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Interior point method for long-term generation scheduling of large-scale hydrothermal systems

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Abstract This paper presents an interior point method for the long-term generation scheduling of large-scale hydrothermal systems. The problem is formulated as a nonlinear programming one due to the nonlinear representation of hydropower production and thermal fuel cost functions. Sparsity exploitation techniques and an heuristic procedure for computing the interior point method search directions have been developed. Numerical tests in case studies with systems of different dimensions and inflow scenarios have been carried out in order to evaluate the proposed method. Three systems were tested, with the largest being the Brazilian hydropower system with 74 hydro plants distributed in several cascades. Results show that the proposed method is an efficient and robust tool for solving the long-term generation scheduling problem.

Keywords Hydrothermal generation scheduling · Long-term operational planning · Nonlinear optimization · Interior point method

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1 Introduction

The operational planning of hydrothermal power systems is designed to provide a reliable and economic operational policy that embraces time frames ranging from the long-term management of hydropower resources to the electric and hydraulic aspects of daily generation dispatch.

Due to the large scale and complexity of this problem, it has usually been approached by decomposition into two hierarchically connected problems, long-term generation scheduling (LTGS) and short-term generation scheduling (STGS).

LTGS covers a planning period of one or more years, divided into monthly intervals, with the first month being divided into weeks. The objective of this planning is to control the water storage in the reservoirs in order to maximize hydroelectric production and, as a result, minimize the cost of non-hydraulic generation sources. LTGS planning provides a weekly generation target for each hydro plant.

STGS covers the next week ahead, usually divided into hourly intervals. Its goal is to obtain a generation scheduling that matches the long-term generation targets for each hydro plant and optimizes the efficiency of hydropower conversion. This planning should comprise a detailed representation of hydro plant operational characteristics and the security operation of the electrical transmission system (Soares and Salmazo 1997).

The present paper deals with the modeling and resolution of the LTGS problem. This, by itself, is very difficult due to aspects such as the long-term horizon to be analyzed, the stochastic nature of water inflows, the operational interconnection between hydro plants in the same cascade, and the nonlinear nature of the functions of hydropower production and non-hydraulic generation cost.

To cope with the stochastic nature of the LTGS problem, one common approach is to consider the randomness of inflows by their probability distribution functions and apply classical optimization techniques based on stochastic dynamic programming (SDP). This approach requires some sort of simplification, however, due to the well known "curse of dimensionality" associated with SDP problems (Bellman 1962). This simplification can be in the representation of the multireservoir system using aggregated composite models (Araavanitidis and Rosing 1970; Saad et al. 1996; Turgeon 1980; Valdes et al. 1995) or in the solution procedure, such as the use of Bender's decomposition (Pereira and Pinto 1991), which linearizes the problem (Ponnambalam 2002).

Another common approach is based on the solution of a deterministic model within the framework of a scenario technique (Dembo 1991; Escudero et al. 1996; Nabona 1993). In this approach, the solution of the stochastic problem is obtained from the solutions of the deterministic model for a set of different scenarios representing the stochastic nature of inflows. The advantage of the scenario approach to the LTGS problem is that it does not require simplification for the handling of multireservoir systems, which is quite important when large-scale hydrothermal power systems are involved.

Thus, in the framework of a scenario technique for LTGS problems, a robust and efficient tool is crucial for solving the deterministic version of the problem since it must be applied several times for different inflow scenarios. Moreover, for the purpose of simulation, the decision process has to be repeated for each time interval in order to update the storage in the reservoirs in an open-loop feedback control framework (Martinez and Soares 2002).

Several techniques have been suggested for the solution of this problem, including nonlinear programming (Gagnon et al. 1974; Hanscom et al. 1980) and network flow approaches (Carvalho and Soares 1987; Lyra and Tavares 1988; Oliveira and Soares 1995; Rosenthal 1981; Sjvelgren et al. 1983). More recently, the option of using interior point methods (IPMs) which are especially efficient for solving large scale optimization problems, has arise.

The first IPM was developed by Dikin (1967). A later IPM for solving Linear Programming (LP) problems was developed by Karmarkar (1984), and results of its use presented by Adler et al. (1989). Since then, IPMs have undergone extensive development and are now being applied for the solution of many optimization problems. Mehrotra (1992) has proposed a predictor-corrector method for obtaining the search directions of an interior point primal-dual method which has proved to be the most efficient of the IPMs for LP.

More recently, IPM-based techniques have been applied in a variety of power systems problems, such as optimal active power dispatch (Oliveira et al. 2003), optimal reactive dispatch (Granville 1994; Torres and Quintana 1998), and security-constrained economic dispatch (Yan and Quintana 1997). A comprehensive survey of the application of IPM to power systems is provided by Quintana et al. (2000).

Research on IPMs for the linearized version of LTGS problem has also been published as reported in (Christoforidis et al. 1996; Ponnambalam et al. 1992; Medina et al. 1999). Ponnambalam et al. (1992) implemented a dual affine algorithm to solve a linear version of the LTGS problem, and Christoforidis et al. (1996) suggested a commercial code assuming a piecewise linear hydro production function. Medina et al. (1999) solved the problem by also assuming linear objective function and constraints. These IPMs have all solved the LTGS problem by simplifying the nonlinear nature of the problem in some way.

The present paper is concerned with the development of an efficient IPM for solving the deterministic version of the LTGS problem for large-scale hydrothermal systems. The problem is formulated precisely by considering nonlinear functions for hydroelectric generation and non-hydraulic operational costs. To overcome difficulties related to the precise representation of nonlinear objective function, a specialized IPM capable of exploiting the sparse structure of the problem and handling its Hessian indefiniteness efficiently is necessary. Thus, two improvements have been implemented: that of (Gondzio and Sarkissian 2003) has been expanded to handle the sparse pattern and nonlinear objective function, while a new heuristic procedure has been developed to handle Hessian indefiniteness.

The optimality of this IPM has been tested in three different case studies, and its performance evaluated. The first of these corresponds to a hydrothermal system composed of a single hydro plant in order to allow a detailed analysis of the optimal solution; the second corresponds to a hydrothermal system composed of 15 hydro plants in a single cascade, whereas the third involves the entire Brazilian hydropower system with 74 hydro plants distributed in several cascades. All these case studies considered a five-year planning horizon which is that adopted by the Brazilian Independent System Operator (ISO). The results, evaluated on the basis of the number of iterations and the computational time attest to the efficiency of the proposed method.

The outline of the paper is as follows. Notation and modeling of the LTGS problem are presented in Sects. 2 and 3. The proposed IPM is presented in Sect. 4 and test results are described in Sect. 5. Finally, conclusions are stated in Sect. 6.

2 Notation

t	Month index.
T	Number of months in the planning period.
i	Hydro plant index.
N	Number of hydro plants.

Ω_i Set of immediate upstream plants for plant i.
Ψ_t Operational cost (dollars).
H_t Total hydroelectric generation (MW).
h_{it} Hydroelectric generation function (MW).
D_t Load demand (MW).
r_{it} Water storage in reservoir (m^3).
$\underline{r}_{it}, \overline{r}_{it}$ Bounds on reservoir storage.
q_{it} Water discharge through turbines (m^3/s).
$\underline{q}_{it}, \overline{q}_{it}$ Bounds on water discharge.
v_{it} Water spillage from reservoir (m^3/s).
k_i Constant factor ($MW/m^3/s/m$).
ϕ_i Forebay elevation function (m).
θ_i Tailrace elevation function (m).
$\varsigma_i(q_{it})$ Penstock head loss function (m).
y_{it} Incremental water inflow (m^3/s).
Δt_t Number of seconds in a month.
n Double the number of all nonnegative variables.

