4. Observe that with the definition of ℓ and $\mathcal{A}$ given in the text, the isoperimetric inequality continues to hold (with the same proof) even when Γ is not simple.

Show that this stronger version of the isoperimetric inequality is equivalent to Wirtinger's inequality, which says that if f is 2π-periodic, of class C^1, and satisfies $\int_0^{2\pi} f(t)\,dt = 0$, then

$$\int_0^{2\pi} |f(t)|^2\,dt \leq \int_0^{2\pi} |f'(t)|^2\,dt$$

with equality if and only if $f(t) = A\sin t + B\cos t$ (Exercise 11, Chapter 3).

[Hint: In one direction, note that if the length of the curve is 2π and γ is an appropriate arc-length parametrization, then

$$2(\pi - \mathcal{A}) = \int_0^{2\pi} \left[x'(s) + y(s)\right]^2\,ds + \int_0^{2\pi} (y'(s)^2 - y(s)^2)\,ds.$$

A change of coordinates will guarantee $\int_0^{2\pi} y(s)\,ds = 0$. For the other direction, start with a real-valued f satisfying all the hypotheses of Wirtinger's inequality, and construct g, 2π-periodic and so that the term in brackets above vanishes.]

11. Show that if $u(x,t) = (f * H_t)(x)$ where H_t is the heat kernel, and f is Riemann integrable, then

$$\int_0^1 |u(x,t) - f(x)|^2\,dx \to 0 \qquad \text{as } t \to 0.$$

12. A change of variables in (8) leads to the solution

$$u(\theta,\tau) = \sum a_n e^{-n^2\tau} e^{in\theta} = (f * h_\tau)(\theta)$$

of the equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \theta^2} \qquad \text{with } 0 \leq \theta \leq 2\pi \text{ and } \tau > 0,$$

with boundary condition $u(\theta,0) = f(\theta) \sim \sum a_n e^{in\theta}$. Here $h_\tau(\theta) = \sum_{n=-\infty}^{\infty} e^{-n^2\tau} e^{in\theta}$. This version of the heat kernel on $[0,2\pi]$ is the analogue of the Poisson kernel, which can be written as $P_r(\theta) = \sum_{n=-\infty}^{\infty} e^{-|n|\tau} e^{in\theta}$ with $r = e^{-\tau}$ (and so $0 < r < 1$ corresponds to $\tau > 0$).

13. The fact that the kernel $H_t(x)$ is a good kernel, hence $u(x,t) \to f(x)$ at each point of continuity of f, is not easy to prove. This will be shown in the next chapter. However, one can prove directly that $H_t(x)$ is "peaked" at $x = 0$ as $t \to 0$ in the following sense:

(a) Show that $\int_{-1/2}^{1/2} |H_t(x)|^2\,dx$ is of the order of magnitude of $t^{-1/2}$ as $t \to 0$. More precisely, prove that $t^{1/2}\int_{-1/2}^{1/2} |H_t(x)|^2\,dx$ converges to a non-zero limit as $t \to 0$.

(b) Prove that $\int_{-1/2}^{1/2} x^2 |H_t(x)|^2\,dx = O(t^{1/2})$ as $t \to 0$.

[Hint: For (a) compare the sum $\sum_{-\infty}^{\infty} e^{-cn^2 t}$ with the integral $\int_{-\infty}^{\infty} e^{-cx^2 t}\,dx$ where $c > 0$. For (b) use $x^2 \leq C(\sin \pi x)^2$ for $-1/2 \leq x \leq 1/2$, and apply the mean value theorem to $e^{-cx^2 t}$.]

2. Let f and g be the functions defined by

$$f(x) = \chi_{[-1,1]}(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Although f is not continuous, the integral defining its Fourier transform still makes sense. Show that

$$\hat{f}(\xi) = \frac{\sin 2\pi\xi}{\pi\xi} \quad \text{and} \quad \hat{g}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2,$$

with the understanding that $\hat{f}(0) = 2$ and $\hat{g}(0) = 1$.

3. The following exercise illustrates the principle that the decay of $\hat{f}$ is related to the continuity properties of f.

