WEAK CONCENTRATION AND WAVE OPERATOR FOR A 3D COUPLED NONLINEAR SCHRÖDINGER SYSTEM

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Abstract. Reported in this paper are results concerning the Cauchy problem and the dynamics for a cubic nonlinear Schrödinger system arising in nonlinear optics. A sharp criterium is given concerned with the dichotomy global existence versus finite time blow-up. When a radial solution blows up in finite time, we prove the concentration in the critical Lebesgue space. Sufficient condition for the scattering and the construction of the wave operator in the energy space are also provided.

1. Introduction and Results

This paper is concerned with the following three-dimensional cubic nonlinear Schrödinger system

\[
\begin{aligned}
\begin{cases}
iu_t + \Delta u + (|u|^2 + \beta |v|^2)u = 0, \\
v_t + \Delta v + (|v|^2 + \beta |u|^2)v = 0,
\end{cases}
\end{aligned}
\]

where \((x,t) \in \mathbb{R}^3 \times \mathbb{R}, u = u(x,t) \text{ and } v = v(x,t)\) are complex-valued functions, and \(\beta > 0\) is a real coupling parameter. System (1.1) appears in many physical situations, especially in nonlinear optics. For instance, when two optical waves of different frequencies copropagate in a medium and interact nonlinearly through the medium, or when two polarization components of a wave interact nonlinearly at some central frequency (see [1]).

Note that when \(v = 0\), (1.1) reduces to the well-known nonlinear Schrödinger equation

\[
iu_t + \Delta u + |u|^2 u = 0,
\]

which is one of the most studied nonlinear dispersive equations. Thus, system (1.1) can also be thought as a perturbation of the single equation (1.2), in the sense that if the solution \(v\) in (1.1) is small in some sense, then the solution \((u,v)\) will behave as a solution of (1.2).

Our main goal in this paper is to study the Cauchy problem and the dynamics associated with (1.1). So, we accomplish (1.1) with the initial conditions

\[
\begin{aligned}
\begin{cases}
u(x,0) = u_0(x), \\
v(x,0) = v_0(x).
\end{cases}
\end{aligned}
\]

To begin with, note that system (1.1) has some conserved quantities. Indeed, define

\[
\begin{aligned}
M_1(u) &= \int |u|^2, \\
M_2(v) &= \int |v|^2, \\
M(u,v) &= M_1(u) + M_2(v),
\end{aligned}
\]

and

\[
E(u,v) = \frac{1}{2} \int (|\nabla u|^2 + |\nabla v|^2) - \frac{1}{4} \int (|u|^4 + 2\beta |uv|^2 + |v|^4).
\]

Then, at least for sufficiently regular solutions, the quantities in (1.4) and (1.5) are conserved by the flow of (1.1). The quantities \(M\) and \(E\) are frequently referred to as the mass and the energy associated with (1.1). Note that \(M\) and \(E\) make sense for \(u,v \in H^1 := H^1(\mathbb{R}^3)\), where \(H^1\) denotes the usual \(L^2\)-based Sobolev space.

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System \((\ref{a1})\) is invariant by scaling: assume that \((u, v)\) is a solution of \((\ref{a1})\) with initial data \((u_0, v_0)\), then
\[
u_t(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad v_t(x, t) = \lambda v(\lambda x, \lambda^2 t),
\]
is also a solution with initial data \((\lambda u_0(\lambda x), \lambda v_0(\lambda x))\), for any \(\lambda > 0\). Now, note that
\[
\|u(\cdot, 0)\|_{L^3} + \|v(\cdot, 0)\|_{L^3} = \|u_0\|_{L^3} + \|v_0\|_{L^3}
\]
and
\[
\|u(\cdot, 0)\|_{\dot{H}^{1/2}} + \|v(\cdot, 0)\|_{\dot{H}^{1/2}} = \|u_0\|_{\dot{H}^{1/2}} + \|v_0\|_{\dot{H}^{1/2}},
\]
where \(\dot{H}^{1/2}\) is the homogeneous \(L^2\)-based Sobolev space of order \(1/2\). Thus, \(L^3 \times L^3\) is the scale-invariant Lebesgue space and \(\dot{H}^{1/2} \times \dot{H}^{1/2}\) is the scale-invariant Sobolev space. Such a spaces are, therefore, called critical spaces.

Let us start with the following local well-posedness result.

**Theorem 1.1** (Local well-posedness in the energy space). For any \((u_0, v_0) \in H^1 \times H^1\), there are \(T_*, T^* > 0\) and a unique solution \((u, v) \in C((-T_*, T^*); H^1 \times H^1)\) of \((\ref{a1})\) satisfying \((u(0), v(0)) = (u_0, v_0)\). Moreover, there holds the blow up alternative:

(i) \(T^* = +\infty\); or

(ii) \(T^* < +\infty\) and
\[
\lim_{t \to T^*} (\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) = +\infty.
\]

A similar statement holds for \(T_*\).

When (i) occurs, we say that the solution is global. When (ii) occurs, we say that the solution blows up in finite time \(T^*\). The proof of Theorem 1.1 is similar to that for the Schrödinger equation \((\ref{a2})\) and it combines Strichartz estimates with the contraction mapping principle (see \[5\]).

Once Theorem 1.1 has been established, a natural question present itself: under what conditions on the initial data \((u_0, v_0)\) do (i) or (ii) hold? For the Schrödinger equation \((\ref{a2})\) this question has been studied in Refs. \[10\] \[11\] (see also Ref. \[5\] and references therein), where the authors established necessary and sufficient condition for the existence of global solutions. These results were obtained taking into account the novel theory put forward in Ref. \[13\]. Since then, much effort has been expended on the study of the dynamics of several dispersive equation. In particular, similar questions for nonlinear Schrödinger-type systems have been addressed (see \[4\] \[6\] \[16\] \[17\] \[18\] \[22\] \[24\] \[25\]).

Here, our approach is different from the papers mentioned above. Our analysis is accomplished directly taking into account the known results for the cubic Schrödinger equation as in \[10\] \[11\]. We first prove the following.

**Theorem 1.2.** Let \((u, v) \in C((-T_*, T^*); H^1 \times H^1)\) be the solution of \((\ref{a1})\) with initial data \((u_0, v_0) \in H^1 \times H^1\), where \(I := (-T_*, T^*)\) is the maximal time interval of existence. Assume that
\[
M(u_0, v_0)E(u_0, v_0) < M(P, Q)E(P, Q).
\]
The following statements hold.

(i) If
\[
M(u_0, v_0)(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2) < M(P, Q)(\|\nabla P\|_2^2 + \|\nabla Q\|_2^2)
\]
then
\[
M(u_0, v_0)(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2) < M(P, Q)(\|\nabla P\|_2^2 + \|\nabla Q\|_2^2)
\]
and the solution exists globally in time, that is, \(I = (-\infty, \infty)\).
In addition, then there exist (and condition (i) this identity has prospects to be applied in other situations.\footnote{virial identity similar to that for the Schrödinger equation (see Theorem 2.1). We believe that the solution is radial. Hence, in order to prove the blow up in finite time we derive a
\begin{equation}
M(u_0, v_0)(\|\nabla u_0\|^2_2 + \|\nabla v_0\|^2_2) > M(P, Q)(\|\nabla P\|^2_2 + \|\nabla Q\|^2_2) \quad (1.10)
\end{equation}
\begin{equation}
M(u_0, v_0)(\|\nabla u(t)\|^2_2 + \|\nabla v(t)\|^2_2) > M(P, Q)(\|\nabla P\|^2_2 + \|\nabla Q\|^2_2). \quad (1.11)
\end{equation}
Moreover, if \( u_0 \) and \( v_0 \) are radial then \( I \) is finite and the solution blows up in finite time.

Here \((P, Q)\) denotes a ground state solution of \((1.1)\) (see Subsection 2.2 below). The proof of Theorem 1.2 follows the same steps as those in Refs. \cite{11, 13}, and the method can be applied to several dispersive equations. Partially, Theorem 1.2 has also appeared in Ref. \cite{16}; however, instead of assuming finite variance of the initial data, we assume that the solution is radial. Hence, in order to prove the blow up in finite time we derive a virial identity similar to that for the Schrödinger equation (see Theorem 2.1). We believe this identity has prospects to be applied in other situations.

Another issue of interest is that of scattering, that is, when the solution of \((1.1)\) approximates a solution of the associated linear system. Roughly speaking, this implies that every nonlinear solution behaves linearly as time evolves. From the mathematical viewpoint this means that there is \((\phi^+, \psi^+) \in H^1 \times H^1\) such that
\begin{equation}
\lim_{t \to \infty} \left\{ \|u(t) - U(t)\phi^+\|_{H^1} + \|u(t) - U(t)\phi^+\|_{H^1} \right\} = 0. \quad (1.12)
\end{equation}
Here and throughout, \(U(t)\) denotes the unitary group associated with the linear Schrödinger equation. A criterium to establish the scattering is given below.

