7. If f is locally integrable on $\mathbb{R}^n$ and $g \in C^k$ has compact support, then $f * g \in C^k$.

14. (**Wirtinger's Inequality**) If $f \in C^1([a,b])$ and $f(a) = f(b) = 0$, then

$$\int_a^b |f(x)|^2\,dx \le \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(x)|^2\,dx.$$

(By a change of variable it suffices to assume $a = 0$, $b = \frac{1}{2}$. Extend f to $[-\frac{1}{2}, \frac{1}{2}]$ by setting $f(-x) = -f(x)$, and then extend f to be periodic on $\mathbb{R}$. Check that f, thus extended, is in $C^1(\mathbb{T})$ and apply the Parseval identity.)

18. Suppose $f \in L^2(\mathbb{R})$.

a. The L^2 derivative f' (in the sense of Exercises 8 and 9) exists iff $\xi\widehat{f} \in L^2$, in which case $\widehat{f'}(\xi) = 2\pi i\xi\widehat{f}(\xi)$.

b. If the L^2 derivative f' exists, then

$$\left[\int |f(x)|^2\,dx\right] \le 4\int |xf(x)|^2\,dx \int |f'(x)|^2\,dx.$$

(If the integrals on the right are finite, one can integrate by parts to obtain $\int |f|^2 = -2\,\mathrm{Re}\int x\overline{f}f'$.)

c. (**Heisenberg's Inequality**) For any $b, \beta \in \mathbb{R}$,

$$\int (x-b)^2|f(x)|^2\,dx \int (\xi-\beta)^2|\widehat{f}(\xi)|^2\,d\xi \ge \frac{\|f\|_2^4}{16\pi^2}.$$

(The inequality is trivial if either integral on the right is infinite; if not, reduce to the case $b = \beta = 0$ by considering $g(x) = e^{-2\pi i\beta x}f(x+b)$.) This inequality, a form of the quantum uncertainty principle, says that f and $\widehat{f}$ cannot both be sharply localized about single points b and β.

2.2.12. Let $1 \le p \le \infty$ and let p' be its dual index.

(a) Prove that Schwartz functions f on the line satisfy the estimate

$$\|f\|_{L^\infty}^2 \le 2\|f\|_{L^p}\|f'\|_{L^{p'}}.$$

(b) Prove that all Schwartz functions f on $\mathbf{R}^n$ satisfy the estimate

$$\|f\|_{L^\infty}^2 \le 2 \sum_{\alpha+\beta=(1,\dots,1)} \|\partial^\alpha f\|_{L^p}\|\partial^\beta f\|_{L^{p'}},$$

where the sum is taken over all multi-indices α and β whose sum is $(1,1,\dots,1)$.

[*Hint:* Part (a): Write $f(x)^2 = \int_{-\infty}^x \frac{d}{dt}f(t)^2\,dt$.]