(3.1) Consider a fluid of constant density in two dimensions with gravity, and suppose that the vorticity $v_x - u_y$ is everywhere constant and equal to ω. Show that the velocity field has the form $(u, v) = (\phi_x + \chi_y, \phi_y - \chi_x)$ where ϕ is harmonic and χ is any function of x, y (independent of t) satisfying $\nabla^2\chi = -\omega$. Show further that

$$\nabla\left(\phi_t + \frac{1}{2}q^2 + \omega\psi + \frac{p}{\rho} + gz\right) = 0$$

where ψ is the stream function for $\mathbf{u}$; i.e., $\mathbf{u} = (\psi_y, -\psi_x)$ and $q^2 = u^2 + v^2$.

(3.3) For steady two-dimensional flow of a fluid of constant density, we have

$$\rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = 0, \quad \nabla\cdot\mathbf{u} = 0.$$

Show that, if $\mathbf{u} = (\psi_y, -\psi_x)$, these equations imply

$$\nabla\psi \times \nabla(\nabla^2\psi) = 0.$$

Thus, show that a solution is obtained by giving a function $H(\psi)$ and then solving $\nabla^2\psi = H'(\psi)$. Show also that the pressure is given by $\frac{p}{\rho} = H(\psi) - \frac{1}{2}(\nabla\psi)^2 +$ const.

(3.4) Prove *Ertel's theorem* for a fluid of constant density: If f is a scalar material invariant, i.e., $\frac{Df}{Dt} = 0$, then $\boldsymbol{\omega}\cdot\nabla f$ is also a material invariant, where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the vorticity field.

(3.5) A steady *Beltrami flow* is a velocity field $\mathbf{u}(\mathbf{x})$ for which the vorticity is always parallel to the velocity, i.e., $\nabla\times\mathbf{u} = f(\mathbf{x})\mathbf{u}$ for some scalar function f. Show that if a steady Beltrami field is also the steady velocity field of an inviscid fluid of constant density, then necessarily f is constant on streamlines. What is the corresponding pressure? Show that

$$\mathbf{u} = (B\sin y + C\cos z, C\sin z + A\cos x, A\sin x + B\cos y)$$

is such a Beltrami field with $f = -1$.

(3.6) Another formula exhibiting a vector field $\mathbf{u} = (u, v, w)$ whose curl is $\boldsymbol{\omega} = (\xi, \eta, \zeta)$, where $\nabla\cdot\boldsymbol{\omega} = 0$, is given by

$$u = z\int_0^1 t\eta(tx, ty, tz)dt - y\int_0^1 t\zeta(tx, ty, tz)dt,$$

$$v = x\int_0^1 t\zeta(tx, ty, tz)dt - z\int_0^1 t\xi(tx, ty, tz)dt,$$

$$w = y\int_0^1 t\xi(tx, ty, tz)dt - x\int_0^1 t\eta(tx, ty, tz)dt.$$

Verify this result. (Note that $\mathbf{u}$ will not in general be divergence-free, e.g., check $\xi = \zeta = 0$, $\eta = x$.)

5.1. Let a closed circuit C of fluid particles be given, at $t=0$, by

$$\boldsymbol{x} = (a\cos s,\ a\sin s,\ 0), \qquad 0 \leqslant s < 2\pi,$$

so that each value of s between 0 and 2π corresponds to a particular fluid particle. Let $C(t)$ be given subsequently by

$$\boldsymbol{x} = (a\cos s + a\alpha t\sin s,\ a\sin s,\ 0), \quad 0 \leqslant s < 2\pi.$$

Find the velocity $\boldsymbol{u}(s, t)$ of each fluid particle, and show that the particles $s=0$ and $s=\pi$ remain at rest. Find the acceleration of each fluid particle, show that

$$\boldsymbol{u} = (\alpha y,\ 0,\ 0),$$

and sketch how the shape of $C(t)$ changes with time.

Now, by definition,

$$\Gamma = \int_{C(t)} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \int_0^{2\pi} \boldsymbol{u} \cdot \frac{\partial \boldsymbol{x}}{\partial s}\,\mathrm{d}s.$$

Calculate the last integral explicitly at time t, confirming that it is independent of t, in accord with Kelvin's circulation theorem.

5.3. Let an ideal fluid be in 2-D motion. By virtue of eqn (5.9) the vorticity ω of any fluid element is conserved. The fluid element must also conserve its volume, and because it is not being stretched in the z-direction its cross-sectional area δS in the x–y plane must therefore be conserved. It follows that the integral

$$\int \omega\,\mathrm{d}S$$

taken over a dyed cross-section S in the x–y plane, must be independent of time. By Stokes's theorem, or by Green's theorem in the plane (A.24), it follows that Γ, the circulation round the dyed circuit which forms the perimeter of S, must also be independent of time.

This is in some respects a nice way of seeing how Kelvin's circulation theorem comes about. It is, however, a wholly 2-D argument, and that theorem is certainly not restricted to 2-D flows. What is the other serious limitation to the above point of view?