Theorem 1.3 \((H^1\text{ scattering})\). Assume that \((u_0, v_0) \in H^1 \times H^1\) satisfies \(\|u_0\|_{H^1} + \|v_0\|_{H^1} \leq B\). Suppose that the IVP \((1.1)-(1.3)\) is globally well-posed in \(H^1 \times H^1\) and \(K := \|u\|_{\ell^2} + \|v\|_{\ell^2} < \infty\). Then, there are \(\phi^+, \psi^+ \in H^1\) such that \((1.12)\) holds.

On the other hand, one can ask about the following question: given \(\phi^+, \psi^+ \in H^1\), under what conditions is there \((u_0, v_0) \in H^1 \times H^1\) such that the corresponding solution given in Theorem 1.1 is global and \((1.12)\) holds? This phenomenon is known as the existence of wave operators and it is studied in many physical situations (see \cite{26, 27}). In this regard, we have the following result.

Theorem 1.4 \((\text{Existence of wave operators})\). Assume \((\phi^+, \psi^+) \in H^1 \times H^1\) and
\begin{equation}
\frac{1}{2} (\|\phi^+\|^2_2 + \|\psi^+\|^2_2) (\|\nabla \phi^+\|^2_2 + \|\nabla \psi^+\|^2_2) < M(P, Q) E(P, Q).
\end{equation}
Then, there exist \((u_0, v_0) \in H^1 \times H^1\) such that the solution \((u(t), v(t))\) of \((1.1)\) with initial condition \((u_0, v_0)\) satisfies
\begin{equation}
\lim_{t \to \infty} \left\{ \|u(t) - U(t)\phi^+\|_{H^1} + \|u(t) - U(t)\phi^+\|_{H^1} \right\} = 0.
\end{equation}
In addition,
\begin{equation}
M(u_0, v_0)(\|\nabla u(t)\|^2_2 + \|\nabla v(t)\|^2_2) < M(P, Q)(\|\nabla P\|^2_2 + \|\nabla Q\|^2_2),
\end{equation}
and
\begin{equation}
M(u, v) = \|\phi^+\|^2_2 + \|\psi^+\|^2_2, \quad E(u, v) = \frac{1}{2} (\|\nabla \phi^+\|^2_2 + \|\nabla \psi^+\|^2_2).
\end{equation}
Finally, we study the weak concentration of the radial finite-time blow up solutions, that is, the concentration of the solution in the critical Lebesgue space $L^3 \times L^3$. The term weak concentration is used to distinguish it from the concentration in $L^2 \times L^2$. Here, similar to the Schrödinger equation (see [11]) we have two possible scenario for the rate of concentration. More precisely, the following holds.

**Theorem 1.5.** Under the assumptions of Theorem 1.2, if part (ii) occurs then either

(i) there is a constant $c > 0$, depending on the solution $(u, v)$, such that

$$
\int_{\{|x| \leq c^2(\|u(t)\|_2^2 + \|v(t)\|_2^2)^{-1}\}} \left( |u(x, t)|^3 + |v(x, t)|^3 \right) dx \geq c^{-3},
$$

where

$$
A_n := \{ |x| \leq \tilde{c}(\|u_0\|_2 + \|v_0\|_2)^{3/2}(\|\nabla u(t_n)\|_2 + \|\nabla v(t_n)\|_2)^{-1/2} \}
$$

and $\tilde{c}$ is a constant independent of $(u, v)$.

Similar considerations apply to $T^*$. We do not really have to use the assumptions in Theorem 1.2. As will be clear below, the proof works if one knows that the solution is radial and blows up at some time $T^*$. To demonstrate Theorem 1.5 we follow closely the arguments in Ref. [11, Theorem 1.2].

In Section 2 we introduce some basic tools and give preliminaries results. The rest of the paper is devoted to prove Theorems 1.2, 1.5.

## 2. Preliminaries and Basic Tools

First of all, let us introduce some notation used throughout. We use $c$ (or $C$) to denote various constants that may vary line by line. Given any positive numbers $A$ and $B$, the notation $A \lesssim B$ means that there exists a positive constant $c$ such that $A \leq cB$. Given a complex number $z$, we use $Re(z)$ and $Im(z)$ to denote, respectively, the real and imaginary parts of $z$. By $a^+$ ($a^-$) we indicate a number slightly bigger (smaller) than $a$. Otherwise is stated, $\int f$ always mean integration of the function $f$ over all $\mathbb{R}^3$.

We use $\| \cdot \|_p$ to denote the $L^p(\mathbb{R}^n)$ norm. If the $L^p$ norm is taken over a set $\Omega \subset \mathbb{R}^n$ we use $\| \cdot \|_{L^p(\Omega)}$. Also, if necessary we use subscript to indicate which variable we are integrating; for instance, $\|f\|_{L^1_tL^5_x}$ stands for the mixed Lebesgue norm $\|f(\cdot, t)\|_{L^5_x} \|\cdot\|_{L^1_t}$, where the integration in $t$ is over $\mathbb{R}$. When $r = q$, for short we denote $\|f\|_{L^r_tL^r_x}$ by $\|f\|_{L^r_tL^r_x}$. If $I \subset \mathbb{R}$, in a similar fashion we define $\|f\|_{L^1_tL^r_x}$. By $H^s = H^s(\mathbb{R}^3)$ we designate the usual $L^2$-based Sobolev space. Also, by $\dot{H}^s = \dot{H}^s(\mathbb{R}^3)$, we represent the homogeneous Sobolev space. The norms in $H^s$ and $\dot{H}^s$ will be denote by $\| \cdot \|_{H^s}$ and $\| \cdot \|_{\dot{H}^s}$, respectively. The Fourier transform of $f$ is given by

$$
\hat{f}(\xi) = \int e^{ix\xi} f(x) dx.
$$
2.1. A Virial Identity. Here we establish a virial identity which will be used to prove Theorem 1.2. Since the result can be proved to a more general system, we consider the following n-dimensional nonlinear Schrödinger system

\[
\begin{align*}
iu_t + \Delta u + (|u|^{2p} + \beta |u|^{p-1}|v|^{p+1})u &= 0, \\
v_t + \Delta v + (|v|^{2p} + \beta |v|^{p-1}|u|^{p+1})v &= 0, \\
\end{align*}
\]

(2.1)

where \( p \geq 1 \) is a real number. Note that system (1.1) is a particular case of (2.1) with \( p = 1 \) and \( n = 3 \). Our main result here reads as follows.

**Theorem 2.1.** Let \((u,v) \in C((-T_*;T^*);H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n))\) be a solution of (2.1) and \( \varphi \in C_0^\infty(\mathbb{R}^n) \). Define

\[ V(t) = \frac{1}{2} \int \varphi(x)(|u|^2 + |v|^2). \]

Then,

\[ V'(t) = \text{Im} \int \nabla \varphi \cdot \nabla u \overline{u} + \text{Im} \int \nabla \varphi \cdot \nabla v \overline{v} \]

(2.2)

and

\[ V''(t) = 2 \sum_{k,j=1}^n \text{Re} \frac{\partial^2 \varphi}{\partial x_k \partial x_j} \left( \partial_{x_k} u \partial_{x_j} \overline{u} + \partial_{x_k} v \partial_{x_j} \overline{v} \right) - \frac{1}{2} \int \Delta^2 \varphi(|u|^2 + |v|^2) \]

\[ + \left( \frac{1}{p+1} - 1 \right) \int \Delta \varphi(|u|^{2p+2} + 2 \beta |u|^{p+1}|v|^{p+1} + |v|^{2p+2}), \]

(2.3)

where \( \partial_{x_k} \) indicates the partial derivative with respect to \( x_k \).

**Proof.** The proof follows the same steps as in Ref. [12, Lemma 2.9]. So we omit the details. \( \Box \)

**Corollary 2.2.** If in addition to the hypothesis of Theorem 2.1, \( u,v \), and \( \varphi \) are radially symmetric then

\[ V''(t) = 2 \int \varphi''(|\nabla u|^2 + |\nabla v|^2) - \frac{1}{2} \int \Delta^2 \varphi(|u|^2 + |v|^2) \]

\[ + \left( \frac{1}{p+1} - 1 \right) \int \Delta \varphi(|u|^{2p+2} + 2 \beta |u|^{p+1}|v|^{p+1} + |v|^{2p+2}). \]

(2.4)

**Proof.** It is easy to see, from the fact that \( u,v \), and \( \varphi \) are radially symmetric, that

\[ \sum_{k,j=1}^n \text{Re} \frac{\partial \varphi}{\partial x_k \partial x_j} \left( \partial_{x_k} u \partial_{x_j} \overline{u} + \partial_{x_k} v \partial_{x_j} \overline{v} \right) = \varphi''(|\nabla u|^2 + |\nabla v|^2). \]

The proof is thus completed. \( \Box \)

2.2. Gagliardo-Nirenberg’s inequality and Ground States. In order to obtain the sharp criterium stated in Theorem 1.2 we need a sharp Gagliardo-Nirenberg-type inequality.