3 Problem formulation

The LTGS problem can be formulated as the following nonlinear programming problem:

$$\text{Min} \sum_{t=1}^{T} \Psi_t \left(D_t - \sum_{i=1}^{N} h_{it} \right) \tag{1}$$

subject to

$$(r_{it} - r_{it-1})/\Delta t_t = \sum_{j\in\Omega_i} (q_{jt} + v_{jt}) - (q_{it} + v_{it}) + y_{it} \tag{2}$$

$$\underline{r}_{it} \le r_{it} \le \overline{r}_{it} \tag{3}$$

$$\underline{q}_{it} \le q_{it} \le \overline{q}_{it} \tag{4}$$

$$v_{it} \ge 0 \tag{5}$$

$$r_{i1},\ \forall i,\ i = 1, 2, \ldots, N,\ \text{given}$$

$$\forall t,\ t = 2, 3, \ldots, T,\ \forall i,\ i = 1, 2, \ldots, N \tag{6}$$

where

$$h_{it} = k_i \left(\phi_i \left(\frac{r_{it} + r_{it-1}}{2} \right) - \theta_i (q_{it} + v_{it}) - \varsigma_i (q_{it}) \right) q_{it}$$

The objective function (1) minimizes the operational cost Ψ_t which represents the minimal cost for complementary non-hydraulic sources, such as thermoelectric generation, imports from neighboring systems, and load shortage. The operational cost Ψ_t obtained from the optimal economic dispatch of these non-hydraulic sources results in a convex increasing operational cost function.

The definition of h_{it} represents a precise modeling of hydro generation as a function of net water head and water discharge through the turbines. The constant k_i depends on water density, gravity acceleration, and average turbine-generator efficiency. Forebay $\phi_i(\cdot)$ and tailrace elevations $\theta_i(\cdot)$ are represented by polynomial functions of storage and release variables, respectively, while forebay elevation $\phi_i(\cdot)$ is calculated on the basis of the average storage during a month. Tailrace elevation $\theta_i(\cdot)$ depends on discharge and spillage variables. Penstock head loss $\varsigma_i(\cdot)$ is a function of water discharge.

The equality constraints in (2) represent the water balance in the reservoir for each month. No time delay for water displacement is being considered in this formulation, since the problem is concerned with LTGS, which encompasses the time interval of a month.

Lower and upper bounds on variables, expressed by constraints (3)–(5), are imposed by the physical operational constraints of the hydro plant, as well as other constraints associated with multiple uses of water, such as irrigation, navigation, and flood control.

One important feature of model (1)–(6) is the precise representation of hydropower output by h_{it}. All nonlinear relations which influence the water head, such as forebay and tailrace elevations, and penstock head loss, were taken into consideration. Therefore, the main objective of LTGS, which is the optimal seasonal management of reservoir storage, can be met precisely, although this may not be assured by models assuming simplified hydropower output functions based on linear and/or piecewise linear approximations such as the IPMs proposed in the literature.

4 Solution technique

4.1 Mathematical formulation

Problem (1)–(6) has the following matricial formulation:

$$\begin{aligned} &\text{Min } f(x) \\ &\text{s.t.} \quad Ax = b \\ &\qquad \underline{x} \le x \le \overline{x} \end{aligned} \tag{7}$$

By introducing slack variable vector s and the transformation $x = \tilde{x} + \underline{x}$, the system (7) can be transformed into:

$$\begin{aligned} &\text{Min } f(x) \\ &\text{s.t.} \quad A\tilde{x} = b - A\underline{x} = \tilde{b} \\ &\qquad \tilde{x} + s = \overline{x} - \underline{x} = \tilde{\overline{x}} \\ &\qquad \tilde{x}, s \ge 0 \end{aligned} \tag{8}$$

The primal of the log-barrier problem (8) can now be written (with a "˜" omitted to provide a cleaner notation) as

$$\begin{aligned} &\text{Min } f(x) - \mu \sum_{i=1}^{n} (\ln s_i + \ln x_i) \\ &\text{s.t.} \quad Ax = b \\ &\qquad x + s = \overline{x} \end{aligned} \tag{9}$$

where μ is a decreasing sequence of positive barrier parameters such that $\lim_{k\to\infty} \mu_k = 0$. The computation of this value is explained in Sect. 4.6. The Lagrangian function associated with (7) is:

$$\mathscr{L} \equiv f(x) + y^t(b - Ax) + w^t(x + s - \overline{x}) - \mu \sum_{i=1}^{n} (\ln s_i + \ln x_i)$$

The following KKT equations can thus be derived:

$$K_\mu(x, y, w, s) \equiv \begin{bmatrix} \mathscr{L}_x \\ \mathscr{L}_y \\ \mathscr{L}_w \\ \mathscr{L}_s \end{bmatrix} = \begin{bmatrix} \nabla f(x) - A^t y + w - \mu X^{-1} e \\ b - Ax \\ x + s - \overline{x} \\ SWe - \mu e \end{bmatrix} = 0$$

where uppercase letters stand for diagonal matrices with entries being the elements of the corresponding lowercase-letter vectors, and $e = (1, 1, \dots, 1)^t$.

Letting $Ze = \mu X^{-1} e$ and defining $v \equiv [x, y, w, z, s]^t$, the Newton direction is obtained by solving the following linear system:

$$K'_\mu(v)\Delta v = -K_\mu(v)$$

$$\begin{bmatrix} -H(x) & A^t & -I & I & 0 \\ A & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & I \\ 0 & 0 & S & 0 & W \\ Z & 0 & 0 & X & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta w \\ \Delta z \\ \Delta s \end{bmatrix} = \begin{bmatrix} r_d \\ r_p \\ r_a \\ r_b \\ r_c \end{bmatrix} \tag{10}$$

where $K_\mu(v) = [-\mathscr{L}_x, \mathscr{L}_y, \mathscr{L}_w, \mathscr{L}_s, \mathscr{L}_z]^t$, $\mathscr{L}_z = XZe - \mu e$, $H(x) = \nabla^2 f(x)$, $r_d = \nabla f(x) - A^t y + w - z$, $r_p = Ax - b$, $r_a = \overline{x} - x - s$, $r_b = \mu e - SWe$ and $r_c = \mu e - XZe$.

To solve the problem, system (10) is reduced by making a number of substitutions, leading to the following normal equation system:

$$(AD^{-1}A^t)\Delta y = r_p + AD^{-1} r_k \tag{11}$$

where $D = H(x) + S^{-1}W + X^{-1}Z$ and $r_k = r_d + S^{-1}(r_b - W r_a) - X^{-1} r_c$.

Matrix $H(x)$ may be indefinite, but it can be transformed to a quasidefinite one, i.e., a matrix that is strongly factorizable and to which a Cholesky-like factorization LDL^t can be applied. For this purpose the procedure described by (Altman and Gondzio 1999) and (Benson et al. 2000) is employed. For interior-point methods, the initial point is interior (nonnegative variables are kept strictly positive) and the step is controlled so that the sequence of points remain interior.

A summary of the necessary calculations for the IPM proposed is provided in Table 1. Some aspects of this IPM, such as the exploitation of sparsity, IPM improvement, initial point, step length, computing μ and stopping criteria are discussed and detailed in Sects. 4.2, 4.3, 4.4, 4.5, 4.6 and 4.7, respectively.

