Coding problems for memory and storage applications

Alexander Barg

University of Maryland

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Introduction: Big Data

Big Data players: Facebook, Instagram, Google, MSFT, etc.; Dropbox, Box, etc.
Companies marketing coding solutions: CleverSafe (RS codes) and others.

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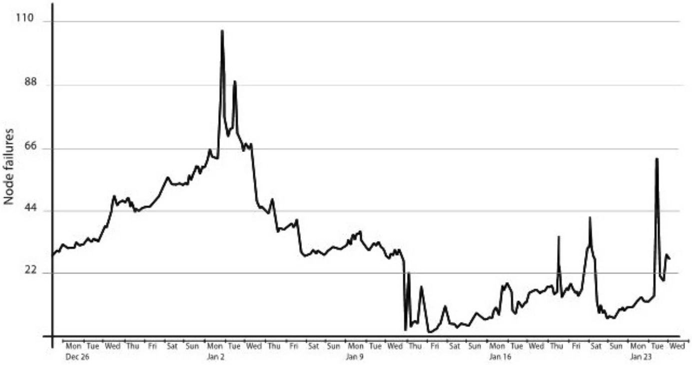
Cluster of machines running Hadoop at Yahoo!

Node failures are the **norm**

Is repair cost a real issue?

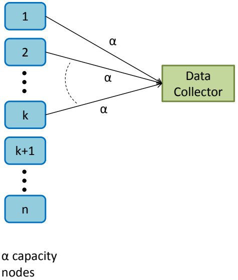


(Average number of failed nodes =20) $\times$15Tb $=$ 300Tb

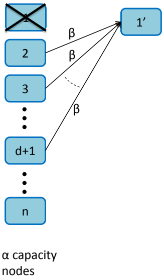
Two approaches to data coding in distributed storage

- Codes with locality
- Regenerating codes

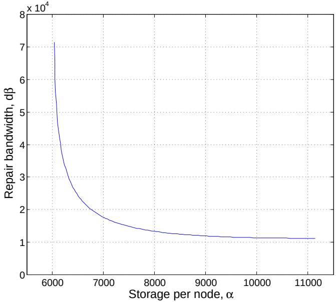
Regenerating codes



Data collection



Node repair



Trade-off

- B symbols are encoded into $n\alpha$ symbols stored in n nodes

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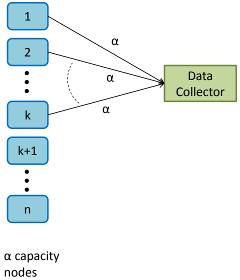

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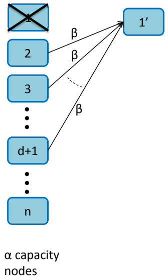

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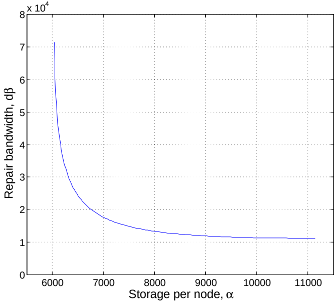

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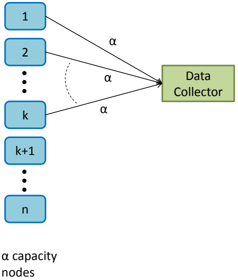

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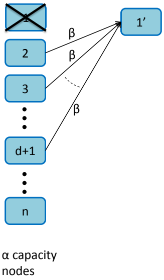

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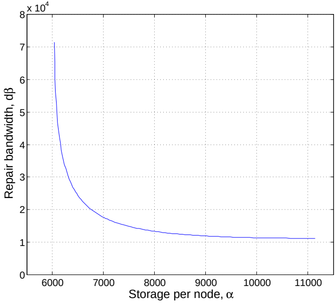
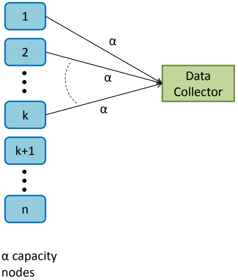


Trade-off

- *B* symbols are encoded into $n\alpha$ symbols stored in *n* nodes
- Downloading the data is possible by accessing any *k* nodes
- Node repair (exact or functional) can be performed by downloading $\beta < \alpha$ symbols from any subset of *d* nodes.

Regenerating codes



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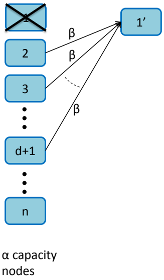
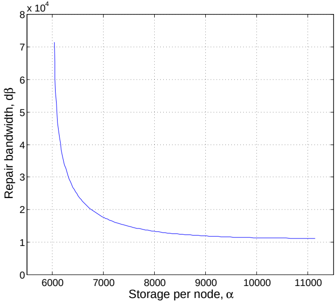


Node repair



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$(n, k, d, \{\alpha, \beta\})$ regenerating codes

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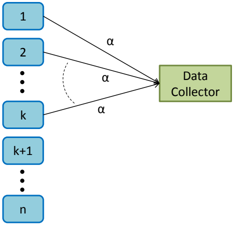
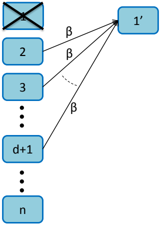


α capacity nodes

Data collection



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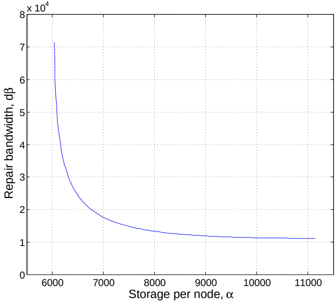

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$(n, k, d, \{\alpha, \beta\})$ regenerating codes

A. Dimakis, P. Godfrey, Y. Wu, M. Wainwright, and **K. Ramchandran**, Network coding for distributed storage systems, 2010

Locally recoverable codes: Plan

In this part we focus on locally recoverable codes

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LRC code: To recover one lost symbol of the encoding it suffices to access a small number r of other symbols.