Sharp constant for the usual Gagliardo-Nirenberg inequality

\[ \|u\|_{L^{p+2}(\mathbb{R}^n)}^{p+2} \leq K_{\text{best}}^{p+2} \|\nabla u\|_{L^2(\mathbb{R}^n)}^{np/2} \|u\|_{L^2(\mathbb{R}^n)}^{2+p(2-n)/2} \]

(2.5)
was first studied in Nagy \cite{21} in the case \( n = 1 \) and then for all \( n \geq 2 \) (with \( 0 < p < 4/(n - 2) \)) in Weinstein \cite{29}. The sharp constant was obtained in terms of the ground state solution of the semilinear elliptic equation

\[
\frac{pn}{4} \Delta \psi - \left( 1 + \frac{p}{4}(2 - n) \right) \psi + \psi^{p+1} = 0.
\]

More precisely,

\[
K_{\text{best}}^{p+2} = \frac{p + 2}{2\|\psi\|_{L^2(\mathbb{R}^n)}^2}.
\]

Similar inequalities involving systems have already appeared in the literature. In what follows we describe a inequality which generalizes (2.5). These results were established in Ref. \cite{8}. Consider the following \( n \)-dimensional elliptic system

\[
\begin{cases}
-\Delta P + P - (|P|^{2p} + \beta |P|^{p-1}|Q|^{p+1})P = 0, \\
-\Delta Q + Q - (|Q|^{2p} + \beta |Q|^{p-1}|P|^{p+1})Q = 0.
\end{cases} \tag{2.6}
\]

Let \( I : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{R} \) be the functional given by

\[
I(u, v) = \frac{1}{2}(\|\nabla u\|^2_2 + \|u\|^2_2 + \|\nabla v\|^2_2 + \|v\|^2_2) - \frac{1}{2(p + 1)}(\|u\|^{2p+2}_{2p+2} + \beta \|uv\|^{p+1}_{p+1} + \|v\|^{2p+2}_{2p+2}), \tag{2.7}
\]

where \( \| \cdot \|_q \) designates the norm in \( L^q(\mathbb{R}^n) \). It is easy to see that critical points of \( I \) are weak solutions of (2.6). Moreover, by elliptic regularity such solutions are indeed smooth.

Recall that a ground state solution of (2.6) is a solution \((P, Q) \neq (0, 0)\) that minimizes \( I \), that is, \( I(P, Q) \leq I(u, v) \) for any solution \((u, v)\) of (2.6). Existence of positive ground state solutions for (2.6) was investigated in Ref. \cite{19}. In particular such solutions can be obtained by solving the minimization problem

\[
m := \inf\{I(u, v) : (u, v) \in \mathcal{N}\}, \tag{2.8}
\]

where \( \mathcal{N} \subset H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \) is the Nehari manifold,

\[
\mathcal{N} = \left\{ (u, v) \neq (0, 0) : \|\nabla u\|^2_2 + \|u\|^2_2 + \|\nabla v\|^2_2 + \|v\|^2_2 = \|u\|^{2p+2}_{2p+2} + 2\beta \|uv\|^{p+1}_{p+1} + \|v\|^{2p+2}_{2p+2} \right\}.
\]

Now we are able to state the following Gagliardo-Nirenberg-type inequality.

**Proposition 2.3.** For all \( p \geq 0 \) and \( u, v \in H^1(\mathbb{R}^n) \), it holds

\[
\|u\|^{2p+2}_{2p+2} + 2\beta \|uv\|^{p+1}_{p+1} + \|v\|^{2p+2}_{2p+2} \leq K_{\text{opt}}(\|u\|^2_2 + \|v\|^2_2)^{p+1-m/2}(\|\nabla u\|^2_2 + \|\nabla v\|^2_2)^{p_{\text{mp}}/2}, \tag{2.9}
\]

where \( K_{\text{opt}} \) is the optimal constant given by

\[
K_{\text{opt}} = \frac{2(p + 1)}{(np)^{np/2}(2p + 2 - np)^{1-np/2}} \left( \|P\|^2_2 + \|Q\|^2_2 \right)^p \tag{2.10}
\]

(\(P, Q\) is any ground state solution of (2.6), and \( m \) is defined in (2.8)).

**Proof.** The proof was established in Ref. \cite{8} Section 3. In particular, it was showed that if \((P, Q)\) is a ground state solution of (2.6), then

\[
m = I(P, Q) = \frac{p}{2p + 2 - np}(\|P\|^2_2 + \|Q\|^2_2). \tag{2.11}
\]

Thus the second equality in (2.10) holds. \( \square \)
Remark 2.4. Uniqueness of ground states for \([2.6]\) is known only for \(\beta\) sufficiently small \([8, 19]\), in which case either \(P\) or \(Q\) must vanish identically. However, the second equality in \((2.10)\) shows that \(K_{opt}\) does not depend on the choice of the ground state.

Remark 2.5. Multiplying the equations in \((2.6)\), respectively, by \(P\) and \(Q\), integrating over \(\mathbb{R}^n\) and using a Pohozaev-type identity, we deduce the following identities (see \([8]\) Section 3)

\[
m = \frac{1}{n}(\|\nabla P\|_2^2 + \|\nabla Q\|_2^2) = \frac{p}{2p + 2}(\|P\|_{2p+2}^{2p+2} + 2\beta\|PQ\|_{p+1}^{p+1} + \|Q\|_{2p+2}^{2p+2}).
\]

(2.12)

In particular combining \((2.11)\) and \((2.12)\), it is inferred that

\[
\|\nabla P\|_2^2 + \|\nabla Q\|_2^2 = \frac{np}{2p + 2 - np}(\|P\|_2^2 + \|Q\|_2^2)
\]

(2.13)

and

\[
\|P\|_{2p+2}^{2p+2} + 2\beta\|PQ\|_{p+1}^{p+1} + \|Q\|_{2p+2}^{2p+2} = \frac{2p + 2}{2p + 2 - np}(\|P\|_2^2 + \|Q\|_2^2).
\]

(2.14)

In the case of one single equation, let us also recall the following.

Lemma 2.6. The following Gagliardo-Nirenberg inequalities hold.

(i) If \(u \in H^1(\mathbb{R}^3)\), then

\[
\|f\|_4^4 \leq C\|\nabla f\|_2^{6(4-p)/(6-p)}\|f\|_p^{4-6(4-p)/(6-p)}.
\]

(2.15)

(ii) If \(u \in H^1(\mathbb{R}^3)\) is radially symmetric, then

\[
\|u\|_{L^4(|x| \geq R)}^4 \leq \frac{C}{R^2}\|u\|_{L^3(|x| \geq R)}^3\|\nabla u\|_{L^2(|x| \geq R)}.
\]

(2.16)

Proof. Part (i) is the usual Gagliardo-Nirenberg inequality \([9]\). Part (ii) is due to Strauss \([28]\) (see also Ref. \([10]\)). \(\square\)

2.3. Strichartz estimates and Leibniz rule. In order to study the Cauchy problem associated with \((1.1)\), let us introduce some notation. Given any \(s \in \mathbb{R}\), we say that a pair \((q, r)\) is \(\dot{H}^s\) admissible if

\[
\frac{2}{q} + \frac{3}{r} = \frac{3}{2} - s.
\]

In what follows, we denote

\[
\|u\|_{S(L^2)} = \sup\{\|u\|_{L^q_tL^r_x}; (q, r) \text{ is } L^2 \text{ admissible, } 2 \leq r \leq 6, 2 \leq q \leq \infty\},
\]

and

\[
\|u\|_{S(H^{1/2})} = \sup\{\|u\|_{L^q_tL^r_x}; (q, r) \text{ is } \dot{H}^{1/2} \text{ admissible, } 3 \leq r \leq 6^-, 4^+ \leq q \leq \infty\}.
\]

We also need to consider the dual norms:

\[
\|u\|_{S'(L^2)} = \inf\{\|u\|_{L^{q'}_tL^{r'}_x}; (q, r) \text{ is } L^2 \text{ admissible, } 2 \leq q \leq \infty, 2 \leq r \leq 6\},
\]

and

\[
\|u\|_{S'(-H^{-1/2})} = \inf\{\|u\|_{L^{q'}_tL^{r'}_x}; (q, r) \text{ is } \dot{H}^{-1/2} \text{ admissible, } \left(\frac{4}{3}\right)^+ \leq q \leq 2^-, 3^+ \leq r \leq 6^-\}.
\]