4.2 Exploiting sparsity

Since the major burden for IPM stems from the solution of normal equations (11), its efficiency depends on the exploitation of the sparse pattern of matrices A and D. Let

$$x = \begin{bmatrix} r & q & v \end{bmatrix}^t$$

Table 1 Primal-Dual IPM for LTGS

IPM Direction Computation

Given $(x^0, s^0, w^0, z^0) > 0$, y^0 free and $\tau \in (0, 1)$;

For $k = 0, 1, \dots$

$\mu^k = \sigma \gamma^k / n$,

where: n is twice the number of all nonnegative variables,

γ^k is the GAP and γ^k/n is the mean GAP.

$r_d^k = \nabla f(x^k) - A^t y^k + w^k - z^k$

$r_p^k = b - Ax^k$

$r_a^k = \overline{x} - x^k - s^k$

$r_b^k = \mu^k e - S^k W^k e$

$r_c^k = \mu^k e - X^k Z^k e$

$r_k^k = r_d^k + S^{-1}(r_b^k - W r_a^k) - (X^k)^{-1} r_c^k$

$D^k = (H(x^k) + (S^k)^{-1} W^k + (X^k)^{-1} Z^k)$

$\Delta y^k = (A(D^k)^{-1} A^t)^{-1} (r_p^k + A(D^k)^{-1} r_k^k)$

$\Delta x^k = (D^k)^{-1} (A^t \Delta y^k - r_k^k)$

$\Delta s^k = r_a^k - \Delta x^k$

$\Delta w^k = (S^k)^{-1} (r_b^k - W^k \Delta s^k)$

$\Delta z^k = (X^k)^{-1} (r_c^k - Z^k \Delta x^k)$

$\alpha_p^k = \text{Min}\left\{ \underset{\partial x_i^k < 0}{\text{Min}} \left(\frac{-x_i^k}{\partial x_i^k} \right), \underset{\partial s_i^k < 0}{\text{Min}} \left(\frac{-s_i^k}{\partial s_i^k} \right) \right\}$

$\alpha_d^k = \text{Min}\left\{ \underset{\partial w_i^k < 0}{\text{Min}} \left(\frac{-w_i^k}{\partial w_i^k} \right), \underset{\partial z_i^k < 0}{\text{Min}} \left(\frac{-z_i^k}{\partial z_i^k} \right) \right\}$

$y^{k+1} = y^k + \alpha_d^k \Delta y^k$

$x^{k+1} = x^k + \alpha_p^k \Delta x^k$

$s^{k+1} = s^k + \alpha_p^k \Delta s^k$

$w^{k+1} = w^k + \alpha_d^k \Delta w^k$

$z^{k+1} = z^k + \alpha_d^k \Delta z^k$

$k \leftarrow k + 1$

Until convergence

then matrices A and D have the following structure

$$A = \begin{bmatrix} \tilde{A} & S & S \end{bmatrix}$$

and

$$D = \begin{bmatrix} D_1 & D_4 & 0 \\ D_4 & D_2 & D_5 \\ 0 & D_5 & D_3 \end{bmatrix}$$

where

$$\tilde{A} = \begin{bmatrix} B_1 & 0 & 0 & \cdots & 0 & 0 \\ -B_2 & B_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B_{T-1} & 0 \\ 0 & 0 & 0 & \cdots & -B_T & B_T \end{bmatrix}$$

$$S = \begin{bmatrix} M & 0 & \cdots & 0 & 0 \\ 0 & M & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M & 0 \\ 0 & 0 & \cdots & 0 & M \end{bmatrix}$$

and

$$D_i = \begin{bmatrix} D_{i(1)} & 0 & \cdots & 0 & 0 \\ 0 & D_{i(2)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & D_{i(T-1)} & 0 \\ 0 & 0 & \cdots & 0 & D_{i(T)} \end{bmatrix}$$

where B_t is an $N \times N$ diagonal matrix for interval t; M is the $N \times N$ network incidence matrix for q or v variables; and $D_{i(t)}$ is an $N \times N$ diagonal matrix obtained from $H(x) + S^{-1}W + X^{-1}Z$.

The approach described in (Gondzio and Sarkissian 2003) has been adapted to LTGS, because IPM can exploit A and D sparse patterns without a new reducing process for the normal equations (11). Only the redefinition of linear algebraic operations is required for the computation of the search direction in terms of the sparse pattern. The two most important computations for the LTGS problem are $\Phi = AD^{-1}A^t$ and its implicit inverse representation LL^t:

$$\Phi = \begin{bmatrix} \Phi_1 & C_1^t & 0 & \cdots & 0 & 0 \\ C_1 & \Phi_2 & C_2^t & \cdots & 0 & 0 \\ 0 & C_2^t & \Phi_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Phi_{T-1} & C_{T-1}^t \\ 0 & 0 & 0 & \cdots & C_{T-1} & \Phi_T \end{bmatrix}$$

which produces the following implicit inverse representation factor:

$$L = \begin{bmatrix} L_1 & 0 & 0 & \cdots & 0 & 0 \\ L_{n,1} & L_2 & 0 & \cdots & 0 & 0 \\ 0 & L_{n,2} & L_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & L_{T-1} & 0 \\ 0 & 0 & 0 & \cdots & L_{n,T-1} & L_T \end{bmatrix}$$

The corresponding linear system $\Phi x = b$ can be solved by computing L using the following relations:

$$\begin{cases} L_1 L_1^t = \Phi_1 \\ L_{n,i} L_i^t = C_i, \quad \forall i = 1, \dots, T \\ L_{n,i-1} L_{n,i-1}^t + L_i L_i^t = \Phi_i, \quad \forall i = 2, \dots, T \end{cases}$$

Given L, $\Phi x = b$ can be solved by:

$$\begin{cases} z_1 = L_1^{-1} b_1 \\ z_i = L_i^{-1}(b_i - (C_{i-1} L_{i-1}^{-t}) z_{i-1}), \quad i = 2, \dots, T \\ x_n = L_n^{-t} z_n \\ x_i = L_i^{-t}(z_i - (L_i^{-1} C_i^t) x_{i+1}), \quad i = 1, \dots, (T-1) \end{cases}$$

It is important to note that $L_{n,i}$ do not need to be computed, resulting in what is known as implicit factorization.

4.3 IPM improvement

One important question related to the non-linearity of function f is that there is no guarantee that the associated Hessian H is always positively defined. A new heuristic procedure has thus been developed using the ideas proposed by Altman and Gondzio (1999). Essentially, the procedure can be seen as a modification of the heuristic procedure of $H + \lambda I$ for Δy direction computation in (11). Also note that the D matrix sparse pattern will have a great influence in the time spent in the resolution of (11), and is directly related to matrix H, which has the following sparse structure:

$$H = \begin{bmatrix} H_{rr} & H_{rq} & 0 \\ H_{qr} & H_{qq} & H_{qv} \\ 0 & H_{vq} & H_{vv} \end{bmatrix}$$

where all submatrices are diagonal. Inspired by the quasi-Newton approaches for nonlinear programming problems, the H matrix can be simplified as:

$$\tilde{H} = \begin{bmatrix} H_{rr} & 0 & 0 \\ 0 & H_{qq} & 0 \\ 0 & 0 & H_{vv} \end{bmatrix}$$

This simplification involves one implicit assumption and two possible interpretations: one physical and the other mathematical. The physical interpretation is that the variable r has two second-order terms $\frac{\partial^2 f}{\partial r_i^2}$ and $\frac{\partial^2 f}{\partial r_i q_i}$, with the contribution of the second term being comparatively smaller than the first so that it can be neglected. This same assumption holds for the variables q and v.