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1. Current solutions
2. Parameters of LRC codes
3. MDS-like codes with the locality property
4. The availability problem: Multiple recovering sets
5. Extensions
   - LRC codes on algebraic curves
   - Cyclic LRC codes
6. Open problems: Bounds on codes; cyclic codes; list decoding

State-of-the-Art Coding technique

RAID: Redundant Array of Independent Disks

RAID 1 – Replication (currently 3x)

- Provides high availability of information
- Can tolerate any 2 disk failures
- Widely used in *Hadoop* and many other systems
- Storage overhead of 200%

RAID 6 uses [6,4,3] RS codes

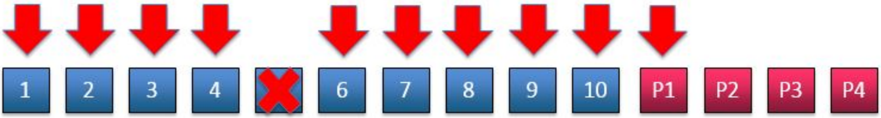
$[n, k]$ RS codes

- Can tolerate any n-k disk failures
- Poor handling of single disk failures (The Repair Problem)

Limitations of Reed-Solomon codes

Example: $[14, 10]$ RS code

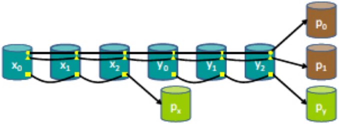
Transmit 10 symbols to recover one lost value



Generates 10x more traffic for recovery of one drive
If large portion of the cluster is RS-coded, this leads to saturation of the network

Other constructions

A combination of local and global parity checks for single and multiple nodes failures



(**C. Huang** at al., Erasure coding in Windows Azure Storage, USENIX Conf. 2012)

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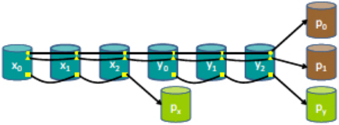

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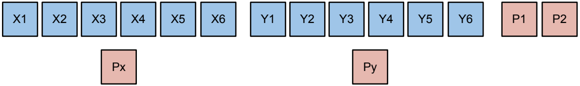
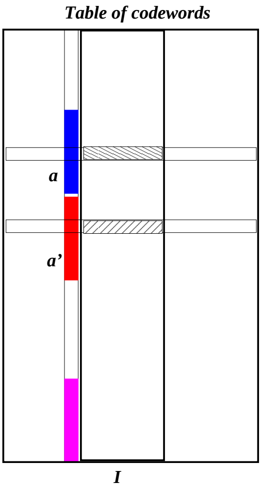
Other similar constructions (Windows Azure code)



Pyramid codes (**C. Huang** et al., 2007)

Locally recoverable codes

The code $\mathcal{C} \subset \mathbb{F}^n$ is locally recoverable with locality r if every symbol can be recovered by accessing some other r symbols in the encoding (recovering set of coordinate i)



(n, k, r) LRC code

Let $a \in \mathbb{F}$; consider the restriction $\mathcal{C}_J$ of $\mathcal{C}$ to a subset $J \subset [n]$.
Let

$$\mathcal{C}_J(a, i) = \{x \in \mathcal{C}_J : x_i = a\}, \quad i \in [n].$$

Definition

Code $\mathcal{C}$ has *locality* r if for every $i \in [n]$ there exists a subset $J_i \subset [n] \backslash i, |J_i| \leq r$ such that

$$\mathcal{C}_{J_i}(a, i) \cap \mathcal{C}_{J_i}(a', i) = \emptyset, \quad a \neq a'$$

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J. Han and **L. Lastras-Montano**, *ISIT* 2007;
C. Huang, M. Chen, and **J. Li**, *Symp. Networks App.* 2007;
F. Oggier and **A. Datta** '10;
P. Gopalan, C. Huang, H. Simitci, and **S. Yekhanin**, *IEEE Trans. Inf. Theory,* Nov. 2012.

Parameters of LRC codes

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Theorem

Let $\mathcal{C}$ be an (n, k, r) LRC code of cardinality q^k over an alphabet of size q, then:
The rate of $\mathcal{C}$ satisfies

$$\frac{k}{n} \le \frac{r}{r+1}. \tag{1}$$

The minimum distance of $\mathcal{C}$ satisfies

$$d \le n - k - \left\lceil \frac{k}{r} \right\rceil + 2. \tag{2}$$

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Note that $r = k$ reduces (2) to the Singleton bound

$$d \le n - k + 1$$

The distance bound

Main idea. Let $\mathcal{C}$ be a q-ary code of length n, size q^k. The distance $d(\mathcal{C})$ equals

$$d(\mathcal{C}) = n - \max_{S\subset[n]}\{|S| : |\mathcal{C}_S| < q^k\}$$

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The Singleton bound (without locality): $|S| = k - 1$

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If $|J_m'| < k - 1$, add to J_m' any $k - 1 - |J_m|$ other coordinates to form the set $L_m \subset [n]$.

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We have

$$|\mathcal{C}_{L_m}| < q^k$$

$$|L_m| = k-1+m = k-1+\left\lfloor \frac{k-1}{r} \right\rfloor = k-2+\left\lceil \frac{k}{r} \right\rceil$$

Cadambe-Mazumdar bound

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Cadambe-Mazumdar bound

(n, k, r) LRC code $\mathcal{C}$

$$\boxed{k \le \min_{s \ge 1}(rs + k_{\text{opt}}^{(q)}(n - s(r+1), d))}$$

Consider the sets of coordinates L_s constructed above, $1 \le s \le \lfloor (k-1)/r \rfloor$.

$$|\mathcal{C}_{L_s}| \le q^{rs}$$

The shortening of the code $\mathcal{C}$ on the coordinates in L_s forms a code of length $n - s(r+1)$ with distance d

Existence (Gilbert-Varshamov) bound

A linear q-ary $[n, k', d]$ code exists if

$$\sum_{i=0}^{d-2} \binom{n-1}{i} (q-1)^i < q^{n-k'}$$

Add $\lceil n/(r+1) \rceil$ local parities

$$k \geq k' - \left\lceil \frac{n}{r+1} \right\rceil$$

Sequences of (R, δ) codes with locality r exist as long as

$$R < \frac{r}{r+1} - \delta \log_q \frac{q-1}{\delta} - (1-\delta) \log_q \frac{1}{1-\delta}$$

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$$R \leq \frac{r}{r+1} - h_q(\delta)$$

Early constructions

1. Optimal $((r+1)\lceil k/r\rceil, k, r)$ LRC code

   Prasanth, Kamath, Lalitha, and **Kumar**, ISIT 2012
   Restricted length

2. Optimal (n, k, r) LRC codes

   Silberstein, Rawat, Koluoglu, and **Vishwanath**, ISIT 2013
   Tamo, Papailiopoulos, and **Dimakis**, ISIT 2013