When a time interval \(I \subset \mathbb{R}\) is given, we use \(S'(L^2; I)\) to inform that the temporal integral is evaluated over \(I\), that is, we replace \(\|\cdot\|_{L^{q'}_tL^{r'}_x}\) by \(\|\cdot\|_{L^{q'}_tL^{r'}_x; I}\). Similarly to the other spaces. We now recall the well-known Strichartz inequalities.
Lemma 2.7. With the above notation we have:

(i) (Linear estimates).
\[ \|U(t)u_0\|_{S(L^2)} \lesssim \|u_0\|_2 \]
\[ \|U(t)u_0\|_{S(H^{1/2})} \lesssim \|u_0\|_{H^{1/2}}. \]

(ii) (Inhomogeneous estimates).
\[ \left\| \int_0^t U(t-t')f(\cdot,t')dt' \right\|_{S(L^2)} \lesssim \|f\|_{S'(L^2)}, \]
\[ \left\| \int_0^t U(t-t')f(\cdot,t')dt' \right\|_{S(H^{1/2})} \lesssim \|D^{1/2}f\|_{S'(L^2)}. \]

Proof. See Refs. [5, 10].

We also need the Leibniz and chain rules for fractional derivatives.

Lemma 2.8. Let \( s \in (0,1) \).

(i) (Fractional chain rule) Suppose \( G \in C^1(\mathbb{C}) \). Let \( 1 < r, r_1, r_2 < \infty \) be such that \( 1/r = 1/r_1 + 1/r_2 \). Then,
\[ \|D^sG(u)\|_r \lesssim \|G'(u)\|_{r_1} \|D^s u\|_{r_2}. \]

(ii) (Leibniz’s rule for fractional derivatives) Let \( 1 < r, p_1, p_2, q_1, q_2 < \infty \) be such that \( 1/r = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 \). Then,
\[ \|D^s(fg)\|_r \lesssim \|f\|_{p_1} \|D^s g\|_{p_2} + \|D^s f\|_{q_1} \|g\|_{q_2}. \]

Proof. See Refs. [7, 14].

Lemma 2.9. Let \( F(u) = |u|^p u \) with \( p > 0 \) and let \( s \in (0,1) \). Then, for \( 1 < q, q_1, q_2 < \infty \) such that \( 1/q = 1/q_1 + p/q_2 \), we have
\[ \|D^s[F(u + v) - F(u)]\|_q \lesssim \|D^s u\|_{q_1} \|v\|_{q_2} + \|D^s v\|_{q_1} \|u + v\|_{q_2}. \]

Proof. See Ref. [15].

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We will use the following lemma.

Lemma 3.1. Let \( I \subset \mathbb{R} \) be an open interval containing the origin. Let \( q > 1, b > 0 \), and \( a \) be real constants. Define \( \gamma = (bq)^{-1/(q-1)} \) and \( f(r) = a - r + br^q \) for \( r \geq 0 \). Let \( G(t) \) be a continuous nonnegative function on \( I \). Assume that \( a < (1 - 1/q)\gamma \) and \( f \circ G \geq 0 \).

(i) If \( G(0) < \gamma \), then \( G(t) < \gamma \), for any \( t \in I \).

(ii) If \( G(0) > \gamma \), then \( G(t) > \gamma \), for any \( t \in I \).

Proof. This lemma was essentially established in Ref. [2]. We present here the minor modifications in the proof. For (i), since \( G \) is continuous and \( G(0) < \gamma \), there exists \( \varepsilon > 0 \) such that \( G(t) < \gamma \) for \( t \in (-\varepsilon, \varepsilon) \). Assume the statement is false. By the continuity of \( G \) we then deduce the existence of \( t^* \in I \) with \( |t^*| \geq \varepsilon \) such that \( G(t^*) = \gamma \). Thus,
\[ f \circ G(t^*) = f(\gamma) = a - \gamma(1 - br^q) = a - \gamma \left(1 - \frac{1}{q}\right) < 0, \]
which contradicts \( \)\( f \circ G \geq 0 \). A similar proof holds for (ii). The lemma is thus proved. \( \square \)
Corollary 3.2. Under the assumptions of Lemma 3.1 with \( a < (1 - 1/q)\gamma \) replaced by \( a < (1 - \delta_1)(1 - 1/q)\gamma \), for some small \( \delta_1 > 0 \); if \( G(0) > \gamma \) then there is a \( \delta_2 = \delta_2(\delta_1) > 0 \) such that \( G(t) > (1 + \delta_2)\gamma \).

Proof. Since \( a - (1 - \delta_1)(1 - 1/q)\gamma > a - (1 - 1/q)\gamma \), we already know from Lemma 3.1 that \( G(t) > \gamma \). Now, under the assumption \( a < (1 - \delta_1)(1 - 1/q)\gamma \), there is a \( \epsilon > 0 \), depending on \( \delta_1 \) such that

\[
f(G(t)) \geq 0 > a - (1 - \delta_1)(1 - 1/q)\gamma = f(\gamma + \epsilon).
\]

The continuity of \( G \) implies that \( G(t) > \gamma + \epsilon \). By choosing \( \delta_2 = \epsilon / \gamma \) we obtain the desired conclusion.

Proof of Theorem 1.2. The idea is to get some control on the gradient of \( u \) and \( v \) using the conserved quantities (1.4) and (1.5) and the Gagliardo-Nirenberg inequality (2.9) with the sharp constant given in (2.10). Indeed, from (2.9) and (2.10) (with the conserved quantities (1.4) and (1.5) and the Gagliardo-Nirenberg inequality (2.9) with the sharp constant given in (2.10)),

\[
2E(u_0, v_0) = \|\nabla u\|_2^2 + \|\nabla v\|_2^2 - \frac{1}{2} (\|u\|_4^4 + 2\beta \|uv\|_2^2 + \|v\|_4^4)
\geq \|\nabla u\|_2^2 + \|\nabla v\|_2^2 - \frac{2}{33/2} \left( \frac{\|u_0\|_2^2 + \|v_0\|_2^2}{\|P\|_2^3 + \|Q\|_2^3} \right) \left( \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right)^{3/2}. \tag{3.1}
\]

Let

\[
a = 2E(u_0, v_0), \quad b = \frac{2}{33/2} \left( \frac{\|u_0\|_2^2 + \|v_0\|_2^2}{\|P\|_2^3 + \|Q\|_2^3} \right)^{1/2}, \quad \text{and} \quad q = \frac{3}{2}.
\]

Define \( f(r) = a - r + br^q \) and \( G(t) = \|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2 \). From Theorem 1.1 \( G \) is continuous. Also, it follows from (3.1) that \( f \circ G \geq 0 \).

In view of (2.13) it is easy to check that \( G(0) < \gamma \) and \( G(0) > \gamma \) are equivalent to (1.8) and (1.9), respectively, where \( \gamma \) is defined as in Lemma 3.1. Moreover, from (2.13) and (2.14), we see that

\[
2E(P, Q) = \|P\|_2^2 + \|Q\|_2^2.
\]

Hence, \( a < (1 - 1/q)\gamma \) is equivalent to (1.7). Lemma 3.1 then gives us the first part of the theorem.

From now on we assume the radial condition and let us prove that the maximal time interval of existence must be finite. As in Refs. \([23, 10, 11]\), we will use the virial identity (2.4). Let \( \varphi(x) = \chi(|x|) \) be a smooth radial function such that

\[
\chi(r) = \begin{cases} 
  r^2, & 0 \leq r \leq 1, \\
  0, & r \geq 3,
\end{cases}
\]

and \( \chi''(r) \leq 2 \) for all \( r \geq 0 \). Define \( \chi_R(r) = R^2 \chi(r/R) \). By definition, it is easily seen that if \( r \leq R \) then \( \Delta \chi_R(r) = 6 \) and \( \Delta^2 \chi_R(r) = 0 \).