The mathematical interpretation of $\tilde{H}$ is that at any point x a good quadratic approximation for the objective function f can be constructed with $\tilde{f} = x^t \tilde{H} x$. Some benefits of this approach include the fact that the Eigenvalues of $\tilde{H}$ are obtained directly, and even an indefinite matrix problem is quickly solved. This is important since there are no guarantee of the convexity of function f. The solution of this problem involves a modification of the procedure used for indefinite matrices presented by Altman and Gondzio (1999), which has been implemented for $\tilde{H}$. This modified procedure consists of the determination of the negative Eigenvalue with the highest module $|\lambda_m|$, then the negative Eigenvalues λ_i, including the one with λ_m, will be recalculated as $\lambda_i = \lambda_i + 1.01|\lambda_m|$.

The objective of this procedure is to conjugate two desirable properties for $\tilde{h}$ in the computation of the direction of Δy:

- Direction Projection: only components associated with positive Eigenvalues will be updated by the original directions proposed by the interior-point method. This will minimize the deviation from the original implicit reduction proposed by the interior point method for the direction Δy.
- Sufficient Decrease: Although there is no guarantee of convexity for f, the alteration of a component with a negative Eigenvalue will always produce a positive definite $\tilde{H}$ for each specified iteration point and Δy can always be computed.

The implicit quadratic approximation based on a consideration of only the diagonal matrices of H can be seen as a quasi-Newton procedure for the computation of the direction Δy. The procedure to preserve the components associated with positive Eigenvalues can be seen as a kind of projection for the original directions given by the IPM.

Another important observation is that the approximation procedures have been effected only for the computation of Hessian matrix H. No such approximations are necessary for the computation of the gradient. Although a predictor-corrector IPM was also implemented, preliminary results have shown that the computational improvements are not as effective in this case.

4.4 Initial point

In this paper, the approach described by Medina et al. (1999) was adopted for the initial point:

$$\begin{aligned}
\tilde{x} &= A^t(AA^t)^{-1}b \\
x_j^0 &= \max\{\tilde{x}_j, \varepsilon_1\} \\
s_j^0 &= \max\{\varepsilon_1, \overline{x}_j - x_j^0\} \\
y^0 &= 0 \\
z_j &= c_j + \varepsilon_2 \quad \text{and} \quad w_j = \varepsilon_2 \quad \text{if } c_j \geq \varepsilon_2 \\
z_j &= -c_j \quad \text{and} \quad w_j = -2c_j \quad \text{if } c_j < -\varepsilon_2 \\
z_j &= c_j + \varepsilon_2 \quad \text{and} \quad w_j = \varepsilon_2 \quad \text{if } 0 \leq c_j \leq \varepsilon_2 \\
z_j &= \varepsilon_2 \quad \text{and} \quad w_j = -c_j + \varepsilon_2 \quad \text{if } -\varepsilon_2 \leq c_j \leq 0
\end{aligned}$$

where $c = \nabla f(x)$ and ε_1, ε_2, ε_3 and ε_4 are defined as

$$\begin{aligned}
\varepsilon_1 &= \max\left\{-\min_{1\leq j\leq n} \tilde{x}_j, \varepsilon_3, \frac{\|b\|}{100}\right\} \\
\varepsilon_2 &= 1 + \varepsilon_4\|c\| \\
\varepsilon_3 &= 100 \\
\varepsilon_4 &= 10
\end{aligned}$$

4.5 Step length

The step length is determined based on the constraints on the nonnegative variables:

$$\hat{\alpha}_p^k = \text{Min}\left\{ \underset{\partial x_i^k<0}{\text{Min}} \left(\frac{-x_i^k}{\partial x_i^k} \right), \underset{\partial s_i^k<0}{\text{Min}} \left(\frac{-s_i^k}{\partial s_i^k} \right) \right\}$$

$$\hat{\alpha}_d^k = \text{Min}\left\{ \underset{\partial w_i^k<0}{\text{Min}} \left(\frac{-w_i^k}{\partial w_i^k} \right), \underset{\partial z_i^k<0}{\text{Min}} \left(\frac{-z_i^k}{\partial z_i^k} \right) \right\}$$

In order to avoid the boundary, the values of $\hat{\alpha}_p^k$ and $\hat{\alpha}_d^k$ are multiplied by a parameter smaller than one to obtain the actual primal and dual step sizes, i.e., $\alpha_p^k = 0.995\hat{\alpha}_p^k$ and $\alpha_d^k = 0.995\hat{\alpha}_d^k$.

4.6 Computation of μ

Once the primal-dual search directions have been employed, the formula for the estimation of μ is provided by Wright (1996):

$$\mu = \frac{x^t z + s^t w}{n} \tag{12}$$

where n is twice the number of all primal nonnegative variables.

4.7 Stopping criteria

The method stops when primal and dual feasibility have been obtained within a given tolerance. The satisfaction of duality gap and dual infeasibility conditions are measured by:

$$\begin{bmatrix} \frac{|x^t z + s^t w|}{1+|b^t y + c^t x + u^t w|} \\ \frac{|\nabla f(x) - A^t y + w - z|}{1+|\nabla f|} \end{bmatrix} \le \varepsilon_1$$

Moreover the satisfaction of the primal infeasibility condition is measured by:

$$\begin{bmatrix} \frac{|Ax-b|}{1+|b|} \\ \frac{|x+s-\overline{x}|}{|\overline{x}|} \end{bmatrix} \le \varepsilon_2$$

where $\varepsilon_1 = 10^{-3}$ and $\varepsilon_2 = 10^{-8}$.

5 Numerical results

5.1 Proposed IPM solutions

The method proposed here was tested for three cases studies differing in relation to the number of hydro plants involved. The first case study considers a single reservoir hydro plant in order to allow a detailed analysis of the optimal solution. The Furnas hydro plant, with an installed capacity of 1,312 MW, was selected since it is the largest reservoir in the

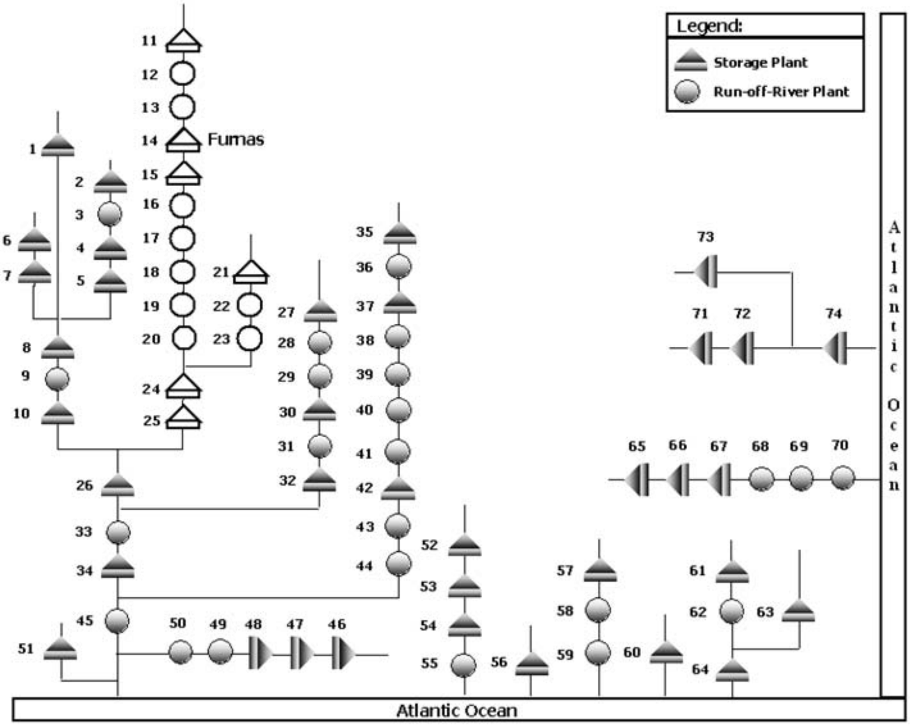


Fig. 1 Hydro plants of the Brazilian hydropower system

upper part of the Grande River cascade, which is one of the most important in the Brazilian hydropower system. The second case study considers the Grand River cascade system as a whole (including the Pardo River Cascade). This system is composed of 15 hydro plants (9 run-off-river plants and 6 associated with reservoirs) and represents an installed capacity of 7,816 MW. The third case study corresponds to the entire Brazilian hydropower system, comprised of 74 hydro plants (32 run-off-river plants and 42 associated with reservoirs) with an installed capacity of 65,667 MW.