   Almost any n, k, r

   Field size $q \sim 2^n$

Reed-Solomon codes

An RS code $\mathcal{C}$ is a linear code of length $n \le q-1$ over the field $\mathbb{F}_q$

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RS code $\mathcal{C}$ encodes messages of k symbols.
Let $V_k(q) = \{f \in \mathbb{F}_q[x] : \deg(f) \le k-1\}$

$$\mathcal{C} : V_k(q) \to \mathbb{F}_q^n$$
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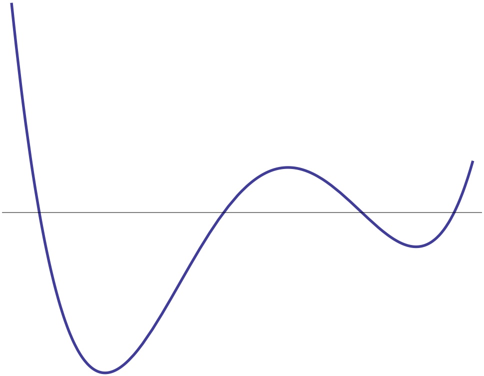
$$\begin{aligned} \mathcal{C} : V_k(q) &\to \mathbb{F}_q^n \\ f &\mapsto ev_A(f) = (f(P_i), i = 1, \dots, n) \end{aligned}$$

Example: Let $q = 8$, $f(x) = 1 + \alpha x + \alpha x^2$

$$f(x) \mapsto (1, \alpha^4, \alpha^6, \alpha^4, \alpha, \alpha, \alpha^6)$$

Reed-Solomon codes



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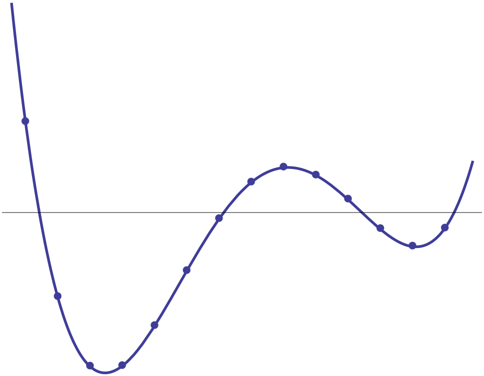

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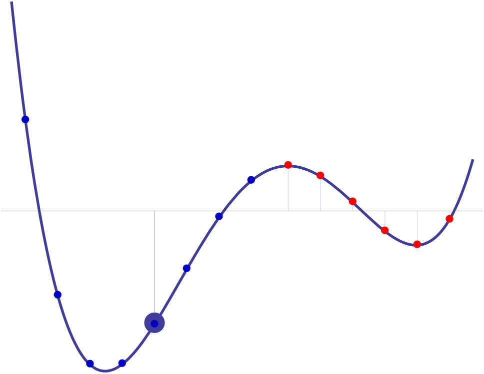

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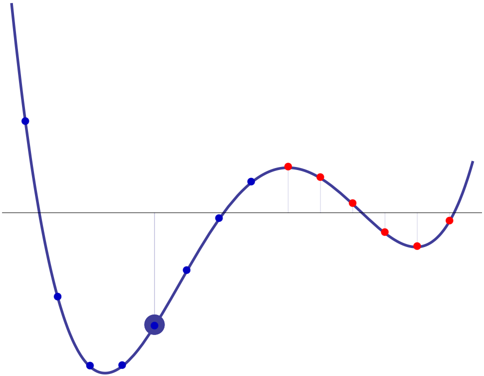

To recover one erased value we need to read *k* other values

LRC codes: Idea of construction

What if we can interpolate low-degree polynomials?

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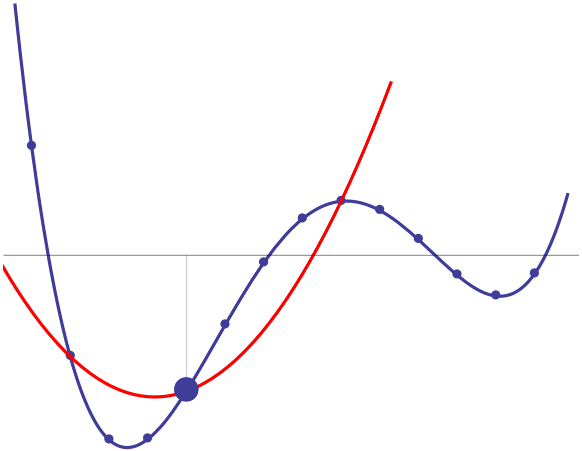

Construction of LRC codes: Limitations

We need a specially chosen set of points A

Restricted set of polynomials

Construction of (n, k, r) LRC codes: Example

Parameters: $n = 9, k = 4, r = 2, q = 13$;

Set of points: A={1,2,3,4,5,6,9,10,12}

$$\mathcal{A} = \{A_1 = \{1, 3, 9\}, A_2 = \{2, 6, 5\}, A_3 = \{4, 12, 10\}\}$$

Message: $a = (a_{0,0}, a_{0,1}, a_{1,0}, a_{1,1}) \in \mathbb{F}_q^k$

Polynomial space:

$$V_k(q) := \{a_{0,0} + a_{1,0}x + a_{0,1}x^3 + a_{1,1}x^4\}$$

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E.g., $a = (1, 1, 1, 1)$, $f_a(x) = 1 + x + x^3 + x^4$; $ev_A(f) = (4, 8, 7, 1, 11, 2, 0, 0, 0)$

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$$\mathcal{A} = \{A_1 = \{1, 3, 9\}, A_2 = \{2, 6, 5\}, A_3 = \{4, 12, 10\}\}$$

Message: $a = (a_{0,0}, a_{0,1}, a_{1,0}, a_{1,1}) \in \mathbb{F}_q^k$

Polynomial space:

$$V_k(q) := \{a_{0,0} + a_{1,0}x + a_{0,1}x^3 + a_{1,1}x^4\}$$

E.g., $a = (1, 1, 1, 1)$, $f_a(x) = 1 + x + x^3 + x^4$; $ev_A(f) = (4, 8, 7, 1, 11, 2, 0, 0, 0)$

Say $c_1 = f_a(1)$ is erased. We access the recovering set A_1 to construct a line $\delta(x) = 2x + 2$ such that $\delta(3) = 8, \delta(9) = 7$.