(a) Suppose that f is a function of moderate decrease on $\mathbb{R}$ whose Fourier transform $\hat{f}$ is continuous and satisfies

$$\hat{f}(\xi) = O\left(\frac{1}{|\xi|^{1+\alpha}}\right) \quad \text{as } |\xi| \to \infty$$

for some $0 < \alpha < 1$. Prove that f satisfies a Hölder condition of order α, that is, that

$$|f(x+h) - f(x)| \leq M|h|^\alpha \quad \text{for some } M > 0 \text{ and all } x, h \in \mathbb{R}.$$

(b) Let f be a continuous function on $\mathbb{R}$ which vanishes for $|x| \geq 1$, with $f(0) = 0$, and which is equal to $1/\log(1/|x|)$ for all x in a neighborhood of the origin. Prove that $\hat{f}$ is not of moderate decrease. In fact, there is no $\epsilon > 0$ so that $\hat{f}(\xi) = O(1/|\xi|^{1+\epsilon})$ as $|\xi| \to \infty$.

[Hint: For part (a), use the Fourier inversion formula to express $f(x+h) - f(x)$ as an integral involving $\hat{f}$, and estimate this integral separately for ξ in the two ranges $|\xi| \leq 1/|h|$ and $|\xi| \geq 1/|h|$.]

4. Bump functions. Examples of compactly supported functions in $\mathcal{S}(\mathbb{R})$ are very handy in many applications in analysis. Some examples are:

(a) Suppose $a < b$, and f is the function such that $f(x) = 0$ if $x \leq a$ or $x \geq b$ and

$$f(x) = e^{-1/(x-a)}e^{-1/(b-x)} \quad \text{if } a < x < b.$$

Show that f is indefinitely differentiable on $\mathbb{R}$.

(b) Prove that there exists an indefinitely differentiable function F on $\mathbb{R}$ such that $F(x) = 0$ if $x \leq a$, $F(x) = 1$ if $x \geq b$, and F is strictly increasing on $[a, b]$.

(c) Let $\delta > 0$ be so small that $a + \delta < b - \delta$. Show that there exists an indefinitely differentiable function g such that g is 0 if $x \leq a$ or $x \geq b$, g is 1 on $[a + \delta, b - \delta]$, and g is strictly monotonic on $[a, a + \delta]$ and $[b - \delta, b]$.

[Hint: For (b) consider $F(x) = c\int_{-\infty}^{x} f(t)\,dt$ where c is an appropriate constant.]

5. Suppose f is continuous and of moderate decrease.

(a) Prove that $\hat{f}$ is continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

(b) Show that if $\hat{f}(\xi) = 0$ for all ξ, then f is identically 0.

[Hint: For part (a), show that $\hat{f}(\xi) = \frac{1}{2}\int_{-\infty}^{\infty}[f(x) - f(x - 1/(2\xi))]e^{-2\pi ix\xi}\,dx$. For part (b), verify that the multiplication formula $\int f(x)\hat{g}(x)\,dx = \int \hat{f}(y)g(y)\,dy$ still holds whenever $g \in \mathcal{S}(\mathbb{R})$.]

7. Prove that the convolution of two functions of moderate decrease is a function of moderate decrease.

[Hint: Write

$$\int f(x-y)g(y)\,dy = \int_{|y|\leq|x|/2} + \int_{|y|\geq|x|/2}.$$

In the first integral $f(x-y) = O(1/(1+x^2))$ while in the second integral $g(y) = O(1/(1+x^2))$.]

8. Prove that f is continuous, of moderate decrease, and $\int_{-\infty}^{\infty} f(y)e^{-y^2}e^{2xy}dy = 0$ for all $x \in \mathbb{R}$, then $f = 0$.

[Hint: Consider $f * e^{-x^2}$.]

9. If f is of moderate decrease, then

$$\int_{-R}^{R} \left(1 - \frac{|\xi|}{R}\right) \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi = (f * \mathcal{F}_R)(x), \tag{14}$$

where the Fejér kernel on the real line is defined by

$$\mathcal{F}_R(t) = \begin{cases} R\left(\dfrac{\sin \pi t R}{\pi t R}\right)^2 & \text{if } t \neq 0, \\ R & \text{if } t = 0. \end{cases}$$

Show that $\{\mathcal{F}_R\}$ is a family of good kernels as $R \to \infty$, and therefore (14) tends uniformly to $f(x)$ as $R \to \infty$. This is the analogue Fejér's theorem for Fourier series in the context of the Fourier transform.