PERTURBED DAMPED PENDULUM: FINDING PERIODIC SOLUTIONS

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ABSTRACT. Using the equation of motion of the damped pendulum, we introduce the averaging method on the study of periodic solutions of dynamical systems with small perturbation. We provide sufficient conditions for the existence of periodic solutions of the perturbed damped pendulum with small oscillations having equations of motion

$$\ddot{\theta} = -a\theta - b\dot{\theta} + \varepsilon f(t, \theta, \dot{\theta}),$$

where $a > 0$, $b > 0$ and ε are real parameters, with $a = g/l$, g the acceleration of the gravity, l the length of the rod and b the damping coefficient. Here the parameters b and ε are small and the smooth function f is T–periodic in t. The averaging theory provides a useful means to study dynamical systems, accessible to Master and PhD students.

1. INTRODUCTION

Systems derived from the pendulum, gives to students important and practical examples of dynamical systems, as we can see in the case of the *weight-driven pendulum clocks*, which has had its historical and dynamical aspects studied by Denny in a recent paper [1]. This system has been revisited by Llibre and Teixeira in [2], who got the same result using averaging theory. These systems, some times, has also been used to introduce mathematical concepts of classical mechanics, as we can see in [3].

In this paper we will use the damped pendulum to introduce some concepts and techniques of averaging theory, which help us to finding periodic solutions of dynamical systems.

We consider a system composed of a point mass m moving in the plane, under gravity force, in which the distance between a point and m is fixed and equal to l. We also consider that the particle suffers resistance to movement proportional at its velocity. This system is called *damped pendulum*.

The position of the pendulum is determined by the angle θ shown in Figure 1. The equation of motion of this system is given by

$$(1) \qquad \ddot{\theta} = -a\sin(\theta) - b\dot{\theta},$$

where $a > 0$ and $b > 0$ are real parameters, with $a = g/l$, g the acceleration of the gravity, l the length of the rod and b the damping coefficient.

We consider only the motion in the vicinity of the equilibrium $(\theta, \dot{\theta}) = (0, 0)$, i.e., we are only interested in small oscillations around this equilibrium. In this case, we can approximate the value of $\sin(\theta)$ to θ, hence, the equation (1) becomes
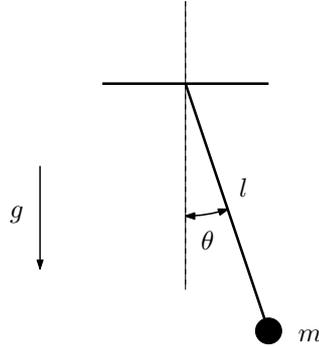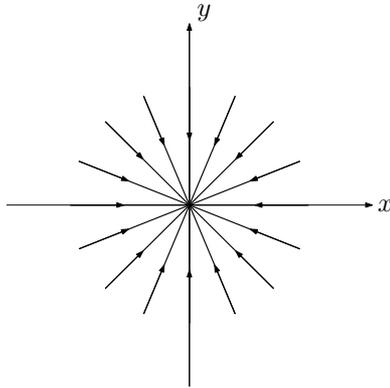
FIGURE 1.   Pendulum.

FIGURE 2.   Attractor Node.

linear; moreover, denoting $(x, y) = (\theta, \dot{\theta})$, the second order differential equation (1) linearized, can be written as the linear system of first order differential equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -ax - by, \end{aligned}$$
(2)

with the following eigenvalues

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4a}}{2} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4a}}{2}.$$

Note that for $b^2 \geq 4a$ the eigenvalues λ_1 and λ_2 are both negative, then the singularity $(x, y) = (0, 0)$ is a attractor node, represented in the Figure 2. Now, for $b^2 < 4a$ both eigenvalues λ_1 and λ_2 have the imaginary part different of zero and negative real part, then the singularity $(x, y) = (0, 0)$ is a attractor focus, represented in the Figure 3. Both cases are topologically equivalent. If we wouldn't consider the damping effect, i.e. $b = 0$, both eigenvalues λ_1 and λ_2 would be purely imaginary, then the singularity $(x, y) = (0, 0)$ would be a center, represented in the Figure 4. This last case is not topologically equivalent to the two cases above.
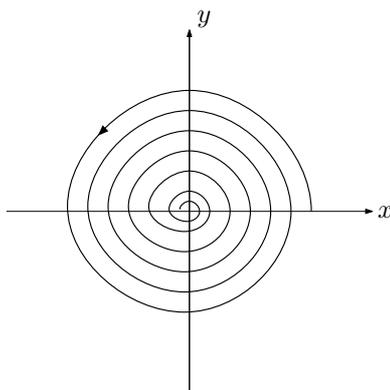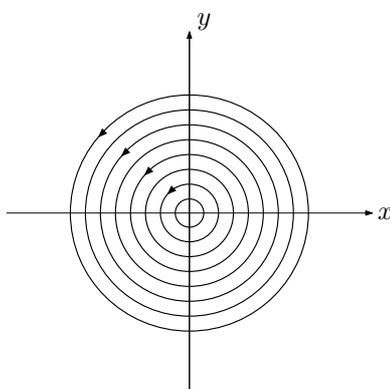
FIGURE 3.   Attractor Focus.