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It works!

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1. $\deg g = r + 1$,
2. There exists a partition $\mathcal{A} = \{A_1, ..., A_{\frac{n}{r+1}}\}$ of A into sets of size $r+1$, such that g is constant on each set A_i in the partition. For all $i = 1, \dots, n/(r+1)$, and any $\alpha, \beta \in A_i$,
$$g(\alpha) = g(\beta).$$

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E.g., $n = 9, r = 2, q = 13$;
$$\mathcal{A} = \{A_1 = \{1, 3, 9\}, A_2 = \{2, 6, 5\}, A_3 = \{4, 12, 10\}\},$$
Then $g(x) = x^3$ is constant on each of the A_i's

Construction of (n, k, r) LRC codes

Given $A \subset \mathbb{F}$, partition $\mathcal{A}$ into $(r+1)$-subsets.

To encode the message $a \in \mathbb{F}^k$, write $a = (a_{ij}, i = 0, \ldots, r-1; j = 0, ..., \frac{k}{r} - 1)$

Define the encoding polynomial

$$f_a(x) = \sum_{i=0}^{r-1} f_i(x) x^i,$$

where

$$f_i(x) = \sum_{j=0}^{\frac{k}{r}-1} a_{ij} g(x)^j, \quad i = 0, ..., r-1$$

A linear code $\mathcal{C}$ is constructed as follows:

$$\begin{aligned} Ev : & \mathbb{F}^k \to \mathbb{F}^n \\ & a \mapsto (f_a(\beta), \beta \in A) \end{aligned}$$

Recovery of erased symbol

Suppose that the location of erased symbol is $\alpha \in A_j$; $A_j \in \mathcal{A}$

To find c_α we rely on the recovering set A_j

Find a polynomial $\delta(x)$ s.t. $\delta(\beta) = c_\beta, \beta \in A_j \backslash \alpha$; $\deg \delta \leq r - 1$:

$$\delta(x) = \sum_{\beta \in A_j \backslash \alpha} c_\beta \prod_{\beta' \in A_j \backslash \{\alpha, \beta\}} \frac{x - \beta'}{\beta - \beta'}$$

Then $c_\alpha = \delta(\alpha)$

Properties of the construction

Theorem

The constructed linear codes are optimal (n, k, r) LRC codes with respect to the "Singleton bound" (2).

Optimality is proved by counting degrees.

Locality: Let $\alpha \in A_j$ be the erased location. Define

$$\partial(x) = \sum_{i=0}^{r-1} f_i(\alpha)x^i$$

By the construction, for all $\beta \in A_j$

$$\partial(\beta) = f_a(\beta)$$

Since $\deg \partial \leq r - 1$, we see that $\partial(x) \equiv \delta(x)$.

Constructing the polynomial $g(x)$

Proposition

Let H be a subgroup of $\mathbb{F}_q^*$ or $\mathbb{F}_q^+$. The annihilator polynomial of H

$$g(x) = \prod_{h \in H} (x - h)$$

is constant on each coset of H.

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Assume that H is a multiplicative subgroup and let $a, a\bar{h}$ be two elements of the coset aH, where $\bar{h} \in H$, then

$$\begin{aligned} g(a\bar{h}) = \prod_{h \in H} (a\bar{h} - h) &= \bar{h}^{|H|} \prod_{h \in H} (a - h\bar{h}^{-1}) \\ &= \prod_{h \in H} (a - h) \\ &= g(a). \end{aligned}$$

Some generalizations

The locator set $A \subset \mathbb{F}$, $A = \sqcup_{i=1}^{m} A_i$. Consider the algebra

$$\mathbb{F}_{\mathcal{A}}[x] = \{f \in \mathbb{F}[x] : f \text{ is constant on } A_i, i = 1, \dots, m;\ \deg f < |A|\}.$$

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The properties of $\mathbb{F}_{\mathcal{A}}[x]$ are summarized as follows:

1. $\dim(\mathbb{F}_{\mathcal{A}}[x]) = m$;
2. Let $\alpha_1, ..., \alpha_m$ be distinct nonzero elements of $\mathbb{F}$, and let g be the polynomial of degree $\deg(g) < |A|$ that satisfies $g(A_i) = \alpha_i$ for all $i = 1, ..., m$. Then the polynomials $1, g, ..., g^{m-1}$ form a basis of $\mathbb{F}_{\mathcal{A}}[x]$.

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General code construction: Let $A \subset \mathbb{F}$, $|A| = n$; $A = \sqcup_{i=1}^{m} A_i$, $|A_i| = r + 1$ for all i. Let Φ be an injective mapping from $\mathbb{F}^k$ to the space of polynomials

$$\mathcal{F}_{\mathcal{A}}^{r} = \oplus_{i=0}^{r-1} \mathbb{F}_{\mathcal{A}}[x] x^i.$$

The evaluation code obtained in this way is an (n, k, r) LRC code.

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3. To improve data availability, replace $[r+1, r, 2]$ local codes with $[r+\rho-1, r]$ MDS codes. Then every c_i is a function of any r out of $r+\rho-1$ coordinates.
   Bound on the distance:

$$d \le n - k + 1 - \left(\left\lceil \frac{k}{r} \right\rceil - 1\right)(\rho - 1) \quad \text{(Kamath e.a., 2013)}$$

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Claim: Taking recovering sets of size $|A_i| = r + \rho - 1$ and a polynomial basis of $\mathbb{F}_{\mathcal{A}}[x]$, we can construct an (n, k, r) LRC code whose distance meets this bound.

