9. In the case of the plucked string, use the formula for the Fourier sine coefficients to show that

$$A_m = \frac{2h}{m^2} \frac{\sin mp}{p(\pi - p)}.$$

For what position of p are the second, fourth, ... harmonics missing? For what position of p are the third, sixth, ... harmonics missing?

10. Show that the expression of the Laplacian

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is given in polar coordinates by the formula

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Also, prove that

$$\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2.$$

1. Consider the Dirichlet problem illustrated in Figure 11.

More precisely, we look for a solution of the steady-state heat equation $\triangle u = 0$ in the rectangle $R = \{(x, y) : \; 0 \leq x \leq \pi, \quad 0 \leq y \leq 1\}$ that vanishes on the vertical sides of R, and so that

$$u(x, 0) = f_0(x) \qquad \text{and} \qquad u(x, 1) = f_1(x)\,,$$

where f_0 and f_1 are initial data which fix the temperature distribution on the horizontal sides of the rectangle.

Use separation of variables to show that if f_0 and f_1 have Fourier expansions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \qquad \text{and} \qquad f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx,$$

then

$$u(x, y) = \sum_{k=1}^{\infty} \left(\frac{\sinh k(1-y)}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k \right) \sin kx.$$

We recall the definitions of the hyperbolic sine and cosine functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

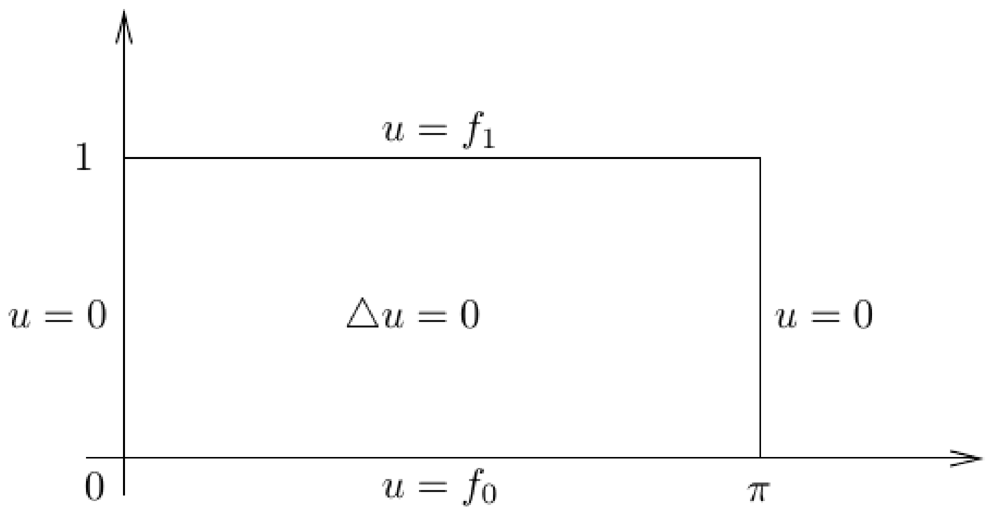

Figure 11. Dirichlet problem in a rectangle

1. Show that $u(x,t) = t^{-1/2}\exp(-x^2/4kt)$ satisfies the heat equation $u_t = ku_{xx}$ for $t > 0$.
2. Show that $u(x,y,t) = t^{-1}\exp[-(x^2+y^2)/4kt]$ satisfies the heat equation $u_t = k(u_{xx} + u_{yy})$ for $t > 0$.
3. Show that $u(x,y) = \log(x^2+y^2)$ satisfies Laplace's equation $u_{xx} + u_{yy} = 0$ for $(x,y) \neq (0,0)$.
4. Show that $u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ satisfies Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$ for $(x,y,z) \neq (0,0,0)$.

7. The voltage v and current i in an electrical cable along the x-axis satisfy the coupled equations

$$i_x + Cv_t + Gv = 0, \qquad v_x + Li_t + Ri = 0,$$

where C, G, L, and R are the capacitance, (leakage) conductance, inductance, and resistance per unit length in the cable. Show that v and i both satisfy the **telegraph equation**

$$u_{xx} = LCu_{tt} + (RC + LG)u_t + RGu.$$

8. Set $u(x,t) = f(x,t)e^{at}$ in the telegraph equation of Exercise 7. What is the differential equation satisfied by f? Show that a can be chosen so that this equation is of the form $f_{xx} = Af_{tt} + Bf$ (with no first-order term), provided that $LC \neq 0$.