By choosing \( \varphi \) as \( \chi_R \) in (2.4), it reduces to

\[
V''(t) = 2 \int \chi''_R(|\nabla u|^2 + |\nabla v|^2) - \frac{1}{2} \int \Delta^2 \chi_R(|u|^2 + |v|^2) \\
- \frac{1}{2} \int \Delta \chi_R(|u|^4 + 2\beta |u|^2|v|^2 + |v|^4). \tag{3.2}
\]

We estimate the terms on the right-hand side of (3.2) as follows:

\[
2 \int \chi''_R(|\nabla u|^2 + |\nabla v|^2) \leq 4 \int (|\nabla u|^2 + |\nabla v|^2),
\]
\[-\frac{1}{2} \int \Delta^2 \chi_R(|u|^2 + |v|^2) = -\frac{1}{2} \int_{|x| \geq R} (|u|^2 + |v|^2) \leq \frac{C}{R^2} \int_{|x| \geq R} (|u|^2 + |v|^2),\]

and
\[-\frac{1}{2} \int \Delta \chi_R(|u|^4 + 2\beta|u|^2|v|^2 + |v|^4) \leq -3 \int_{|x| \leq R} (|u|^4 + 2\beta|u|^2|v|^2 + |v|^4) + C \int_{|x| \geq R} (|u|^4 + 2\beta|u|^2|v|^2 + |v|^4) \leq -3 \int_{\mathbb{R}^3} (|u|^4 + 2\beta|u|^2|v|^2 + |v|^4) + C\beta \int_{|x| \geq R} (|u|^4 + |v|^4),\]

where we have used Young’s inequality in the last line. Thus, using the conserved quantities (1.4) and (1.5), we can write
\[
V''(t) \leq 12E(u_0, v_0) - 2(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \frac{C}{R^2} \int_{|x| \geq R} (|u|^2 + |v|^2) + C\beta \int_{|x| \geq R} (|u|^4 + |v|^4).\]

In view of (2.16) and (1.4), we get
\[
C\beta \int_{|x| \geq R} (|u|^4 + |v|^4) \leq \frac{C\beta}{R^2} (\|u_0\|_2^3 \|\nabla u\|_2 + \|v_0\|_2^3 \|\nabla v\|_2) \leq \frac{C\beta}{R^4} \|u_0\|_2^6 + \varepsilon \|\nabla u\|_2^2 + \frac{C\beta}{R^4} \|v_0\|_2^6 + \varepsilon \|\nabla v\|_2^2,
\]

where we used the $\epsilon$-Young’s inequality. The parameter $\epsilon$ will be chosen later. Plugging this estimate in (3.3) yields
\[
V''(t) \leq 12E(u_0, v_0) - (2 - \epsilon)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \frac{C}{R^2} M(u_0, v_0).
\]

Multiplying (3.5) by $M(u_0, v_0)$ we obtain
\[
M(u_0, v_0) V''(t) \leq 12E(u_0, v_0)M(u_0, v_0) - (2 - \epsilon)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)M(u_0, v_0) + \frac{C}{R^2} M^2(u_0, v_0) + \frac{C\beta}{R^4} (\|u_0\|_2^6 + \|v_0\|_2^6)M(u_0, v_0).\]

Since $E(u_0, v_0)M(u_0, v_0) < E(P, Q)M(P, Q)$ there is $\delta_1 > 0$ such that
\[
E(u_0, v_0)M(u_0, v_0) < (1 - \delta_1)E(P, Q)M(P, Q),
\]

which in turn is equivalent to the condition $a < (1 - \delta_1)\gamma(1 - 1/q)$ in Corollary 3.2. As a result, we deduce the existence of a $\delta_2 > 0$ such that
\[
M(u_0, v_0)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) > (1 + \delta_2)(\|\nabla P\|_2^2 + \|\nabla Q\|_2^2)M(P, Q),
\]
Replacing (3.7) and (3.8) in (3.6), and using that
\[ E(P,Q) = \left( \frac{1}{6} \right) \left( \| \nabla P \|_2^2 + \| \nabla Q \|_2^2 \right) \]
(see (2.13)), we have
\[ M(u_0,v_0) \leq \left[ 2(1-\delta_1) - (2-\epsilon)(1+\delta_2) \right] \left( \| \nabla P \|_2^2 + \| \nabla Q \|_2^2 \right) M(P,Q) 
+ \frac{C}{R^2} M^2(u_0,v_0) + \frac{C \beta \epsilon}{R^4} \left( \| u_0 \|_6^2 + \| v_0 \|_6^2 \right) M(u_0,v_0). \]  
(3.9)

We now can choose \( \epsilon > 0 \) small enough and \( R > 0 \) large enough such that the right-hand side of (3.9) is bounded by a strictly negative constant. Consequently, the maximal interval of existence must be finite. This proves the theorem. \( \Box \)

When a solution blows up in finite time (not necessarily radial) then a scaling argument gives a lower bound on the growth of the gradient in (1.6) (a blow-up rate). Indeed, assume that \((u,v)\) blows up in the finite time \( T^* \). Define,
\[ \tilde{u}(x,s) = \lambda u(\lambda x, t + \lambda^2 s), \quad \tilde{v}(x,s) = \lambda v(\lambda x, t + \lambda^2 s), \]
where
\[ \lambda = \lambda(t) = \frac{1}{\left( \| \nabla u(t) \|_2 + \| \nabla v(t) \|_2 \right)^{2/3}}. \]
It is easy to see that \((\tilde{u}, \tilde{v})\) is a solution of (1.1) and
\[ \| \nabla \tilde{u}(0) \|_2 + \| \nabla \tilde{v}(0) \|_2 = \lambda^{1/2} \left( \| \nabla u(t) \|_2 + \| \nabla v(t) \|_2 \right) = 1. \]
By observing that Theorem 1.1 also holds here, we deduce that there is \( s_0 > 0 \) such that \( t + \lambda(t)^2 s_0 \leq T^* \). This in turn implies the existence of a constant \( c > 0 \) such that
\[ \| \nabla u(t) \|_2 + \| \nabla v(t) \|_2 \geq \frac{c}{(T^*-t)^{1/4}}. \]  
(3.10)

As a consequence, we can also prove the blow up in the supercritical Lebesgue space \( L^p \times L^p \), for \( p > 3 \). More precisely, we have.

**Proposition 3.3.** Assume \((u_0,v_0) \in H^1 \times H^1\) and that the corresponding solution \((u(t),v(t))\) blows up in finite time \( T^* \).

(i) If \( 3 < p < 4 \), then
\[ \| u(t) \|_p + \| v(t) \|_p \geq \frac{C}{(T^*-t)^{(p-3)/2p}}. \]
(ii) If \( p \geq 4 \), then
\[ \| u(t) \|_p + \| v(t) \|_p \geq \frac{C}{(T^*-t)^{(p-2)/4p}}. \]

**Proof.** The proof is close to that for the Schrödinger equation. The interested reader will find the details in Ref. [5, Corollary 6.5.14]. \( \Box \)

4. **THE CAUCHY PROBLEM AND PROOF OF THEOREM 1.3**

We start this section by establishing the global well-posedness for small data in the critical Sobolev space. The proof of Theorem 1.3 will follow in a similar fashion. As a consequence, we will also prove that when a solution blows up in finite time \( T^* \), it also blows up in a suitable Strichartz norm (see Corollary 4.2 below).
Theorem 4.1 (Global well-posedness in the critical space). Assume that \((u_0, v_0) \in \dot{H}^{1/2} \times \dot{H}^{1/2}\) satisfies \(\|u_0\|_{\dot{H}^{1/2}} + \|v_0\|_{\dot{H}^{1/2}} \leq A\). There is \(\delta > 0\) such that if \(\|U(t)u_0\|_{S(\dot{H}^{1/2})} + \|U(t)v_0\|_{S(\dot{H}^{1/2})} \leq \delta\) then the initial-value problem (1.1)-(1.3) is globally well-posed in \(\dot{H}^{1/2} \times \dot{H}^{1/2}\). In addition there exists \(c > 0\) such that
\[
\|u\|_{S(\dot{H}^{1/2})} + \|v\|_{S(\dot{H}^{1/2})} \leq 2\left(\|U(t)u_0\|_{S(\dot{H}^{1/2})} + \|U(t)v_0\|_{S(\dot{H}^{1/2})}\right),
\]
\[
\|D^{1/2}u\|_{S(L^2)} + \|D^{1/2}v\|_{S(L^2)} \leq 2c\left(\|u_0\|_{\dot{H}^{1/2}} + \|v_0\|_{\dot{H}^{1/2}}\right).
\]

Proof. The proof is based on the contraction mapping principle. Thus, using the equivalent integral formulation, the proof reduces to showing that the operator \(\Gamma = (\Phi, \Psi)\) given by
\[
\Phi(u, v) = U(t)u_0 - i \int_0^t U(t - t')(|u|^2u + \beta|v|^2v)dt',
\]
\[
\Psi(u, v) = U(t)v_0 - i \int_0^t U(t - t')(|v|^2v + \beta|u|^2u)dt',
\]
is a contraction on a suitable space. The details are similar to those in the proof of Theorem 1.3 below (see also Proposition 2.1 in Ref. [10]).