Availability problem

"Hot data" accessed simultaneously by a very large number of users

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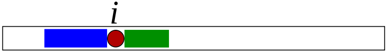
Multiple recovering sets: Definition

Every symbol in data encoding appears in several disjoint (orthogonal) parity checks

$\mathcal{C} \subset \mathbb{F}^n$ a code of length n

Every coordinate is recoverable from the codeword symbols in several recovering sets:



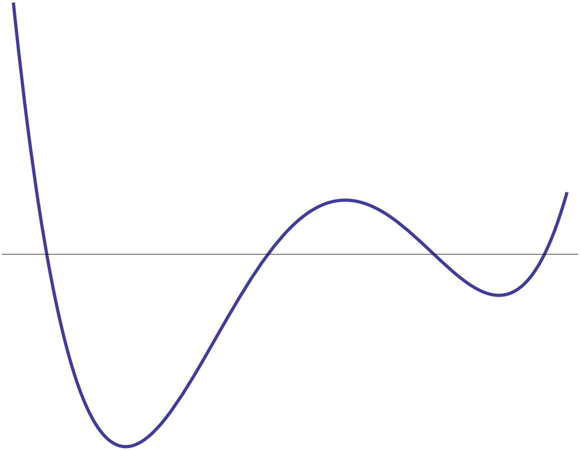
Multiple recovering sets: Definition

Let $\mathcal{C}(a,i) = \{x \in \mathcal{C} : x_i = a\}, a \in \mathbb{F}, i \in [n]$

The code $\mathcal{C}$ has two disjoint recovering sets if for every $i \in [n]$ there are subsets $R_i^1, R_i^2 \subset [n]\backslash\{i\}, R_i^1 \cap R_i^2 = \emptyset$ such that

$$\mathcal{C}(a,i)_{R_i^j} \cap \mathcal{C}(a',i)_{R_i^j} = \emptyset, \quad a \neq a';\ j = 1,2$$

Multiple recovering sets: Idea of construction



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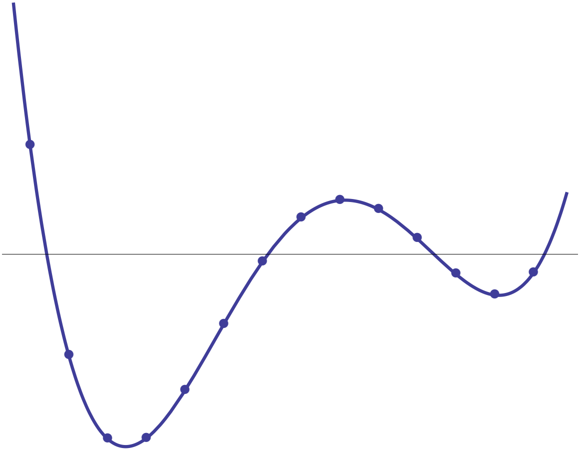

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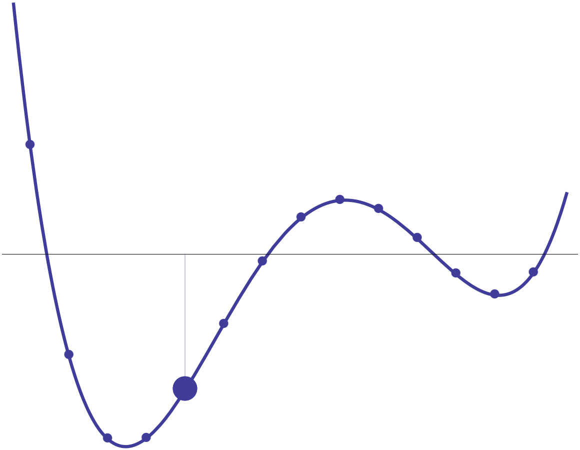

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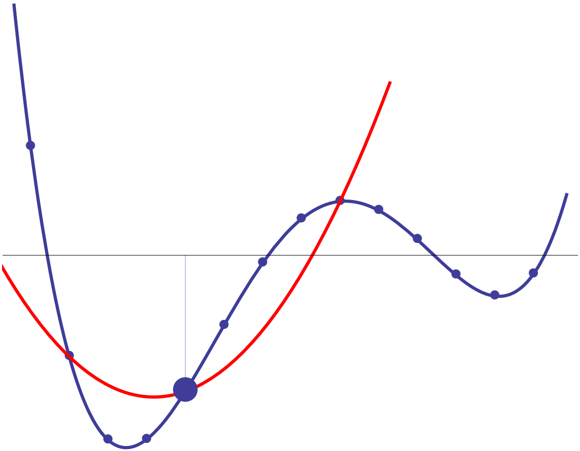

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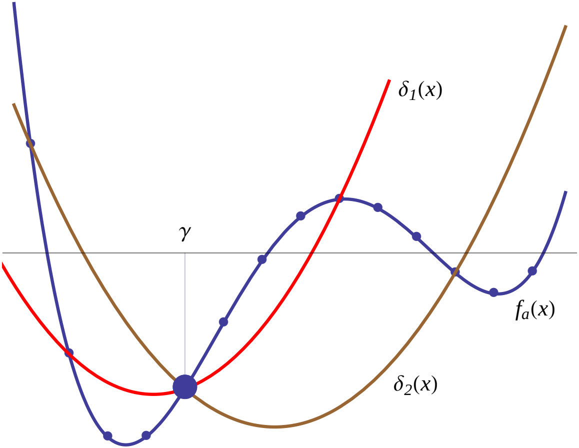

$f_a(\gamma)$ can be found by interpolating $\delta_1(x)$ as well as $\delta_2(x)$

Multiple recovering sets: Example

Take $\mathbb{F} = \mathbb{F}_{13}$; $G, H \le \mathbb{F}^*$; $G = \langle 5 \rangle, H = \langle 3 \rangle$

$$\mathcal{A}_G = \{\{1, 5, 12, 8\}, \{2, 10, 11, 3\}, \{4, 7, 9, 6\}\}$$
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We construct an LRC $(12, 4, \{2,3\})$, distance ≥ 6, code $\mathcal{C} : \mathbb{F}^4 \to \mathbb{F}^{12}$

$$a = (a_0, a_1, a_2, a_3) \mapsto f_a(x) = a_0 + a_1 x + a_2 x^4 + a_3 x^6$$

$$f_a(x) = \sum_{i=0}^{2} f_i(x) x^i, \text{ where } f_0(x) = a_0 + a_2 x^4,\ f_1(x) = a_1,\ f_2(x) = a_3 x^4; f_i \in \mathbb{F}_{\mathcal{A}}[x]$$

$$f_a(x) = \sum_{j=0}^{1} g_j(x) x^j \text{ where } g_0(x) = a_0 + a_3 x^6, g_1(x) = a_1 + a_2 x^3; g_j \in \mathbb{F}_{\mathcal{A}_H}[x]$$

E.g., $f_a(1)$ can be recovered by computing $\delta_1(x), x \in \{5,12,8\}$ OR $\delta_2(x), x \in \{3,9\}$