For each case study, a five years horizon with two different inflow scenarios was considered. The five-year horizon is what is adopted by the Brazilian ISO. The studies adopt a constant load demand equal to the installed capacity of the specific hydropower system, thus assuring more or less the same hydro and thermal proportions for each case study. The month of May, the beginning of the hydrological year for the Brazilian system, was adopted as the beginning of the planning horizon, and initial and final reservoir storages were set for the maximum. The operational cost function used was a quadratic function adjusted to the economic dispatch of Brazilian non-hydraulic sources (thermal generation, import from neighboring systems, load shortage), given by:

$$\Psi_t = 0.04\left(D_t - \sum_{i=1}^{N} h_{it}\right)^2 \tag{13}$$

Figure 1 illustrates the topology of the Brazilian hydropower system and Tables 2 and 3 present some of the characteristics of these plants. The Furnas hydro plant is represented by

Table 2 Brazilian hydro plants with reservoirs

Number	Power capacity (MW)	Storage capacity (hm^3)	Discharge capacity (m^3/s)
1	1192	17724	893
2	510	12792	532
4	240	239	495
5	210	878	483
6	143	3680	240
7	375	1496	545
8	2280	17027	2748
10	1710	12584	2277
11	46	792	200
14	1312	22950	1516
15	478	4039	1235
21	80	554	87
24	1488	6150	2637
25	1398	11027	2492
26	3444	21046	8421
27	140	3135	703
30	264	7407	1158
32	807	13370	1975
34	1540	20001	7977
35	97	7010	348
37	414	8795	630
42	608	10550	1322
46	1676	5779	1173
47	1260	2950	1081
48	1332	6775	1262
51	210	7352	372
52	288	2442	224
53	690	4971	440
54	1140	3340	1118
56	220	1589	93
57	140	3646	370
60	260	179	35
61	1200	54400	1033
63	1087	1850	3850
64	4000	45500	5963
65	396	19528	827
66	1050	34116	3833
67	1500	10782	2820
71	86	4732	117
72	50	436	63
73	28	1236	60
74	222	888	338

Table 3 Brazilian Run-off-river plants

Number	Power capacity (MW)	Discharge capacity (m^3/s)
3	390	596
9	638	2275
12	52	214
13	180	466
16	1104	1817
17	424	964
18	210	1376
19	380	1419
20	328	1781
22	108	134
23	32	161
28	144	144
29	132	655
31	347	9228
33	1551	7476
36	80	336
38	43	464
39	73	515
40	81	633
41	84	619
43	554	2423
44	372	2483
45	12600	10229
49	1050	1559
50	1240	1791
55	1450	1356
58	140	370
59	180	217
62	902	3046
68	400	1971
69	1423	1770
70	3000	2418

number 14, the Grand River cascade (including the Pardo River cascade) is represented by the plants numbered 11–25. Information about the Brazilian hydropower system considered here can be obtained at http://www.ccee.org.br/precos/downloads/index.jsp.

Two five-year inflow periods were selected from the historical records, the driest one from 1952 to 1957, hereafter called the "dry period", and the wettest one from 1980 to 1985, hereafter called the "wet period".

The optimal solution for the dry period of the first case study is shown in Figs. 2, 3, and 4, corresponding to the release (discharge plus spillage), storage and generation trajectories, respectively.

Fig. 2 Release trajectory of Furnas hydro plant during the dry period

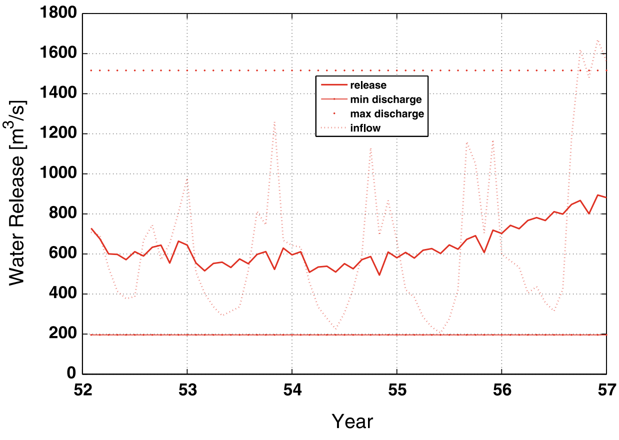


Fig. 3 Storage trajectory of Furnas hydro plant during the dry period

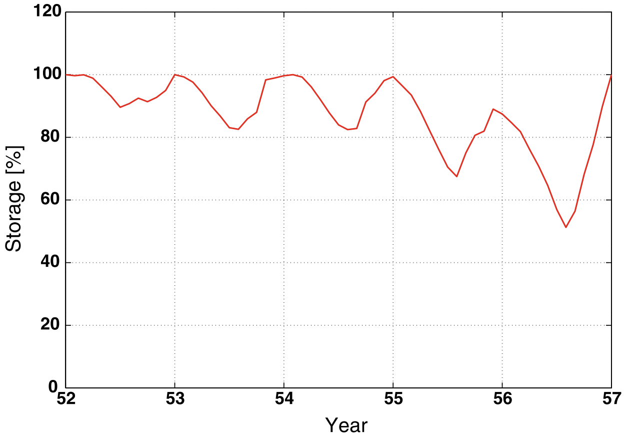
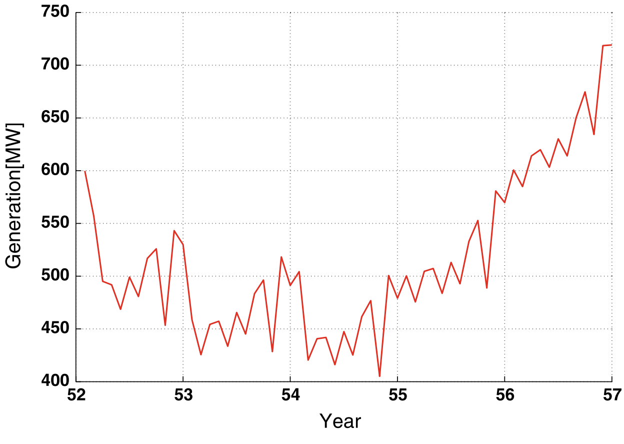