Proof of Theorem 1.3. Since \((u, v)\) is a fixed point of the integral equations (4.1)-(4.2), it follows that
\[
u(t) = U(t)v_0 - i \int_0^t U(t - t')(|v|^2v + \beta|u|^2u)dt',
\]
\[
\|u\|_{S(L^2)} + \|\nabla u\|_{S(L^2)} + \|v\|_{S(L^2)} + \|\nabla v\|_{S(L^2)} < \infty.
\]
Indeed, since \(\|u\|_{L^1_x} + \|v\|_{L^1_x} \leq K\), we can decompose \([0, \infty) = \bigcup_{j=1}^{J_0} I_j\) such that \(\|u\|_{L^1_{I_j}} + \|v\|_{L^1_{I_j}} < \delta\), where \(\delta\) is a small constant to be determined later. It is easy to see that \(J_0 \sim K^5\). In view of Lemma 2.7, we have
\[
\|\nabla u\|_{S(L^2; I_j)} \lesssim \|U(t)\nabla u_0\|_{S(L^2; I_j)} + \|\nabla(|u|^2u)\|_{L_{I_j}^{10/7} L_x^{10/7}} + \|\nabla(|v|^2u)\|_{L_{I_j}^{10/7} L_x^{10/7}}
\]
\[
\lesssim \|u_0\|_{H^1} + I_1 + I_2.
\]
Let us estimate \(I_2\). We first note that
\[
\|\nabla(|u|^2u)\|_{L_{10/7}^{10/7}} \lesssim \|u\|_{L^3_x}^2 \|\nabla u\|_{L^{10/3}_x} + \|\nabla u\|_{L^{10/3}_x} \|\nabla u\|_{L^3_x} \|u\|_{L^5_x}.
\]
Hence, applying Hölder’s inequality, we get
\[
I_2 \lesssim \|u\|_{L^3_x}^2 \|\nabla u\|_{L^{10/3}_x L_x^{10/3}} + \|\nabla u\|_{L^{10/3}_x L_x^{10/3}} \|\nabla u\|_{L^3_x} \|u\|_{L^5_x}.
\]
Similarly,
\[
I_1 \lesssim \|u\|_{L^3_x}^2 \|\nabla u\|_{L^{10/3}_x L_x^{10/3}}.
\]
From (4.5)-(4.7),
\[
\|\nabla u\|_{S(L^2; I_j)} \lesssim \|u_0\|_{H^1} + 2\delta^2 \|\nabla u\|_{S(L^2; I_j)} + \delta^2 \|\nabla v\|_{S(L^2; I_j)},
\]
Using the same arguments, we can see that
\[
\|\nabla v\|_{S(L^2; I_j)} \lesssim \|v_0\|_{H^1} + 2\delta^2 \|\nabla v\|_{S(L^2; I_j)} + \delta^2 \|\nabla u\|_{S(L^2; I_j)}.
\]
Summing (4.8) and (4.9),
\[ \|\nabla u\|_{S(L^2;I_j)} + \|\nabla v\|_{S(L^2;I_j)} \lesssim B + 3\delta^2(\|\nabla u\|_{S(L^2;I_j)} + \|\nabla v\|_{S(L^2;I_j)}). \]

By choosing \( \delta \) small enough, we obtain
\[ \|\nabla u\|_{S(L^2;I_j)} + \|\nabla v\|_{S(L^2;I_j)} \lesssim B. \quad (4.10) \]

A summation in \( j \) gives
\[ \|\nabla u\|_{S(L^2)} + \|\nabla v\|_{S(L^2)} \lesssim BK^5. \]

In view of Hölder’s inequality, we also have \( \|u\|_{S(L^2)} + \|v\|_{S(L^2)} \lesssim BK^5 \). The claim is thus proved.

Now define,
\[ \phi^+ = u_0 - i \int_0^\infty U(-t')(|u|^2u + \beta|v|^2u)dt', \]
\[ \psi^+ = v_0 - i \int_0^\infty U(-t')(|v|^2v + \beta|u|^2v)dt'. \]

By using the last claim, it is not difficult to see that \( \phi^+, \psi^+ \in H^1 \).

Noting that
\[ u(t) - U(t)\phi^+ = -i \int_t^\infty U(t-t')(|u|^2u + \beta|v|^2u)dt', \]
\[ v(t) - U(t)\psi^+ = -i \int_t^\infty U(t-t')(|v|^2v + \beta|u|^2v)dt', \]
and using the previous arguments,
\[ \|u(t) - U(t)\phi^+\|_{H^1} \lesssim \|u\|_{L_{[t,\infty]}^2}^2 \cdot \|u\|_{L_{[t,\infty]}^{10/3}L_{x}^{10/3}}^2 + \|v\|_{L_{[t,\infty]}^2}^2 \cdot \|u\|_{L_{[t,\infty]}^{10/3}L_{x}^{10/3}} + \|\nabla u\|_{L_{[t,\infty]}^{10/3}L_{x}^{10/3}} \cdot \|u\|_{L_{[t,\infty]}^5}^2 \quad (4.11) \]
\[ + \|\nabla v\|_{L_{[t,\infty]}^{10/3}L_{x}^{10/3}} \cdot \|v\|_{L_{[t,\infty]}^5}^2 \cdot \|u\|_{L_{[t,\infty]}^5} \]

The right-hand side of (4.11) goes to zero, as \( t \to \infty \), in view of the above claim. Similarly we show that \( \|v(t) - U(t)\psi^+\|_{H^1} \to 0 \), as \( t \to \infty \). This proves the theorem. \( \square \)

**Corollary 4.2 (Blow up of the Strichartz norm).** Assume \( T^* < \infty \) in Theorem 1.1. Then,
\[ \Lambda_{T^*} := \|u\|_{L_{[0,T^*]}^5L_{x}^5} + \|v\|_{L_{[0,T^*]}^5L_{x}^5} = \infty. \quad (4.12) \]

**Proof.** Let \((u, v)\) be any solution of (1.1)–(1.3) defined in the interval \([0, \tilde{T}]\). We claim that if \( \Lambda_{\tilde{T}} < \infty \), then we can extend the solution beyond \( \tilde{T} \). Indeed, by mimicking the proof of Theorem 1.3 one can see that \((u(t), v(t)) \in H^1 \times H^1\) for all \( t \in [0, \tilde{T}] \). In particular, \((u(\tilde{T}), v(\tilde{T})) \in H^1 \times H^4\). An application of Theorem 1.1 implies the existence of \( \epsilon_0 > 0 \) such that the solution can be extended to \([0, \tilde{T} + \epsilon_0]\). Hence, if \( T^* < \infty \), (4.12) must hold. \( \square \)
5. Proof of Theorem 1.4

Following the steps of Theorem 1.3 it suffices to prove that the integral equations

\[
\begin{align*}
u(t) &= U(t)\phi^+ - i \int_{t}^{\infty} U(t - t')(|u|^2u + \beta|v|^2u)dt', \\
v(t) &= U(t)\psi^+ - i \int_{t}^{\infty} U(t - t')(|v|^2v + \beta|u|^2v)dt',
\end{align*}
\]

has a global solution. We first show the existence of a solution for \( t \geq T \), where \( T \) is sufficiently large. Note that if \( \delta \) is sufficiently small we can choose, in view of Theorem 4.1 a \( T > 0 \) sufficiently large such that

\[
\|U(t)u_0\|_{\dot{H}^{1/2}[T,\infty)} + \|U(t)v_0\|_{\dot{H}^{1/2}[T,\infty)} < \delta.
\]

Thus, as in Theorem 4.1, we can solve the integral equations in \( \dot{H}^{1/2} \times \dot{H}^{1/2} \), for \( t \geq T \). Moreover, the solution \((u, v)\) is small in the sense that

\[
\|u\|_{\dot{H}^{1/2}[T,\infty)} + \|v\|_{\dot{H}^{1/2}[T,\infty)} \leq 2\delta. \tag{5.1}
\]

We now claim that \((u(t), v(t)) \in H^1 \times H^1 \). Indeed, taking into account (5.1), as in (4.10), we obtain

\[
\|\nabla u\|_{L^2([T,\infty))} + \|\nabla v\|_{L^2([T,\infty))} \leq 2c(\|\nabla \phi^+\|_2 + \|\nabla \psi^+\|_2).
\]

Dropping the gradient, we deduce a similar estimate. In the same way,

\[
\lim_{T \to \infty} \|\nabla(u(t) - U(t)\phi^+)\|_{L^2([T,\infty))} = \lim_{T \to \infty} \|\nabla(v(t) - U(t)\psi^+)\|_{L^2([T,\infty))} = 0.
\]