Multiple recovering sets

General Construction: $A = \{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{F}, |A| = n$;

$$A = \overbrace{\sqcup_{i\ge 0} R_i^1}^{\mathcal{A}} = \overbrace{\sqcup_{j\ge 0} R_j^2}^{\mathcal{A}'}; \quad |R_i^1| = r+1, |R_j^2| = s+1$$

$$f_a(x) = \sum_{i=0}^{k-1} a_i g_i(x), \quad g_i(x) \in \mathcal{F}_{\mathcal{A}}^r \cap \mathcal{F}_{\mathcal{A}'}^s$$

Evaluation map: $(a_1, \dots, a_k) \overset{\mathcal{C}}{\mapsto} (f_a(\alpha_1), \dots, f_a(\alpha_n))$

Theorem: Assume that the partitions $\mathcal{A}, \mathcal{A}'$ are *orthogonal*. Then

$$Eval(f : f \in \mathcal{F}_{\mathcal{A}}^r \cap \mathcal{F}_{\mathcal{A}'}^s), x \in A$$

gives an $(n, k, \{r, s\})$ LRC code with distance $\ge n - m + 1$, where m is the largest degree in $\mathcal{F}_{\mathcal{A}}^r \cap \mathcal{F}_{\mathcal{A}'}^s$.

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Orthogonal partitions can be obtained from the structure of additive or multiplicative subgroups of $\mathbb{F}_q$

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2. Take $G, H \le \mathbb{F}_q^+$, e.g., $G \cong H \cong (\mathbb{Z}_2)^2$; $\mathbb{F}_{16}^+ = (\mathbb{Z}_2)^2 \times (\mathbb{Z}_2)^2$

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$$G = \{0000, 0001, 0010, 0011\} \text{ and } H = \{0000, 0100, 1000, 1100\}$$
$$\mathcal{A}_G = \{\{0,1,\alpha,\alpha^4\},\{\alpha^5,\alpha^{10},\alpha^2,\alpha^8\},\{\alpha^6,\alpha^{13},\alpha^{11},\alpha^{12}\},\{\alpha^7,\alpha^9,\alpha^{14},\alpha^3\}\}$$
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Proposition: Two subgroups G, H define orthogonal coset partitions if they intersect trivially: $G \cap H = \mathrm{id}$

Remarks

There are other ways of constructing codes with multiple (e.g., two) recovering sets:

Product codes, Bipartite-graph codes

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A family of optimal locally recoverable codes, with **I. Tamo**, arXiv:1311.3284 (*IT Trans.*, no. 8, 2014)

Bounds on the parameters

Theorem

Let $\mathcal{C}$ be an (n,k,r,t) LRC code with t disjoint recovering sets of size r. Then the rate of $\mathcal{C}$ satisfies

$$\frac{k}{n} \le \frac{1}{\prod_{j=1}^{t}(1+\frac{1}{jr})} \approx t^{-\frac{1}{r}}$$

The minimum distance of $\mathcal{C}$ is bounded above as follows:

$$d \le n - \sum_{i=0}^{t} \left\lfloor \frac{k-1}{r^i} \right\rfloor . \qquad (Tamo - B, 2014)$$

$$d \le n - k - \left\lceil \frac{t(k-1)+1}{t(r-1)+1} \right\rceil + 2 \qquad (Rawat\ e.a., 2014)$$

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It is likely that these bounds are not final

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Asymptotic GV bound with locality:

$$R \geq \frac{r}{r+1} - h_q(\delta)$$

Extensions

Reed-Solomon codes can be extended in two ways:

- Codes on algebraic curves
- Cyclic codes and subfield subcodes

Algebraic codes

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RS codes: $n \le q-1$, $1 \le k \le n$, $d = n-k+1$

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Curves to the rescue!

AG codes in error correction

1. Gilbert-Varshamov bound

An $[n, k, d]$ code exists if

$$\sum_{i=0}^{d-2} \binom{n-1}{i} (q-1)^i < q^{n-k}$$

Let $R = k/n, \delta = d/n$, take logs and divide by n:

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2. Tsfasman-Vlăduţ-Zink bound

There exist explicit sequences of codes on algebraic curves with the parameters

$$R \ge 1 - \delta - \frac{1}{\sqrt{q} - 1}$$

RS type codes

Given $A \subset \mathbb{F}$, partition it into $(r+1)$-subsets.

To encode the message $a \in \mathbb{F}^k$, write $\underline{a} = (a_{ij}, i = 0, \dots, r-1; j = 0, ..., \frac{k}{r} - 1)$

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Evaluation code $\mathcal{C}$

$$\begin{aligned} Ev : &\mathbb{F}^k \to \mathbb{F}^n \\ &a \mapsto (f_a(P), P \in A) \end{aligned}$$

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They should really be

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Geometric interpretation

$$A := \{1,2,3,4,5,6,9,10,12\} \subset \mathbb{F}_{13}$$

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$$A = \{A_1 = \{1,3,9\}, A_2 = \{2,6,5\}, A_3 = \{4,12,10\}\}$$

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$$\begin{array}{cccc} & 1 & 2 & 4 \\ X & 3 & 6 & 12 \\ & 9 & 5 & 10 \end{array}$$

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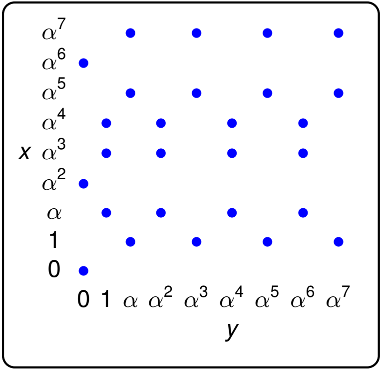
LRC codes on curves

Consider the set of pairs $(x, y) \in \mathbb{F}_9$ that satisfy the equation $x^3 + x = y^4$



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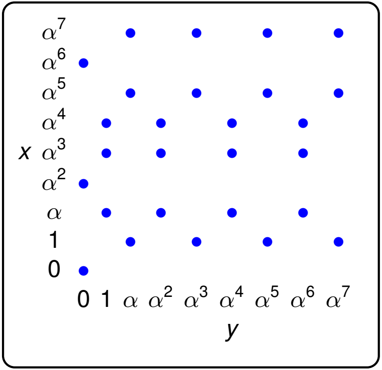

27 points of the Hermitian curve over $\mathbb{F}_9$; $\alpha^2 = \alpha + 1$

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Take the space of functions $V := \langle 1, y, y^2, x, xy, xy^2 \rangle$