Fig. 4 Generation trajectory of Furnas hydro plant during the dry period



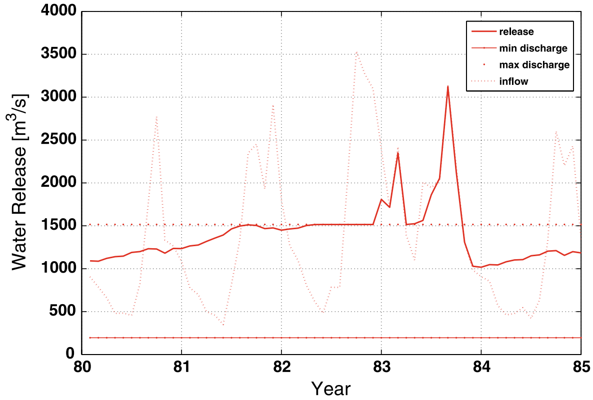


Fig. 5 Release trajectory of Furnas hydro plant during the wet period

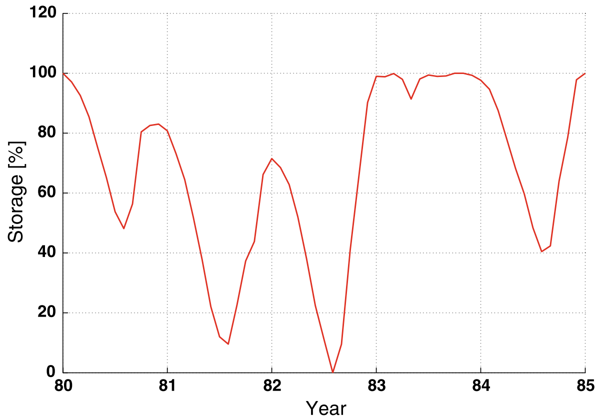


Fig. 6 Storage trajectory of Furnas hydro plant during the wet period

The optimality of the solution can be confirmed by the following features:

1. The release trajectory in Fig. 2 is always below the maximum discharge, indicating the total avoidance of spillage.
2. The storage trajectory in Fig. 3 reaches the maximum almost every year at the beginning of the dry season (May for this system), which means that water head and hydropower conversion efficiency are maximized.
3. Storage varies throughout the year due to the seasonal behavior of inflows and the more stable behavior of releases due to the quadratic nature of the objective function with respect to hydropower generation.
4. The generation trajectory in Fig. 4 presents an overall shape similar to the release trajectory, although with greater oscillation, due to variations in storage and water head.

Figures 5, 6 and 7 show the optimal solution for case study 1 during the wet period. Here again, various features of optimality can be identified:

1. Even discharging at maximum level during the years of 1982 and 1983, spillage could not be completely avoided, as shown in Fig. 5.
2. Spillage only occurs when storage and discharge are at the limits, as shown in Figs. 5 and 6.

Fig. 7 Generation trajectory of Furnas hydro plant during the wet period

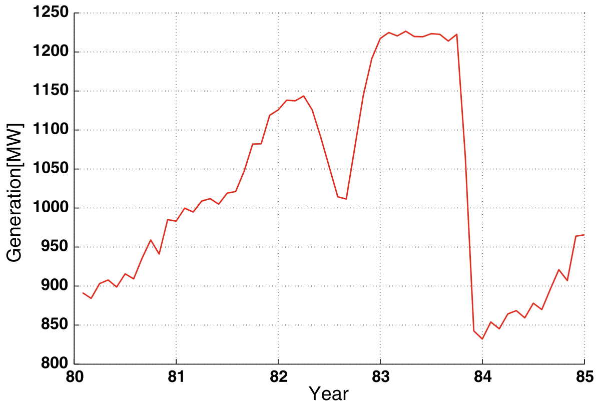
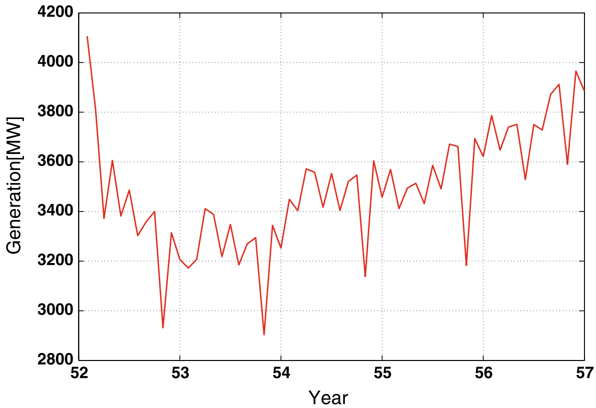


Fig. 8 Generation trajectory of Grande River hydro system during the dry period



3. Storage is reduced to the minimum in November 1982 in order to minimize future spillages, as shown in Fig. 6.
4. The generation trajectory in Fig. 7 increases until May 1983, drops in November 1982, and is then maintained close to the maximum until another even steeper drop close to May 1984.

Figures 8 and 9 show the generation trajectories for the second case study during the wet and dry periods. The shapes of the curves here are similar to those of the curves for the first case study, which reflects the fact that the characteristics of the Furnas hydro plant mirror those of the cascade system where it is located.

Figures 10 and 11 show the generation trajectories of the third case study also during the wet and dry periods. Again, the resultant curves are similar to those in the previous case studies, indicative of the importance of the Grande River cascade in the Brazilian Hydropower System.

All the test results were obtained with the proposed IPM implemented in Matlab 7.0 running on an 2.0 GHz Intel Pentium PC with 1 Giga of RAM equipped with Windows XP Professional software.

The computation time for all of these case studies is shown in Table 4, as well as the number of iterations necessary. In general, the computation time and the number of iterations

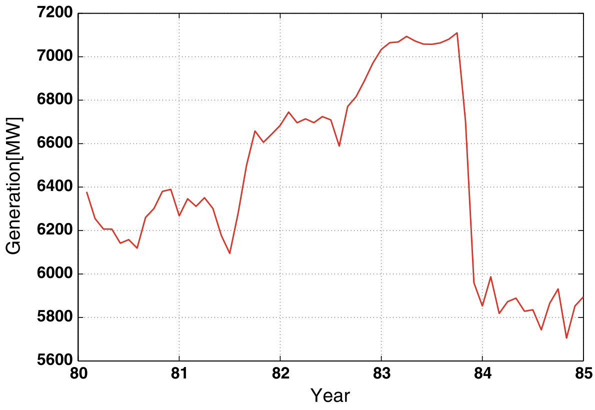


Fig. 9 Generation trajectory of Grande River hydro system during the wet period

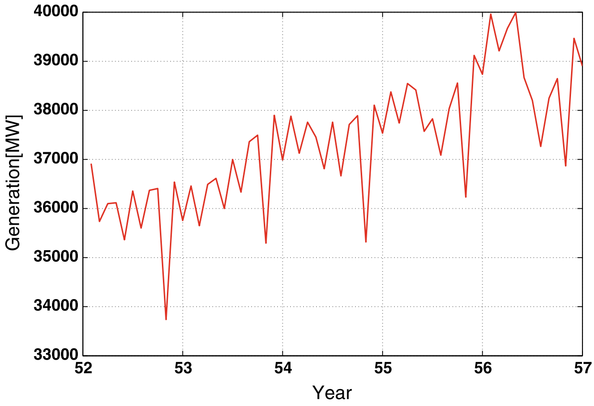


Fig. 10 Generation trajectory of the Brazilian Hydropower System during the dry period

Fig. 11 Generation trajectory of the Brazilian Hydropower System during the wet period

for the wet period are greater than for dry period, since optimal solutions involving spillage are more difficult to establish than those without it.

Table 5 shows the number of hydro plants considered in each case study and the corresponding hydro scheduling problem dimensions.

