9. In the case of the plucked string, use the formula for the Fourier sine coefficients to show that

$$A_m = \frac{2h}{m^2} \frac{\sin mp}{p(\pi - p)}.$$

For what position of p are the second, fourth, ... harmonics missing? For what position of p are the third, sixth, ... harmonics missing?

10. Show that the expression of the Laplacian

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is given in polar coordinates by the formula

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Also, prove that

$$\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2.$$

1. Consider the Dirichlet problem illustrated in Figure 11.

More precisely, we look for a solution of the steady-state heat equation $\triangle u = 0$ in the rectangle $R = \{(x,y) : \ 0 \leq x \leq \pi, \quad 0 \leq y \leq 1\}$ that vanishes on the vertical sides of R, and so that

$$u(x,0) = f_0(x) \qquad \text{and} \qquad u(x,1) = f_1(x)\,,$$

where f_0 and f_1 are initial data which fix the temperature distribution on the horizontal sides of the rectangle.

Use separation of variables to show that if f_0 and f_1 have Fourier expansions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \qquad \text{and} \qquad f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx,$$

then

$$u(x,y) = \sum_{k=1}^{\infty} \left(\frac{\sinh k(1-y)}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k \right) \sin kx.$$

We recall the definitions of the hyperbolic sine and cosine functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

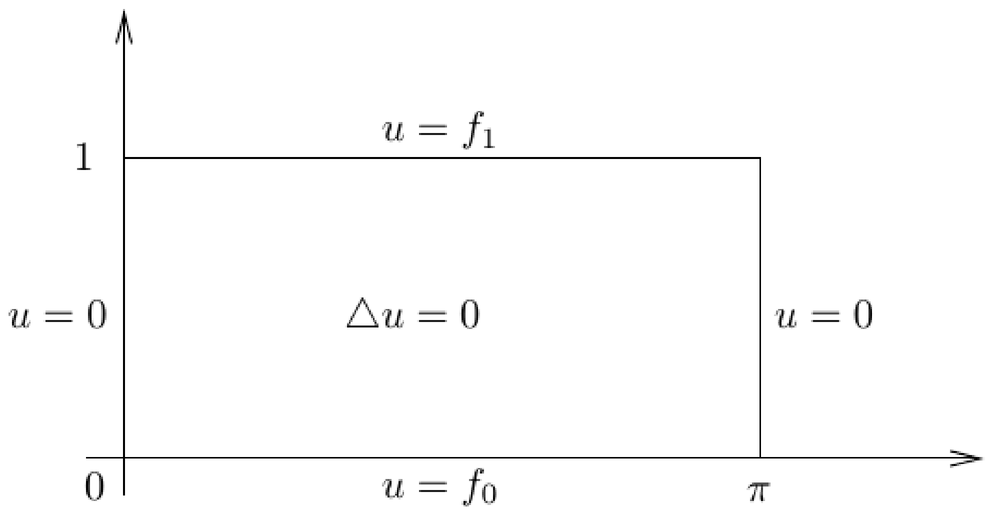

Figure 11. Dirichlet problem in a rectangle

1. Suppose f is 2π-periodic and integrable on any finite interval. Prove that if $a, b \in \mathbb{R}$, then

$$\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx = \int_{a-2\pi}^{b-2\pi} f(x)\,dx.$$

Also prove that

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx.$$

2. In this exercise we show how the symmetries of a function imply certain properties of its Fourier coefficients. Let f be a 2π-periodic Riemann integrable function defined on $\mathbb{R}$.

(a) Show that the Fourier series of the function f can be written as

$$f(\theta) \sim \hat{f}(0) + \sum_{n\geq 1} [\hat{f}(n) + \hat{f}(-n)] \cos n\theta + i[\hat{f}(n) - \hat{f}(-n)] \sin n\theta.$$

(b) Prove that if f is even, then $\hat{f}(n) = \hat{f}(-n)$, and we get a cosine series.

(c) Prove that if f is odd, then $\hat{f}(n) = -\hat{f}(-n)$, and we get a sine series.

(d) Suppose that $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbb{R}$. Show that $\hat{f}(n) = 0$ for all odd n.

(e) Show that f is real-valued if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n.

5. On the interval $[-\pi, \pi]$ consider the function

$$f(\theta) = \begin{cases} 0 & \text{if } |\theta| > \delta, \\ 1 - |\theta|/\delta & \text{if } |\theta| \leq \delta. \end{cases}$$

Thus the graph of f has the shape of a triangular tent. Show that

$$f(\theta) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1 - \cos n\delta}{n^2 \pi \delta} \cos n\theta.$$

6. Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$.

(a) Draw the graph of f.

(b) Calculate the Fourier coefficients of f, and show that

$$\hat{f}(n) = \begin{cases} \dfrac{\pi}{2} & \text{if } n = 0, \\[2ex] \dfrac{-1 + (-1)^n}{\pi n^2} & \text{if } n \neq 0. \end{cases}$$

(c) What is the Fourier series of f in terms of sines and cosines?

(d) Taking $\theta = 0$, prove that

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

7 Problems

1. One can construct Riemann integrable functions on $[0,1]$ that have a dense set of discontinuities as follows.

(a) Let $f(x)=0$ when $x<0$, and $f(x)=1$ if $x\geq 0$. Choose a countable dense sequence $\{r_n\}$ in $[0,1]$. Then, show that the function

$$F(x)=\sum_{n=1}^{\infty}\frac{1}{n^2}f(x-r_n)$$

is integrable and has discontinuities at all points of the sequence $\{r_n\}$. [Hint: F is monotonic and bounded.]

(b) Consider next

$$F(x)=\sum_{n=1}^{\infty}3^{-n}g(x-r_n),$$

where $g(x)=\sin 1/x$ when $x\neq 0$, and $g(0)=0$. Then F is integrable, discontinuous at each $x=r_n$, and fails to be monotonic in any subinterval of $[0,1]$. [Hint: Use the fact that $3^{-k}>\sum_{n>k}3^{-n}$.]

(c) The original example of Riemann is the function

$$F(x)=\sum_{n=1}^{\infty}\frac{(nx)}{n^2},$$

where $(x)=x$ for $x\in(-1/2,1/2]$ and (x) is continued to $\mathbb{R}$ by periodicity, that is, $(x+1)=(x)$. It can be shown that F is discontinuous whenever $x=m/2n$, where $m,n\in\mathbb{Z}$ with m odd and $n\neq 0$.