A={27 points of the Hermitian curve over $\mathbb{F}_9$}; $n = 27, k = 6$

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E.g., message $(1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5)$

$$F(x, y) = 1 + \alpha y + \alpha^2 y^2 + \alpha^3 x + \alpha^4 xy + \alpha^5 xy^2$$

$$F(0, 0) = 1 \text{ etc.}$$

LRC codes on curves

x \ y	0	1	α	α^2	α^3	α^4	α^5	α^6	α^7
α^7			α		α^7		α^5		0
α^6	α^2								
α^5			α^6		α^4		α^2		0
α^4		α^7		α^3		α^5		α^5	
α^3		α^3		α^7		α		α	
α^2	α^3								
α		0		0		0		0	
1			1		α^6		α^4		0
0	1								

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x	α^3		α^3		α^7		α		α	
	α^2	α^3								
	α		$\cancel{0}$		0		0		0	
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		0	1	α	α^2	α^3	α^4	α^5	α^6	α^7

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α^3		α^3		α^7		α		α	
α^2	α^3								
α		?		0		0		0	
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0	1								

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Find $f(x) : f(\alpha^4) = \alpha^7, f(\alpha^3) = \alpha^3$

$$\Rightarrow \quad f(x) = \alpha x - \alpha^2$$

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$$f(\alpha) = 0 = F(P)$$

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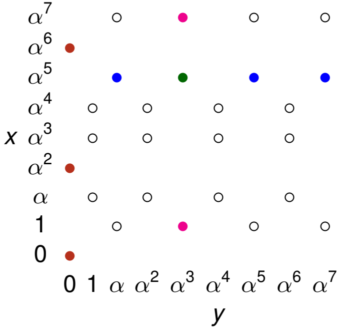
It is also possible to take $g(P) = x$ (projection on x); we obtain LRC codes with locality q_0

Two recovering sets



Polynomial basis $\{x^i y^j, i = 0, 1, \ldots, r_1 - 1, j = 0, 1, \ldots, r_2 - 1\}$

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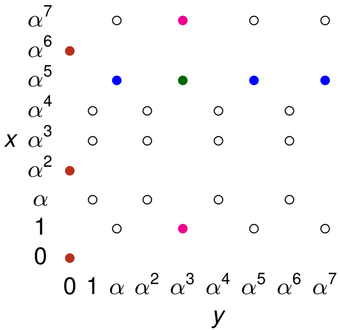

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$(24, 6, \{2, 3\})$ LRC(2) code over $\mathbb{F}_9$

General LRC codes on curves

Map of curves

X, Y smooth projective absolutely irreducible curves over $\Bbbk$

$$g : X \to Y$$

rational separable map of degree $r+1$.

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Lift the points of Y

$S = \{P_1, \ldots, P_s\} \subset Y(\mathbb{k})$; $Q_\infty = \pi^{-1}(\infty)$, where $\pi : Y \to \mathbb{P}^1_{\mathbb{k}}$. Assume that there is a partition of points

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$$g(P_{ij}) = P_j \text{ for all } i, j.$$

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Construct LRC codes

Evaluation codes constructed on the set A have the locality property with parameter r.

Asymptotically good sequences of codes

Let $q = q_0^2$, where q_0 is a prime power. Take Garcia-Stichtenoth towers of curves:

$$x_0 := 1;\ X_1 := \mathbb{P}^1, \Bbbk(X_1) = \Bbbk(x_1);$$
$$X_l : z_l^{q_0} + z_l = x_{l-1}^{q_0+1}, x_{l-1} := \frac{z_{l-1}}{x_{l-2}} \in \Bbbk(X_{l-1}) \text{ (if } l \geq 3\text{)},$$

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There exist families of q-ary LRC codes with locality r whose *rate and relative distance* satisfy

$$R \geq \frac{r}{r+1}\Big(1 - \delta - \frac{3}{\sqrt{q}+1}\Big), \qquad r = \sqrt{q} - 1$$
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$^{*)}$Recall the TVZ bound without locality: $R \geq 1 - \delta - \frac{1}{\sqrt{q}-1}$

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Locally recoverable codes on algebraic curves, with **I. Tamo** and **S. Vlăduţ**, arXiv:1501.04904

What next?

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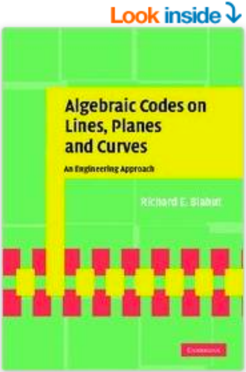
Look inside
Algebraic Codes on
Lines, Planes
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An Engineering Approach
Richard E. Blahut

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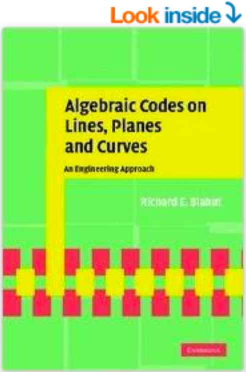
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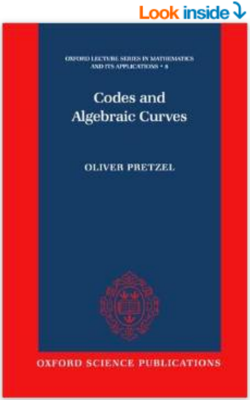
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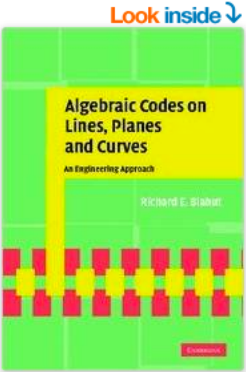
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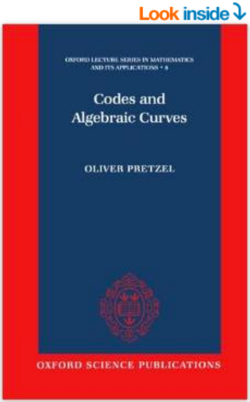
An Engineering Approach
Richard E. Blahut


Look inside
Codes and Algebraic Curves
OLIVER PRETZEL
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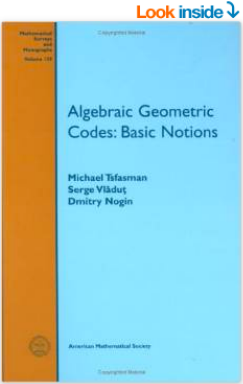
Look inside
Algebraic Geometric Codes: Basic Notions
Michael Tsfasman
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Another connection: Cyclic codes and Binary cyclic codes

