**8.** Verify that $\frac{1}{2i}\sum_{n\neq 0}\frac{e^{inx}}{n}$ is the Fourier series of the $2\pi$-periodic **sawtooth** function illustrated in Figure 6, defined by $f(0)=0$, and

$$f(x)=\begin{cases} -\frac{\pi}{2}-\frac{x}{2} & \text{if } -\pi<x<0, \\ \frac{\pi}{2}-\frac{x}{2} & \text{if } 0<x<\pi. \end{cases}$$

Note that this function is not continuous. Show that nevertheless, the series converges for every $x$ (by which we mean, as usual, that the symmetric partial sums of the series converge). In particular, the value of the series at the origin, namely 0, is the average of the values of $f(x)$ as $x$ approaches the origin from the left and the right.

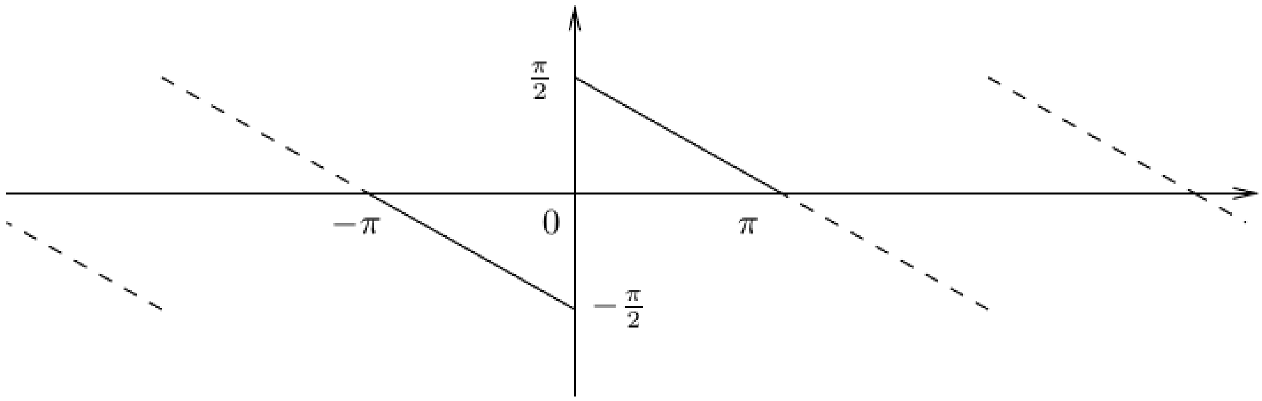


**Figure 6.** The sawtooth function

**9.** Let $f(x)=\chi_{[a,b]}(x)$ be the characteristic function of the interval $[a,b]\subset[-\pi,\pi]$, that is,

$$\chi_{[a,b]}(x)=\begin{cases} 1 & \text{if } x\in[a,b], \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the Fourier series of $f$ is given by

$$f(x)\sim\frac{b-a}{2\pi}+\sum_{n\neq 0}\frac{e^{-ina}-e^{-inb}}{2\pi in}\,e^{inx}.$$

The sum extends over all positive and negative integers excluding 0.

(b) Show that if $a\neq-\pi$ or $b\neq\pi$ and $a\neq b$, then the Fourier series does not converge absolutely for any $x$. [Hint: It suffices to prove that for many values of $n$ one has $|\sin n\theta_0|\geq c>0$ where $\theta_0=(b-a)/2$.]

(c) However, prove that the Fourier series converges at every point $x$. What happens if $a=-\pi$ and $b=\pi$?

**13.** The purpose of this exercise is to prove that Abel summability is stronger than the standard or Cesàro methods of summation.

(a) Show that if the series $\sum_{n=1}^{\infty} c_n$ of complex numbers converges to a finite limit $s$, then the series is Abel summable to $s$. [Hint: Why is it enough to prove the theorem when $s = 0$? Assuming $s = 0$, show that if $s_N = c_1 + \cdots + c_N$, then $\sum_{n=1}^{N} c_n r^n = (1-r)\sum_{n=1}^{N} s_n r^n + s_N r^{N+1}$. Let $N \to \infty$ to show that

$$\sum c_n r^n = (1-r)\sum s_n r^n.$$

Finally, prove that the right-hand side converges to 0 as $r \to 1$.]

(b) However, show that there exist series which are Abel summable, but that do not converge. [Hint: Try $c_n = (-1)^n$. What is the Abel limit of $\sum c_n$?]

(c) Argue similarly to prove that if a series $\sum_{n=1}^{\infty} c_n$ is Cesàro summable to $\sigma$, then it is Abel summable to $\sigma$. [Hint: Note that

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n,$$

and assume $\sigma = 0$.]