1. Suppose u_1 and u_2 are both solutions of the linear differential equation $L(u) = f$, where $f \neq 0$. Under what conditions is the linear combination $c_1 u_1 + c_2 u_2$ also a solution of this equation?
2. Consider the nonlinear (ordinary) differential equation $u' = u(1 - u)$.
   a. Show that $u_1(x) = e^x/(1 + e^x)$ and $u_2(x) = 1$ are solutions.
   b. Show that $u_1 + u_2$ is not a solution.
   c. For which values of c is cu_1 a solution? How about cu_2?
3. Give examples of linear differential operators L and M for which it is not true that $L(M(u)) = M(L(u))$ for all u. (Hint: At least one of L and M must have nonconstant coefficients.)
4. What form must G have for the differential equation $u_{tt} - u_{xx} = G(x, t, u)$ to be linear? Linear and homogeneous?
5. a. Show that for $n = 1, 2, 3, \ldots$, $u_n(x, y) = \sin(n\pi x)\sinh(n\pi y)$ satisfies

$$u_{xx} + u_{yy} = 0, \qquad u(0, y) = u(1, y) = u(x, 0) = 0.$$

   b. Find a linear combination of the u_n's that satisfies $u(x, 1) = \sin 2\pi x - \sin 3\pi x$.
   c. Show that for $n = 1, 2, 3, \ldots$, $\widetilde{u}_n(x, y) = \sin(n\pi x)\sinh n\pi(1 - y)$ satisfies

$$u_{xx} + u_{yy} = 0, \qquad u(0, y) = u(1, y) = u(x, 1) = 0.$$

   d. Find a linear combination of the $\widetilde{u}_n$'s that satisfies $u(x, 0) = 2\sin \pi x$.
   e. Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad u(0, y) = u(1, y) = 0,$$
$$u(x, 0) = 2\sin \pi x, \quad u(x, 1) = \sin 2\pi x - \sin 3\pi x.$$

1. Derive pairs of ordinary differential equations from the following partial differential equations by separation of variables, or show that it is not possible.
   a. $yu_{xx} + u_y = 0$.
   b. $x^2u_{xx} + xu_x + u_{yy} + u = 0$.
   c. $u_{xx} + u_{xy} + u_{yy} = 0$.
   d. $u_{xx} + u_{xy} + u_y = 0$.
2. Derive sets of three ordinary differential equations from the following partial differential equations by separation of variables.
   a. $yu_{xx} + xu_{yy} + xyu_{zz} = 0$.
   b. $x^2u_{xx} + xu_x + u_{yy} + x^2u_{zz} = 0$.
3. Use the results in the text to solve

$$u_{tt} = 9u_{xx}, \qquad u(0,t) = u(1,t) = 0,$$
$$u(x,0) = 2\sin \pi x - 3\sin 4\pi x, \qquad u_t(x,0) = 0 \qquad (0 < x < 1).$$

4. Use the results in the text to solve

$$u_t = \tfrac{1}{10}u_{xx}, \qquad u_x(0,t) = u_x(\pi,t) = 0,$$
$$u(x,0) = 3 - 4\cos 2x \qquad (0 < x < \pi).$$

   Determine a value of t_0 so that $|u(x,t) - 3| < 10^{-4}$ for $t > t_0$.

5. By separation of variables, derive the solutions $u_n(x,y) = \sin n\pi x \sinh n\pi y$ of

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(1,y) = u(x,0) = 0$$

   that were discussed in Exercise 5a, §1.2.
6. By separation of variables, derive the family

$$u_{mn}^{\pm}(x,y,z) = \sin m\pi x \cos n\pi y \exp\left(\pm\sqrt{m^2+n^2}\,\pi z\right)$$

   of the problem

$$\nabla^2 u = 0, \qquad u(0,y,z) = u(1,y,z) = u_y(x,0,z) = u_y(x,1,z) = 0.$$

7. Use separation of variables to find an infinite family of independent solutions of

$$u_t = ku_{xx}, \qquad u(0,t) = 0, \qquad u_x(l,t) = 0,$$

   representing heat flow in a rod with one end held at temperature zero and the other end insulated.