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Generator matrix

$$G = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \alpha & \alpha^2 & \dots & \alpha^{14} \\ 1 & \alpha^2 & \alpha^4 & \dots & \alpha^{2\cdot 14} \\ 1 & \alpha^3 & \alpha^6 & \dots & \alpha^{3\cdot 14} \end{pmatrix}$$

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$$\underline{c} = (c_1, \dots, c_{15}); \quad c(x) = \sum_{i=1}^{15} c_i x^{i-1} : \quad c(\alpha^i) = 0, i = 1, \dots, 14$$

BCH codes: Subfield subcodes of RS codes

- Consider the subset of vectors of the RS code with coordinates 0 or 1
- $c(x) = \sum_{i=1}^{n} x^i : c(\alpha^j) = 0$
- They form a BCH code, a binary cyclic code of length $2^m - 1$
- This construction is called a Subfield Subcode
  Observation 1: expand parity-check matrix
  Observaion 2: conjugate roots

Cyclic codes

- Consider an $[n|(q-1), k=n-d+1, d]$ RS code $\mathcal{C}$ over $\mathbb{F}_q$

  $A=(1,\alpha,\dots,\alpha^{n-1})$ where $\alpha^n=1$

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  $$G=\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \alpha & \alpha^2 & \dots & \alpha^{n-1} \\ \vdots & & \vdots & & \vdots \\ 1 & \alpha^{k-1} & \alpha^{2(k-1)} & \dots & \alpha^{(k-1)(n-1)} \end{pmatrix}$$

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- Consider a subfield subcode $\mathcal{D}\subset\mathcal{C}$,
  $\mathcal{D}:=\{(c_0,\dots,c_{n-1})\in\mathcal{C} : c_j\in\mathbb{F}_p, 0\le j\le n-1\}$
  Zeros of $\mathcal{D}$: $\{(\alpha,\alpha^p,\dots,\alpha^{p^{m-1}\mathrm{mod} n}),\dots\}$

Cyclic codes: Example

- RS code $\mathcal{C}$ of length $n = 15, k = 8, d = 8, q = 2^4$
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  Generator polynomial $g(x) = \prod_{i=1}^{t}(x - \alpha^i)$, $\dim(\mathcal{C}) = n - \deg(g) = 8$

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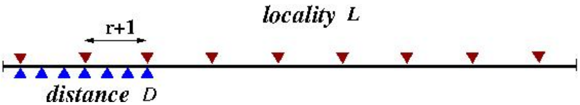
$$k = 6; d = 8 = n - k\frac{r+1}{r} + 2$$

Cyclic LRC codes

Main idea: Suppose that the zeros are arranged as follows:



The cyclic code with zeros $\{D \cup L\}$ has distance $\geq |D|$ and locality r.

Cyclic LRC codes: Details

The following result describes the cyclic case of the main construction.

Theorem (RS-type cyclic LRC codes): Let α be a primitive n-th root of unity, where $n|(q-1)$; let $l, 0 \le l \le r$ be an integer. Consider the following sets of elements of $\mathbb{F}_q$:

$$L = \{\alpha^i, i \bmod (r+1) = l\},$$

and

$$D = \Big\{\alpha^{j+s}, s = 0, \dots, n - \frac{k}{r}(r+1)\Big\},$$

where $\alpha^j \in L$. The cyclic code with the defining set of zeros $L \cup D$ is an optimal $^{*)}$ (n, k, r) q-ary cyclic LRC code.

$^{*)}$ Singleton-like optimality; see (1)

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Locality of $\mathcal{C}$:

$$r = d(\mathcal{C}^\perp) = d^\perp(\mathcal{C})$$

In the cyclic case Locality=Dual distance

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What about binary codes?

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Example:
Code $\mathcal{C}$ over $\mathbb{F}_{16}$ has zeros $Z = \{\alpha, \alpha^2, \alpha^3, \alpha^4\} \cup \{\alpha, \alpha^4, \alpha^7, \alpha^{10}, \alpha^{13}\}$.

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Binary subcode $\mathcal{D} \subset \mathcal{C}$: zeros Z and all conjugates
The locality of D may decrease; the distance may increase. The dimension becomes smaller.

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Let $\mathcal{C}$ be a cyclic code over $\mathbb{F}_{q^m}$; let $\mathcal{D}$ be the subfield subcode of $\mathcal{C}$

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$$r(\mathcal{D}) \leq r(\mathcal{C})$$

Subfield subcodes

The analysis: Ideas.

- Take a subfield subcode D of the code $\mathcal{C}$ constructed in the RS-like LRC codes Theorem.
- Locality of D = distance of $D^{\perp}$
- Let $q = 2^m$, $T_m(x) = x + x^2 + \cdots + x^{2^{m-1}}, x \in \mathbb{F}_q$

$$T_m(\mathcal{C}) := \{(T_m(c_1), \ldots, T_m(c_n)), \underline{c} \in \mathcal{C}\}$$

Theorem (Delsarte '74, Sidelnikov '71): $D = T_m(\mathcal{C}^{\perp})$

- Analyze the locality of D using $d(D^{\perp})$ (techniques: irreducible cyclic codes)

Some examples

n	k	d	$Z(\mathcal{D})$	r	w	$Z((\mathcal{C}')^{\perp})$	$d^{\perp}$	SH	LP	locator field
35	20	3	$\{1, 15\}$	$r \le 3$	4	$\{0, 1, 7, 15\}$	4	$k \le 25$	$k \le 29$	$\mathbb{F}_{2^{12}}$
45	33	3	$\{1\}$	$r \le 7$	8	$\{0, 1, 3, 5, 9, 15, 21\}$	8	$k \le 37$	$k \le 39$	$\mathbb{F}_{2^{12}}$
27	7	6	$\{1, 9\}$	$r = 1$	2	$\{0, 3\}$	2			$\mathbb{F}_{2^{18}}$
63	36	3	$\{1, 9, 11, 15, 23\}$	$r \le 3$	4	$\{0, 1, 7, 9, 11, 15, 21, 23\}$	4			$\mathbb{F}_{2^{6}}$

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Cyclic LRC codes and their subfield subcodes, with **I. Tamo, S. Goparaju,** and **R. Calderbank**, arXiv:1502.01414.