(d) Give an example of a series that is Abel summable but not Cesàro summable. [Hint: Try $c_n = (-1)^{n-1}n$. Note that if $\sum c_n$ is Cesàro summable, then $c_n/n$ tends to 0.]

The results above can be summarized by the following implications about series:

$$\text{convergent} \Longrightarrow \text{Cesàro summable} \Longrightarrow \text{Abel summable},$$

and the fact that none of the arrows can be reversed.

**16.** The Weierstrass approximation theorem states: Let $f$ be a continuous function on the closed and bounded interval $[a,b] \subset \mathbb{R}$. Then, for any $\epsilon > 0$, there exists a polynomial $P$ such that

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon.$$

Prove this by applying Corollary 5.4 of Fejér's theorem and using the fact that the exponential function $e^{ix}$ can be approximated by polynomials uniformly on any interval.

**17.** In Section 5.4 we proved that the Abel means of $f$ converge to $f$ at all points of continuity, that is,

$$\lim_{r\to 1} A_r(f)(\theta) = \lim_{r\to 1} (P_r * f)(\theta) = f(\theta), \qquad \text{with } 0 < r < 1,$$

whenever $f$ is continuous at $\theta$. In this exercise, we will study the behavior of $A_r(f)(\theta)$ at certain points of discontinuity.

An integrable function is said to have a **jump discontinuity** at $\theta$ if the two limits

$$\lim_{\substack{h \to 0 \\ h > 0}} f(\theta + h) = f(\theta^+) \qquad \text{and} \qquad \lim_{\substack{h \to 0 \\ h > 0}} f(\theta - h) = f(\theta^-)$$

exist.

(a) Prove that if $f$ has a jump discontinuity at $\theta$, then

$$\lim_{r\to 1} A_r(f)(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}, \qquad \text{with } 0 \le r < 1.$$

[Hint: Explain why $\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta)\,d\theta = \frac{1}{2\pi}\int_{0}^{\pi} P_r(\theta)\,d\theta = \frac{1}{2}$, then modify the proof given in the text.]

(b) Using a similar argument, show that if $f$ has a jump discontinuity at $\theta$, the Fourier series of $f$ at $\theta$ is Cesàro summable to $\frac{f(\theta^+)+f(\theta^-)}{2}$.

**18.** If $P_r(\theta)$ denotes the Poisson kernel, show that the function

$$u(r,\theta) = \frac{\partial P_r}{\partial \theta},$$

defined for $0 \le r < 1$ and $\theta \in \mathbb{R}$, satisfies:

(i) $\triangle u = 0$ in the disc.

(ii) $\lim_{r\to 1} u(r,\theta) = 0$ for each $\theta$.

However, $u$ is not identically zero.

**19.** Solve Laplace's equation $\triangle u = 0$ in the semi infinite strip

$$S = \{(x, y) : 0 < x < 1,\ 0 < y\},$$

subject to the following boundary conditions

$$\begin{cases} u(0, y) = 0 & \text{when } 0 \leq y, \\ u(1, y) = 0 & \text{when } 0 \leq y, \\ u(x, 0) = f(x) & \text{when } 0 \leq x \leq 1 \end{cases}$$

where $f$ is a given function, with of course $f(0) = f(1) = 0$. Write

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

and expand the general solution in terms of the special solutions given by

$$u_n(x, y) = e^{-n\pi y} \sin(n\pi x).$$

Express $u$ as an integral involving $f$, analogous to the Poisson integral formula (6).

**8.** Exercise 6 in Chapter 2 dealt with the sums

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Similar sums can be derived using the methods of this chapter.

(a) Let $f$ be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$. Use Parseval's identity to find the sums of the following two series:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

In fact, they are $\pi^4/96$ and $\pi^4/90$, respectively.

(b) Consider the $2\pi$-periodic odd function defined on $[0, \pi]$ by $f(\theta) = \theta(\pi - \theta)$. Show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

**Remark.** The general expression when $k$ is even for $\sum_{n=1}^{\infty} 1/n^k$ in terms of $\pi^k$ is given in Problem 4. However, finding a formula for the sum $\sum_{n=1}^{\infty} 1/n^3$, or more generally $\sum_{n=1}^{\infty} 1/n^k$ with $k$ odd, is a famous unresolved question.

**10.** Consider the example of a vibrating string which we analyzed in Chapter 1. The displacement $u(x,t)$ of the string at time $t$ satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \tau/\rho.$$

The string is subject to the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x),$$

where we assume that $f \in C^1$ and $g$ is continuous. We define the total **energy** of the string by

$$E(t) = \frac{1}{2}\rho\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\tau\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx.$$

The first term corresponds to the "kinetic energy" of the string (in analogy with $(1/2)mv^2$, the kinetic energy of a particle of mass $m$ and velocity $v$), and the second term corresponds to its "potential energy."

