11. Suppose that u is the solution to the heat equation given by $u = f * \mathcal{H}_t$ where $f \in \mathcal{S}(\mathbb{R})$. If we also set $u(x, 0) = f(x)$, prove that u is continuous on the closure of the upper half-plane, and vanishes at infinity, that is,

$$u(x,t) \to 0 \qquad \text{as } |x| + t \to \infty.$$

[Hint: To prove that u vanishes at infinity, show that (i) $|u(x,t)| \le C/\sqrt{t}$ and (ii) $|u(x,t)| \le C/(1+|x|^2) + Ct^{-1/2}e^{-cx^2/t}$. Use (i) when $|x| \le t$, and (ii) otherwise.]

12. Show that the function defined by

$$u(x,t) = \frac{x}{t}\mathcal{H}_t(x)$$

satisfies the heat equation for $t > 0$ and $\lim_{t\to 0} u(x,t) = 0$ for every x, but u is *not* continuous at the origin.

[Hint: Approach the origin with (x,t) on the parabola $x^2/4t = c$ where c is a constant.]

21. Suppose that f is continuous on $\mathbb{R}$. Show that f and $\hat{f}$ cannot both be compactly supported unless $f = 0$. This can be viewed in the same spirit as the uncertainty principle.

7. Consider the time-dependent heat equation in $\mathbb{R}^d$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}, \qquad \text{where } t > 0, \tag{15}$$

with boundary values $u(x,0) = f(x) \in \mathcal{S}(\mathbb{R}^d)$. If

$$\mathcal{H}_t^{(d)}(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t} = \int_{\mathbb{R}^d} e^{-4\pi^2 t|\xi|^2} e^{2\pi i x\cdot\xi}\, d\xi$$

is the d-dimensional **heat kernel**, show that the convolution

$$u(x,t) = (f * \mathcal{H}_t^{(d)})(x)$$

is indefinitely differentiable when $x \in \mathbb{R}^d$ and $t > 0$. Moreover, u solves (15), and is continuous up to the boundary $t = 0$ with $u(x,0) = f(x)$.

The reader may also wish to formulate the d-dimensional analogues of Theorem 2.1 and 2.3 in Chapter 5.

8. In Chapter 5, we found that a solution to the steady-state heat equation in the upper half-plane with boundary values f is given by the convolution $u = f * \mathcal{P}_y$ where the Poisson kernel is

$$\mathcal{P}_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2} \qquad \text{where } x \in \mathbb{R} \text{ and } y > 0.$$

More generally, one can calculate the d-dimensional Poisson kernel using the Fourier transform as follows.

(a) The **subordination principle** allows one to write expressions involving the function e^{-x} in terms of corresponding expressions involving the function e^{-x^2}. One form of this is the identity

$$e^{-\beta} = \int_0^\infty \frac{e^{-u}}{\sqrt{\pi u}} e^{-\beta^2/4u}\, du$$

when $\beta \geq 0$. Prove this identity with $\beta = 2\pi|x|$ by taking the Fourier transform of both sides.

(b) Consider the steady-state heat equation in the upper half-space $\{(x, y) : x \in \mathbb{R}^d,\ y > 0\}$

$$\sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with the Dirichlet boundary condition $u(x, 0) = f(x)$. A solution to this problem is given by the convolution $u(x, y) = (f * P_y^{(d)})(x)$ where $P_y^{(d)}(x)$ is the d-dimensional Poisson kernel

$$P_y^{(d)}(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-2\pi|\xi|y}\, d\xi.$$

Compute $P_y^{(d)}(x)$ by using the subordination principle and the d-dimensional heat kernel. (See Exercise 7.) Show that

$$P_y^{(d)}(x) = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \frac{y}{(|x|^2 + y^2)^{(d+1)/2}}.$$

10. Let $u(x,t)$ be a solution of the wave equation, and let $E(t)$ denote the energy of this wave

$$E(t) = \int_{\mathbb{R}^d} \left| \frac{\partial u}{\partial t}(x,t) \right|^2 + \sum_{j=1}^{d} \int_{\mathbb{R}^d} \left| \frac{\partial u}{\partial x_j}(x,t) \right|^2 \, dx.$$

We have seen that $E(t)$ is constant using Plancherel's formula. Give an alternate proof of this fact by differentiating the integral with respect to t and showing that

$$\frac{dE}{dt} = 0.$$

[Hint: Integrate by parts.]