Table 4 IPM performance

Case study	Period	Time (s)	Iterations
1	dry	1.94	18
	wet	2.13	19
2	dry	6.81	27
	wet	8.23	31
3	dry	157.28	92
	wet	147.70	85

Table 5 Case study information

Case study	Hydro plants	Variables	Equality constraints	Installed capacity [MW]
1	15	180	60	1,312
2	11–26	2,700	900	7,816
3	1–74	13,320	4,440	65,667

Computational tests indicate significant IPM differences in CPU time depending on how the sub-matrices from matrix D are dealt with (Sect. 4.2). If only the sub-matrices D_1, D_2 and D_3 are evaluated, the same optimal solutions are obtained, but in a much shorter computational time than if all sub-matrices are considered.

For the third case study, which corresponds to the entire Brazilian hydropower system, with its 74 hydro plants and 42 reservoirs, the resulting problem has 13,320 variables, 4,400 equality constraints and 22,200 inequality constraints. Even in this case LTGS solution was reached in less than 3 minutes using Matlab, although with other programming languages such as C or FORTRAN this time can be reduced even further.

5.2 Comparison with other approaches

To better understand the contribution of the proposed IPM, the results in Sect. 5.1 are compared with two other approaches: a nonlinear method and a linear one. The nonlinear method corresponds to the Fmincon Matlab function which is based on a sequential quadratic programming (SQP) approach. In this method, a quadratic programming (QP) subproblem is solved at each iteration. An estimate of the Hessian of the Lagrangian is updated at each iteration using the BFGS formula (see references Fletcher and Powell 1963 and Goldfarb 1970). A line search is performed using a merit function similar to that proposed by Fletcher (1970). The QP subproblem is solved using an active set strategy similar to that described in (Coleman and Li 1994).

The linear method corresponds to the use of a linear programming package to solve the linearized version of the nonlinear problem, as described in detail by (Barros et al. 2003). For this approach, the Linprog Matlab command in large-scale mode, that is based on LIPSOL (Linear Interior Point Solver, Zhang 1995), a variant of Mehrotra's predictor-corrector algorithm (Mehrotra 1992), was adopted. In order to linearize the nonlinear function (1) the hydro generation function h_{it} is given as follows:

$$h_{it} = \xi_{it} q_{it} \tag{14}$$

Table 6 Comparison of methods for Furnas hydro plant in the dry and wet periods

Period	Method	Time (s)	Iterations	Objective function
dry	Linprog	0.19	14	1.436×10^8
	Fmincon	535.88	219	1.018×10^8
	IPM	1.94	17	1.018×10^8
wet	Linprog	0.14	12	3.309×10^7
	Fmincon	474.81	219	1.585×10^7
	IPM	1.69	18	1.585×10^7

The productivity ξ_{it} is now considered a fixed parameter instead of being a function of the decision variables, as considered in the formulation (1)–(6) where:

$$\xi_{it} = k_i \left(\phi_i \left(\frac{r_{it} + r_{it-1}}{2} \right) - \theta_i (q_{it} + v_{it}) - \varsigma_i (q_{it}) \right) \tag{15}$$

This linearization procedure has been adopted by (Barros et al. 2003; Medina et al. 1999; Ponnambalam et al. 1992). In (Barros et al. 2003) the average productivity obtained from the long-term operational records has been used as the fixed value of ξ_{it}. Here, the average productivity is calculated using the nonlinear solution given by the proposed IPM for each of the case studies.

The next step in the linearization procedure of the LTGS model consists in the use of the following objective function:

$$\text{Min} \sum_{t=1}^{T} \left| D_t - \sum_{i=1}^{N} \xi_{it} q_{it} \right| \tag{16}$$

which is equivalent to:

$$\underset{\gamma_t \geq 0}{\text{Min}} \sum_{t=1}^{T} \gamma_t \tag{17}$$

subject to

$$-\gamma_t \leq D_t - \sum_{i=1}^{N} \xi_{it} q_{it} \leq \gamma_t \tag{18}$$

Equations (17)–(18) together with (2)–(6) correspond to the linearized model.

Table 6 shows the results for the case study with Furnas hydro plant using the linear model (Linprog), the nonlinear model (Fmincon), and the proposed IPM. It can be seen that the proposed IPM achieves the same final value of the objective function achieved by the Fmincon method in a much faster way (280 times). One of the reasons for this savings in CPU time is the sparsity exploitation specially developed for the LTGS problem by the proposed IPM, as detailed in Sect. 4.2.

Another important result from Table 6 is that the linear model gave a quite worst final objective function value compared with the nonlinear models. This difference shows that the linearizations of the hydropower generation function, done by (14), and the objective function, done by (16), lead to a great simplification of the model. Indeed, as shown in

Fig. 12 Linear and nonlinear storage trajectories for Furnas hydroplant during the dry period

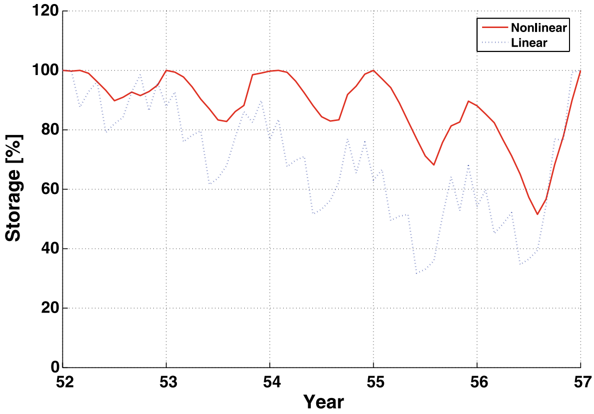


Fig. 13 Linear and nonlinear generation trajectories for Furnas hydroplant during the dry period

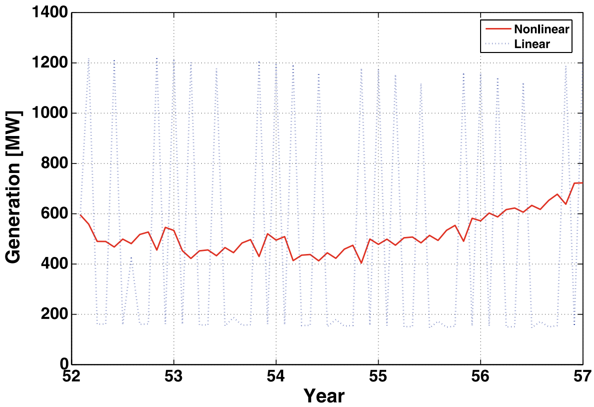
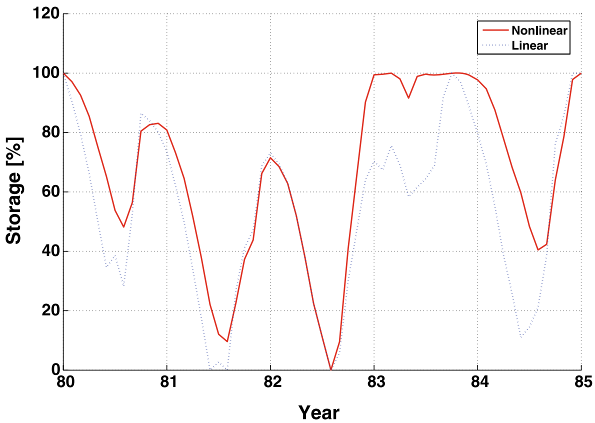


Fig. 14 Linear and nonlinear storage trajectories for Furnas hydroplant during the wet period