This shows that

\[
\begin{align*}
\lim_{t \to \infty} \|\nabla u(t)\|_2 &= \lim_{t \to \infty} \|\nabla U(t)\phi^+\|_2 = \|\nabla \phi^+\|_2, \tag{5.2} \\
\lim_{t \to \infty} \|\nabla v(t)\|_2 &= \lim_{t \to \infty} \|\nabla U(t)\psi^+\|_2 = \|\nabla \psi^+\|_2, \tag{5.3}
\end{align*}
\]

and

\[
M(u(t), v(t)) = \|\phi^+\|^2_2 + \|\psi^+\|^2_2. \tag{5.4}
\]

Next, since \( \|u(t) - U(t)\phi^+\|_{H^1} \to 0 \), as \( t \to \infty \) and (see Ref. [1] Corollary 2.3.7),

\[
\lim_{t \to \infty} \|U(t)\phi^+\|_4 = 0,
\]

we see from

\[
\begin{align*}
\|u(t)\|_4 &\leq \|u(t) - U(t)\phi^+\|_4 + \|U(t)\phi^+\|_4 \\
&\leq c\|u(t) - U(t)\phi^+\|_{H^1} + \|U(t)\phi^+\|_4,
\end{align*}
\]

that \( \lim_{t \to \infty} \|u(t)\|_4 = 0 \). Similarly, \( \lim_{t \to \infty} \|v(t)\|_4 = 0 \). The Young inequality then implies

\[
\lim_{t \to \infty} (\|u(t)\|^4_4 + 2\beta\|u(t)v(t)\|^2_2 + \|v(t)\|^4_4) = 0. \tag{5.5}
\]

Thus, in view of (5.2)-(5.5),

\[
E(u(t), v(t)) = \lim_{t \to \infty} E(u(t), v(t)) = \lim_{t \to \infty} \frac{1}{2}(\|\nabla u(t)\|^2_2 + \|\nabla v(t)\|^2_2) = \frac{1}{2}(\|\nabla \phi^+\|^2_2 + \|\nabla \psi^+\|^2_2).
\]
Using the hypothesis we promptly see that, for \( t \geq T \),
\[
M(u(t), v(t))E(v(t), u(t)) < M(P, Q)E(P, Q).
\]
In addition, using (2.13),
\[
\lim_{t \to \infty} M(u(t), v(t))(\| \nabla u(t) \|^2 + \| \nabla v(t) \|^2) = (\| \phi^+ \|^2 + \| \psi^+ \|^2)(\| \nabla \phi^+ \|^2 + \| \nabla \psi^+ \|^2)
\]
\[
< 2M(P, Q)E(P, Q)
\]
\[
< \frac{1}{3}M(P, Q)(\| \nabla P \|^2 + \| \nabla Q \|^2),
\]
where we have used that \( E(P, Q) = (1/6)(\| \nabla P \|^2 + \| \nabla Q \|^2) \). Thus, for \( T \) large enough, \( M(u(T), v(T))(\| \nabla u(T) \|^2 + \| \nabla v(T) \|^2) < M(P, Q)(\| \nabla P \|^2 + \| \nabla Q \|^2) \). From Theorem 1.2
the solution can be extended globally. The proof is thus completed.

6. Proof of Theorem 1.5

In this section we prove Theorem 1.5. Let us start by establishing the nonexistence
of a strong limit at the blow-up time in the critical Lebesgue space even for non-radial
solutions.

Proposition 6.1. Let \((u_0, v_0) \in H^1 \times H^1\) and assume that the corresponding solution
of the Cauchy problem (1.1)-(1.3) blows up in finite time \( T^* \). Let \( \{t_n\} \) be a sequence of
times converging to \( T^* \), as \( n \to \infty \). Then, \( \{(u(t_n), v(t_n))\} \) does not have a strong limit in
\( L^3 \times L^3 \), as \( n \to \infty \).

Proof. The proof follows the same strategy in Ref. [20], where the authors proved the
nonexistence of a strong limit in \( L^2 \) for the Schrödinger equation. So we leave out the
details. \( \square \)

Let \( \phi \in C_0^\infty(\mathbb{R}^3) \) be a radial cut-off function such that
\[
\phi(x) = \begin{cases}
1, & |x| \leq 1, \\
0, & |x| \geq 2.
\end{cases}
\]
Let \((u, v)\) be a solution of (1.1). For a fixed \( R = R(t) \), to be chosen later, define
\[
u_1(x, t) = \phi \left( \frac{x}{R} \right) u(x, t), \quad \nu_2(x, t) = \left( 1 - \phi \left( \frac{x}{R} \right) \right) u(x, t).
\]
Let \( \psi \in C_0^\infty(\mathbb{R}^3) \) be another positive radial cut-off function such that \( \hat{\psi}(0) = 1 \) and \( \psi = 0 \)
for \( |x| \geq 1 \). For a fixed \( \rho = \rho(t) \), also to be chosen later, define \( u_{1L} \) and \( u_{1H} \) via their
Fourier transform by
\[
\tilde{u}_{1L}(\xi, t) = \hat{\psi} \left( \frac{\xi}{\rho} \right) \tilde{u}_1(\xi, t), \quad \tilde{u}_{1H}(\xi, t) = \left( 1 - \hat{\psi} \left( \frac{\xi}{\rho} \right) \right) \tilde{u}_1(\xi, t).
\]
In a similar manner, we define \( v_1, v_2, v_{1L}, \) and \( v_{1H} \).

Lemma 6.2. Let \((u(t), v(t)) \in H^1 \times H^1\) be a solution of (1.1) that blows up in finite time
\( T^* \). Let
\[
R(t) = c_1(\| u_0 \|^2 + \| v_0 \|^2)^{3/2}(\| \nabla u(t) \|^2 + \| \nabla v(t) \|^2)^{-1/2}
\]
and
\[
\rho(t) = c_2(\| \nabla u(t) \|^2 + \| \nabla v(t) \|^2),
\]
for suitable chosen constants \( c_1 > 0 \) and \( c_2 > 0 \). Under the above notation, we have.
(i) There is a constant \(c > 0\) such that, as \(t \to T^*\),
\[
\|u_{1L}\|_3 + \|v_{1L}\|_3 \geq c.
\]
(ii) If \(\|u_1\|_3 + \|v_1\|_3 \leq c^*,\) then there are \(c > 0\) and \(x_0 = x_0(t)\) such that, as \(t \to T^*\),
\[
\|u_1\|_{L^3(|x-x_0(t)| \leq \rho^{-1})} + \|v_1\|_{L^3(|x-x_0(t)| \leq \rho^{-1})} \geq \frac{c}{(c^*)^3}.
\]
Moreover, we have \(|x_0(t)| \leq c(c^*)^6 \rho(t)^{-1} \).

Before proving Lemma 6.2, we establish the following.

Lemma 6.3. Under the above notation,
\[
C(\|u_{1H}\|_4^4 + \|v_{1H}\|_4^4) \leq \frac{1}{4(\beta + 1)}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2),
\]
where the constant \(C > 0\) appears in the proof of Lemma 6.2 below.

Proof. From the embedding \(H^{3/4}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)\) (see for instance Ref. [3, Theorem 6.5.1]), we obtain
\[
C(\|u_{1H}\|_4^4 + \|v_{1H}\|_4^4) \leq C\|u_{1H}\|_{H^{3/4}}^4 + \|v_{1H}\|_{H^{3/4}}^4
\]
\[
= C \left( \int \left| \xi \right|^{3/4}(1 - \hat{\psi}(\xi/\rho))\hat{u}_1(\xi,t) \right|^2 d\xi \right)^2
\]
\[
+ C \left( \int \left| \xi \right|^{3/4}(1 - \hat{\psi}(\xi/\rho))\hat{v}_1(\xi,t) \right|^2 d\xi \right)^2.
\]

