3)

Substitution of this result in the diffraction traveltime in equation A-1 yields, after some tedious algebraic manipulations, the final expression for the reflection traveltime of a dipping reflector as a function of \( y \) and \( h \):

\[
t_{\text{ref}} = \frac{2}{v} \sqrt{\frac{h^2 + (my+z_0)^2}{1+m^2}}.
\]

(A-4)

To simplify the expressions below, we introduce the following notations: The depth \( d \) of the reflector point vertically below the midpoint \( y \) (see Figure 1) is given by

\[
d = my + z_0,
\]

(A-5)

and the distance between this reflector point and source or receiver is

\[
r = \sqrt{d^2 + h^2}.
\]

(A-6)

With these notations, the horizontal coordinate (equation A-3) of the reflection point can be written as

\[
x = \frac{d(d - (1+m^2)z_0) - m^2h^2}{m(1+m^2)d},
\]

(A-7)

and the traveltime (equation A-4) takes the form

\[
t_{\text{ref}} = \frac{2}{v} \sqrt{\frac{r}{1+m^2}} = \frac{2}{v} r \cos \theta.
\]

(A-8)

Equation A-8 can be alternatively obtained from the application of a number of algebraic manipulations to the 2D version of the expression of Levin (1971).

Reflector image

The reflection event located at the traveltime (equation A-8) in the data is now to be migrated using the (wrong) migration velocity \( v_m \). Its position in a time-migrated section can be constructed as the envelope of the isochrons for all points on the reflection traveltime curve (equation A-4). Any of these isochrons is described by the formula

\[
t(x,y,h) = \frac{2b}{v_m} \sqrt{1 - \frac{(x-y)^2}{a^2}},
\]

(A-9)

where \( t \) is the vertical time \( t = 2z/v_m \), \( a = v_m t_{\text{dif}}/2 \), and \( b = \sqrt{a^2 - h^2} \). Moreover, \( x \) describes the horizontal coordinate of the image to be constructed.

Geophysicists are accustomed to interpreting equation A-9 geometrically in connection with common-offset migration, i.e., for constant \( h \). Then, this equation describes a lower half-ellipse with semi-axes \( a \) and \( b \) for each point \((y,t_{\text{dif}})\) on the reflection traveltime curve. The envelope of all these ellipses constructs the reflector image as a function of \( x \).

Here, we are interested in the position of the image in an image gather, i.e., for a constant \( y \). Therefore, we can assume, without loss of generality, that the coordinate origin is located at the position of the present image gather, i.e., \( x = 0 \). Under this condition, \( z_0 \) is the true depth of the reflector at the considered image point. The envelope of all isochrons (equation A-9) for all points \((y,t_{\text{dif}})\) on the reflection traveltime curve will then provide the reflector image in the image gather, i.e., as a function of \( h \).

Upon substitution of formula A-4 for \( t_{\text{ref}} \) in equation A-9, the family of isochrons reads

\[
r(0,y,h) = \frac{2}{v_m} \frac{pq}{\gamma \sqrt{1 + m^2}},
\]

(A-10)

where \( \gamma \) is the ratio between the migration and true medium velocities as defined by Al-Yahya (1989), i.e.,

\[
\gamma = \frac{v_m}{v}.
\]

(A-11)

Moreover, we have introduced the notations

\[
p = \sqrt{r^2 - (1+m^2)y^2}
\]

(A-12)

and

\[
q = \sqrt{r^2 - (1+m^2)h^2}.
\]

(A-13)

As mentioned above, the reflector image is given by the envelope of this family of isochrons. The envelope can be computed again by setting to zero the derivative of formula A-10 with respect to \( y \), solving the resulting equation for \( \gamma \), and substituting this result in equation A-10. In this way, the envelope condition reads

\[
\frac{dt}{dy} = \frac{(1+m^2)\gamma qr^2}{\gamma qr^2 \sqrt{1+m^2}} = 0.
\]

(A-14)
Unfortunately, this expression cannot be analytically solved for $y$. We therefore have to look for an approximate envelope. For this purpose, we use the Taylor series up to third order in $m$ instead of the true derivative $A-14$. Consequently, all following formulas are valid if $m^4 \ll 1$, that is, for reflector dips up to approximately 25°. Because the resulting Taylor series is quite large, we refrain from stating it here. It can be solved for $y$ to yield, up to third order in $m$,

$$y = \gamma^2 (h^2 + z_0^2) z_0^2 \left( \frac{m}{v} \right)^2 + \left( \frac{1}{2} \gamma^2 \frac{h^2}{z_0^2} + 2h^2 \right) \left( \frac{m^2}{v^2} \right) - \gamma^2 z_0 \left( h^2 + 3 \gamma^2 h^2 \right) \left( \frac{m^2}{v^2} \right).$$

(A-15)

Here, we have introduced yet another abbreviating notation:

$$z = z(h, \gamma) = \sqrt{(\gamma^2 - 1)h^2 + \gamma^2 z_0^2}.$$  

(A-16)

For a horizontal reflector, $z$ can be interpreted as an apparent depth as a function of offset and migration velocity.

Instead of developing equation $A-14$ into a Taylor series with respect to $m$, valid for small dips (i.e., $m \ll 1$), one could conceive of other possible approximations, such as small offsets (i.e., $h/2z_0 \ll 1$), midpoints close to the considered image point (i.e., $y/2z_0 \ll 1$), or even velocity ratios close to unity (i.e., $\gamma - 1 \ll 1$). Unfortunately, none of these Taylor series could be solved for the stationary value of $y$.

Substituting the stationary value $A-15$ in equation $A-10$ yields the approximation position of the reflector image in the image gather as a function of half-offset $h$:

$$t_{ig}(h) = \frac{2}{v_m} z + (1 - \gamma^2) \frac{h^2 + z_0^2 (h^2 - \gamma^2 z_0^2)}{v_m^2 z_0^2} \left( \frac{m}{v} \right)^2,$$

(A-17)

again up to third order in $m$.

Of course, the depth $z_0$ of the reflector at the image point is unknown at this stage. It has to be replaced by its vertical time coordinate, i.e., $z_0 = v_t/2 = v_d/2 \gamma$. This leads to the final expression for the position of the reflector image in the image gather as stated in equation 2.