Figs. 12, 13, 14 and 15, the linear model provides a solution which produces lower storage trajectories and much more unstable generation trajectories. With lower storages, the hydro plant operates with lower productivity, producing less hydropower generation for the same

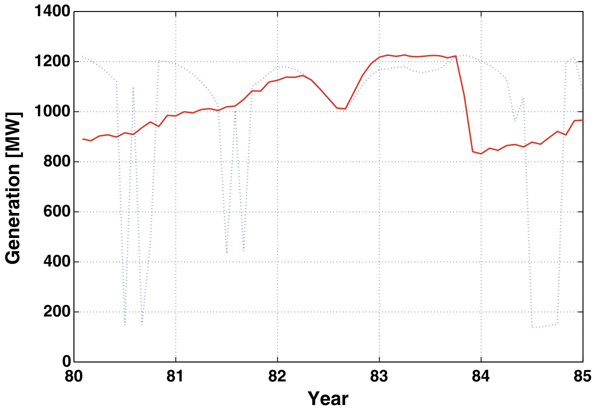


Fig. 15 Linear and nonlinear generation trajectories for Furnas hydroplant during the wet period

Table 7 Comparison of methods for the Grande River cascade during the dry and wet periods

Period	Method	Time (s)	Iterations	Objective function
dry	Linprog	1.28	21	4.071×10^9
	Fmincon	–	–	–
	IPM	7.55	28	2.993×10^9
wet	Linprog	5.07	78	4.759×10^8
	Fmincon	–	–	–
	IPM	7.25	29	3.363×10^8

water discharge. Moreover, with more unstable generation, the monotonically increasing objective function of (1) will provide higher values.

The linear and nonlinear models have also been compared for the Grande River cascade case study and the results are presented in Table 7. For this case study, the Fmincon method was not able to solve the nonlinear model due to the fact that this method do not exploit the sparsity as the developed IPM. As a consequence, the Fmincon, which requires a large amount of memory, cannot be used. As shown in Table 7, the proposed IPM provides better solutions than Linprog in a small additional amount of computational time. Similar to the case of Furnas hydro plant, the differences on objective function values are due to differences on reservoir and generation trajectories. Figures 16 and 17 show the differences on generation trajectories between the linear and nonlinear models.

Finally, Table 8 presents the results for the Brazilian hydropower system. Here again the same behavior of the Grand river cascade case is observed. The differences on the objective function values are due to differences on operation as shown in Figs. 18 and 19.

The comparisons with other approaches have shown that linear models cannot adequately cope with the nonlinear characteristics of the LTGS problem since they provide higher operating costs due to lower reservoir trajectories (lower productivity) and more unstable generation trajectories. On the other hand, nonlinear approaches would require an adequate sparsity exploitation in order to cope with large-scale hydrothermal systems.

Fig. 16 Linear and nonlinear generation trajectories for the Grande River cascade during the dry period

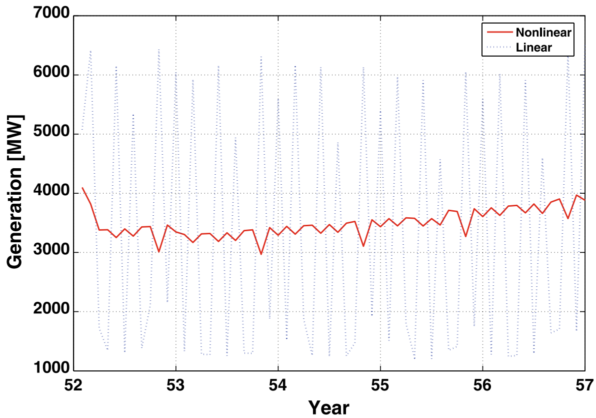


Fig. 17 Linear and nonlinear generation trajectories for the Grande River cascade during the wet period

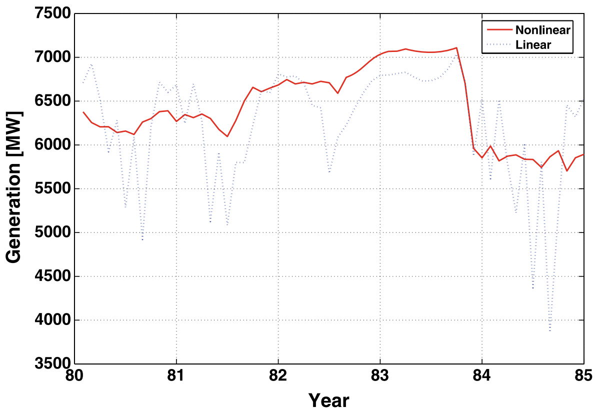


Table 8 Comparison of methods for the Brazilian hydropower system during the dry and wet periods

Period	Method	Time (s)	Iterations	Objective function
dry	Linprog	31.48	76	1.584×10^{11}
	Fmincon	–	–	–
	IPM	140.91	92	1.283×10^{11}
wet	Linprog	41.31	101	4.631×10^{10}
	Fmincon	–	–	–
	IPM	147.03	86	3.842×10^{10}

6 Conclusion

This paper has presented an interior point method for long-term generation scheduling of large-scale hydrothermal systems. The problem was formulated and solved using an interior point method as a nonlinear programming model which allows precise representation of hydropower output and operational cost functions.

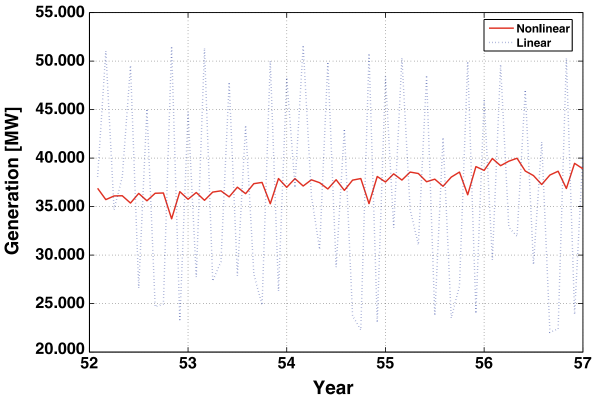


Fig. 18 Linear and nonlinear generation trajectories for the Brazilian hydropower system during the dry period

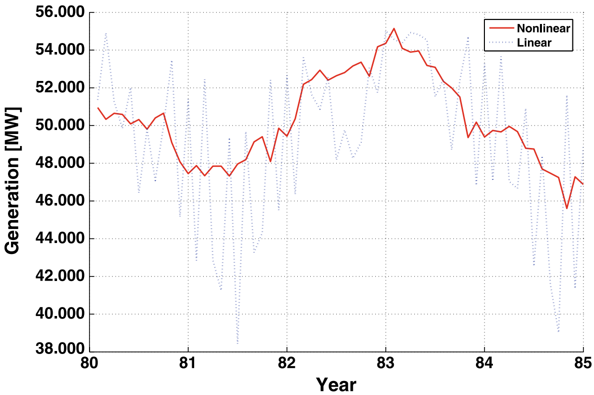


Fig. 19 Linear and nonlinear generation trajectories for the Brazilian hydropower system during the wet period

The method was applied in three different case studies. In the first, composed of a single hydro plant, a detailed analysis of the optimal solution is revealed. The second was comprised of 15 hydro plants located in a single cascade, while the third involved 74 hydro plants in the Brazilian hydropower system. A planning period of 5 years and two different hydrological scenarios (dry and wet years) were considered for numerical evaluations.

Comparison with a linearized and a nonlinear model were performed. The results attest to the adequacy and efficiency of the proposed method for long-term scheduling of large-scale hydrothermal power systems.

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