Note, from the mean value theorem, that
\[
|1 - \hat{\psi}(\xi)| = |\hat{\psi}(0) - \hat{\psi}(\xi)| \leq C \min\{1, |\xi|\}. \quad (6.1)
\]
If \(|\xi| \leq \rho\), we have
\[
|\xi|^{3/4}|1 - \hat{\psi}(\xi/\rho)| \leq |\xi|^{3/4}\left|\frac{\xi}{\rho}\right| \leq \rho^{3/4}\left|\frac{\xi}{\rho}\right| = \frac{|\xi|}{\rho^{1/4}}. \quad (6.2)
\]
On the other hand, if \(|\xi| \geq \rho\),
\[
|\xi|^{3/4}|1 - \hat{\psi}(\xi/\rho)| \leq |\xi|^{3/4} \leq \rho^{3/4}\left|\frac{\xi}{\rho^{1/4}}\right| = \frac{|\xi|}{\rho^{1/4}}. \quad (6.3)
\]
Using (6.2) and (6.3) we then see that
\[
C(\|u_{1H}\|_4^4 + \|v_{1H}\|_4^4) \leq C \left( \int \left| \frac{\xi}{\rho^{1/4}}\hat{u}_1(\xi,t) \right|^2 d\xi \right)^2
\]
\[
+ C \left( \int \left| \frac{\xi}{\rho^{1/4}}\hat{v}_1(\xi,t) \right|^2 d\xi \right)^2
\]
\[
\leq C \frac{1}{\rho}(\|\xi\hat{u}_1\|_2^4 + \|\xi\hat{v}_1\|_2^4)
\]
\[
\leq C \frac{1}{\rho}(\|\nabla u\|_2^4 + \|\nabla v\|_2^4)
\]
From the definition of \(u_1\) and \(v_1\), and the fact that \(\|\nabla u(t)\|_2 + \|\nabla v(t)\|_2 \to \infty\), as \(t \to T^*\) (see Theorem 1.1), it is easy to see that, for \(t \sim T^*\), \(\|\nabla u\|_2 + \|\nabla v\|_2 \leq C(\|\nabla u\|_2^4 + \|\nabla v\|_2^4)\) (here and throughout, \(t \sim T^*\) means that \(t\) is sufficiently close to \(T^*\)). Thus,
An application of Lemma 2.6 (ii) yields
\[ C(\|u_{1H}\|_4^4 + \|v_{1H}\|_4^4) \leq C \frac{1}{\rho}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) \]
\[ \leq C \frac{1}{\rho}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^2 \]
\[ \leq \frac{1}{4(\beta + 1)}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2), \]
where the "suitable" constant \( c_2 \) in Lemma 6.2 has been chosen large enough so that the last inequality holds. \( \square \)

We now prove Lemma 6.2.

Proof of Lemma 6.2. In view of Theorem 1.1, we know that \( \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \to \infty \), as \( t \to T^* \). From (1.5), we have
\[ \frac{\|u\|_4^4 + 2\beta \|uv\|_2^2 + \|v\|_4^4}{\|\nabla u\|_2^2 + \|\nabla v\|_2^2} = 2 - \frac{4E(u_0, v_0)}{\|\nabla u\|_2^2 + \|\nabla v\|_2^2}. \]
Hence, for \( t \sim T^* \),
\[ \|u\|_4^4 + 2\beta \|uv\|_2^2 + \|v\|_4^4 \geq \|\nabla u\|_2^2 + \|\nabla v\|_2^2. \] (6.4)
In view of Young’s inequality and (6.4), we obtain
\[ \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \leq (\beta + 1)(\|u\|_4^4 + \|v\|_4^4) \]
\[ \leq C(\beta + 1)(\|u_{1L}\|_4^4 + \|u_{1H}\|_4^4 + \|u_2\|_4^4 + \|v_{1L}\|_4^4 + \|v_{1H}\|_4^4 + \|v_2\|_4^4). \] (6.5)
An application of Lemma 2.6 (ii) yields
\[ C(\|u_2\|_4^4 + \|v_2\|_4^4) \leq C\|u_2\|^{4+}_{L^4(|x| \geq R)} + C\|v_2\|^{4}_{L^4(|x| \geq R)} \]
\[ \leq C \frac{R}{2^2} \|u_0\|_2^2 \|\nabla u\|_2 + C \frac{R}{2^2} \|v_0\|_2^2 \|\nabla v\|_2 \]
\[ \leq C \frac{R}{2^2} (\|u_0\|_2 + \|v_0\|_2)^3(\|\nabla u\|_2 + \|\nabla v\|_2) \]
\[ \leq C \frac{1}{c_1} (\|\nabla u\|_2^2 + \|\nabla v\|_2^2) \]
\[ \leq \frac{1}{4(\beta + 1)}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2). \] (6.6)
Here the constant \( c_1 \) has also been chosen in such a way that the last inequality holds.

A combination of (6.4) - (6.6) with Lemma 6.3 gives
\[ \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \leq C(\beta + 1)(\|u_{1L}\|_4^4 + \|v_{1L}\|_4^4). \] (6.7)
Next, an application of the Gagliardo-Nirenberg inequality (2.15) leads to
\[ \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \leq C(\beta + 1)(\|u_{1L}\|_3^3 \|\nabla u_{1L}\|_2^2 + \|v_{1L}\|_3^3 \|\nabla v_{1L}\|_2^2) \]
\[ \leq C(\beta + 1)(\|u_{1L}\|_3 + \|v_{1L}\|_3)^2(\|\nabla u\|_2^2 + \|\nabla v\|_2^2). \] (6.8)
Therefore, inequality (6.8) promptly imply part (i) of the lemma.
To prove part (ii), we note that from (6.7),
\[
\| \nabla u \|_2^2 + \| \nabla v \|_2^2 \leq c(\| u_{1L} \|_\infty \| u_{1L} \|_3^3 + \| v_{1L} \|_\infty \| v_{1L} \|_3^3)
\]
\[
\leq c(c^*)^3 \sup_{x \in \mathbb{R}^3} \left( \| (\tilde{\psi}(:/\rho)) \| u_1(x) + \| (\tilde{\psi}(:/\rho)) \| v_1(x) \right). \tag{6.9}
\]
By choosing \( x_0 = x_0(t) \) such that \( 1/2 \) of this supremum is attained, and using Hőlder’s inequality we arrive to
\[
\| \nabla u \|_2^2 + \| \nabla v \|_2^2 \leq c(c^*)^3 |\rho|^3 \int \psi(\rho(x_0 - y))(u_1(y,t) + v_1(y,t)) dy
\]
\[
\leq c(c^*)^3 |\rho|^3 \int_{|\rho| |x_0 - y| \leq 1} (|u_1(y,t)| + |v_1(y,t)|) dy \tag{6.10}
\]
\[
\leq c(c^*)^3 |\rho| \left( \| u_1 \|_{L^3(|x_0 - x| \leq \rho^{-1})} + \| v_1 \|_{L^3(|x_0 - x| \leq \rho^{-1})} \right).
\]
The definition of \( \rho \) yields the inequality claimed in part (ii). To see that \( |x_0(t)| \leq c(c^*)^6 |\rho(t)|^{-1} \) one proceeds as in Ref. [11, Proposition 3.1]. \( \square \)

Finally, we prove Theorem 1.5.

\textbf{Proof of Theorem 1.5.} Since \( \| u_{1L} \|_3 + \| v_{1L} \|_3 \leq \| u_1 \|_3 + \| v_1 \|_3 \), it follows from Lemma 6.2 that \( \| u_1(t) \|_3 + \| v_1(t) \|_3 \) is bounded from below. We now divide the proof into two cases.

\textbf{Case 1.} If \( \| u_1(t) \|_3 + \| v_1(t) \|_3 \) is not bounded. In this case, there is a sequence \( t_n \to T^* \) such that \( \| u_1(t_n) \|_3 + \| v_1(t_n) \|_3 \to \infty \). Since
\[
\| u_1(t_n) \|_3^3 + \| v_1(t_n) \|_3^3 = \int_{|x| \leq 2R(t_n)} \left| \phi \left( \frac{x}{R(t_n)} \right) u(x,t_n) \right|^3 + \left| \phi \left( \frac{x}{R(t_n)} \right) v(x,t_n) \right|^3 dx
\]
\[
\leq \int_{|x| \leq R(t_n)} \left| u(x,t_n) \right|^3 + \left| v(x,t_n) \right|^3 dx,
\]
(6.11)

taking the limit in (6.11), as \( n \to \infty \), we obtain (1.14).

\textbf{Case 2.} If there is \( c^* > 0 \) such \( \| u_1(t) \|_3 + \| u_1(t) \|_3 \leq c^* \). Here, Lemma 6.2 part (ii) implies that
\[
\frac{c}{(c^*)^3} \leq \left( \int_{|x-x_0| \leq \rho^{-1}} |u_1(x,t)|^3 dx \right)^{1/3} + \left( \int_{|x-x_0| \leq \rho^{-1}} |v_1(x,t)|^3 dx \right)^{1/3}
\]
\[
\leq \left( \int_{|x| \leq |x_0| + \rho^{-1}} |u_1(x,t)|^3 dx \right)^{1/3} + \left( \int_{|x| \leq |x_0| + \rho^{-1}} |v_1(x,t)|^3 dx \right)^{1/3} \tag{6.12}
\]
Since for \( t \sim T^* \), \( |x_0| + \rho^{-1} \leq c(c^*)^6 \rho^{-1} + \rho^{-1} = c(c^*)^6 \rho^{-1} \). Inequality (6.12) gives (1.13) and the proof of the theorem is completed. \( \square \)

\textbf{Remark 6.4.} Note that Case 1 in the proof of Theorem 1.5 includes the case when either \( \| u_1(t) \|_3 \) or \( \| v_1(t) \|_3 \) is unbounded. It should be interesting to know when only one of these quantities is unbounded.

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