Show that the total energy of the string is conserved, in the sense that $E(t)$ is constant. Therefore,

$$E(t) = E(0) = \frac{1}{2}\rho\int_0^L g(x)^2\,dx + \frac{1}{2}\tau\int_0^L f'(x)^2\,dx.$$

**11.** The inequalities of Wirtinger and Poincaré establish a relationship between the norm of a function and that of its derivative.

(a) If $f$ is $T$-periodic, continuous, and piecewise $C^1$ with $\int_0^T f(t)\,dt = 0$, show that

$$\int_0^T |f(t)|^2\,dt \le \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2\,dt,$$

with equality if and only if $f(t) = A\sin(2\pi t/T) + B\cos(2\pi t/T)$. [Hint: Apply Parseval's identity.]

(b) If $f$ is as above and $g$ is just $C^1$ and $T$-periodic, prove that

$$\left|\int_0^T \overline{f(t)}g(t)\,dt\right|^2 \le \frac{T^2}{4\pi^2}\int_0^T |f(t)|^2\,dt \int_0^T |g'(t)|^2\,dt.$$

(c) For any compact interval $[a,b]$ and any continuously differentiable function $f$ with $f(a) = f(b) = 0$, show that

$$\int_a^b |f(t)|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2\,dt.$$

Discuss the case of equality, and prove that the constant $(b-a)^2/\pi^2$ cannot be improved. [Hint: Extend $f$ to be odd with respect to $a$ and periodic of period $T = 2(b-a)$ so that its integral over an interval of length $T$ is 0. Apply part a) to get the inequality, and conclude that equality holds if and only if $f(t) = A\sin(\pi\frac{t-a}{b-a})$].

**12.** Prove that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

[Hint: Start with the fact that the integral of $D_N(\theta)$ equals $2\pi$, and note that the difference $(1/\sin(\theta/2)) - 2/\theta$ is continuous on $[-\pi,\pi]$. Apply the Riemann-Lebesgue lemma.]

**16.** Let $f$ be a $2\pi$-periodic function which satisfies a Lipschitz condition with constant $K$; that is,

$$|f(x) - f(y)| \le K|x - y| \qquad \text{for all } x, y.$$

This is simply the Hölder condition with $\alpha = 1$, so by the previous exercise, we see that $\hat{f}(n) = O(1/|n|)$. Since the harmonic series $\sum 1/n$ diverges, we cannot say anything (yet) about the absolute convergence of the Fourier series of $f$. The outline below actually proves that the Fourier series of $f$ converges absolutely and uniformly.

(a) For every positive $h$ we define $g_h(x) = f(x + h) - f(x - h)$. Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2\,dx = \sum_{n=-\infty}^{\infty} 4|\sin nh|^2|\hat{f}(n)|^2,$$

and show that

$$\sum_{n=-\infty}^{\infty} |\sin nh|^2|\hat{f}(n)|^2 \le K^2h^2.$$

(b) Let $p$ be a positive integer. By choosing $h = \pi/2^{p+1}$, show that

$$\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|^2 \le \frac{K^2\pi^2}{2^{2p+1}}.$$

(c) Estimate $\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|$, and conclude that the Fourier series of $f$ converges absolutely, hence uniformly. [Hint: Use the Cauchy-Schwarz inequality to estimate the sum.]

(d) In fact, modify the argument slightly to prove Bernstein's theorem: If $f$ satisfies a Hölder condition of order $\alpha > 1/2$, then the Fourier series of $f$ converges absolutely.

**18.** Here are a few things we have learned about the decay of Fourier coefficients:

(a) if $f$ is of class $C^k$, then $\hat{f}(n) = o(1/|n|^k)$;

(b) if $f$ is Lipschitz, then $\hat{f}(n) = O(1/|n|)$;

(c) if $f$ is monotonic, then $\hat{f}(n) = O(1/|n|)$;

(d) if $f$ is satisfies a Hölder condition with exponent $\alpha$ where $0 < \alpha < 1$, then $\hat{f}(n) = O(1/|n|^\alpha)$;

(e) if $f$ is merely Riemann integrable, then $\sum |\hat{f}(n)|^2 < \infty$ and therefore $\hat{f}(n) = o(1)$.

Nevertheless, show that the Fourier coefficients of a continuous function can tend to 0 arbitrarily slowly by proving that for every sequence of nonnegative real numbers $\{\epsilon_n\}$ converging to 0, there exists a continuous function $f$ such that $|\hat{f}(n)| \geq \epsilon_n$ for infinitely many values of $n$.

[Hint: Choose a subsequence $\{\epsilon_{n_k}\}$ so that $\sum_k \epsilon_{n_k} < \infty$.]