FIGURE 4.   Center.

To apply the averaging method, we take the damping coefficient b as a perturbation parameter of the system assuming that $b = \varepsilon \bar{b}$. Then, in this case, the unperturbed system is given by system (2) with $b = 0$, with the following eigenvalues

$$\lambda_1 = i\omega \quad \text{and} \quad \lambda_2 = -i\omega,$$

where $\omega = \sqrt{a}$.

2. STATEMENTS OF THE MAIN RESULTS

The objective of this paper is to provide a system of equations whose simple zeros provide periodic solutions of the perturbed damped pendulum with equations of motion

(3) $$\ddot{x} = -ax + \varepsilon \left(-\bar{b}\dot{x} + f(t, x, \dot{x}) \right),$$

where ε is a small parameter and the smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is T–periodic in t.

2.1. **Non-autonomous perturbation.** Given $\mathbf{z} = (x_0, y_0) \in \mathbb{R}^2$, we define the real functions

(4)
$$
\mathcal{F}_1(\mathbf{z}) = \left(\frac{\bar{b}\sin(2T\omega)}{4\omega} - \frac{\bar{b}T}{2} \right) x_0 + \frac{\bar{b}\sin^2(T\omega)}{2\omega^2} y_0 - \frac{1}{\omega} \int_0^T \sin(s\omega) f(s, e^{As}\mathbf{z})\, ds,
$$
$$
\mathcal{F}_2(\mathbf{z}) = \frac{\bar{b}\sin^2(T\omega)}{2} x_0 + \left(-\frac{\bar{b}\sin(2T\omega)}{4\omega} - \frac{\bar{b}T}{2} \right) y_0 + \int_0^T \cos(s\omega) f(s, e^{As}\mathbf{z})\, ds.
$$

A zero (x_0^*, y_0^*) of the system of equations

(5)
$$
\mathcal{F}_1(x_0, y_0) = 0, \quad \mathcal{F}_2(x_0, y_0) = 0,
$$

such that

$$
\det \left(\left. \frac{\partial(\mathcal{F}_1, \mathcal{F}_2)(\mathbf{z})}{\partial \mathbf{z}} \right|_{\mathbf{z}=(x_0^*, y_0^*)} \right) \neq 0,
$$

is called a *simple zero* of system (5).

Our first main result on the periodic solutions of the damped pendulum with small perturbation (3) is the following

Theorem 1. *Assume that the function f of the damped pendulum with small perturbation with equations of motion (2) is smooth and T–periodic in t, then for $\varepsilon \neq 0$ sufficiently small and for every simple zero $(x_0^*, y_0^*) \neq (0,0)$ of the system (3), the damped pendulum with small non-autonomous perturbation (3) has a periodic solution $x(t, \varepsilon)$, such that $(x(0, \varepsilon), \dot{x}(0, \varepsilon)) \to (x_0^*, y_0^*)$ when $\varepsilon \to 0$.*

Theorem 1 is proved in section 4. Its proof is based in the averaging theory for computing periodic solutions, which has been explored in section 3.

Corollary 2. *Assume that the function f of Theorem 1 has period $T = (p/q)(2\pi/\omega)$, with p and q relatively prime positive integers. Then for $\varepsilon \neq 0$ sufficiently small and for every simple zero $(x_0^*, y_0^*) \neq (0,0)$ of the system*

(6)
$$
\tilde{\mathcal{F}}_1(\mathbf{z}) = p\pi\bar{b}x_0 + \int_0^{\frac{2p\pi}{\omega}} \sin(s\omega) f(s, e^{As}\mathbf{z})\, ds = 0,
$$
$$
\tilde{\mathcal{F}}_2(\mathbf{z}) = \frac{p\pi\bar{b}}{\omega} y_0 - \int_0^{\frac{2p\pi}{\omega}} \cos(s\omega) f(s, e^{As}\mathbf{z})\, ds = 0,
$$

the damped pendulum with small non-autonomous perturbation (3) has a periodic solution $x(t, \varepsilon)$, such that $(x(0, \varepsilon), \dot{x}(0, \varepsilon)) \to (x_0^*, y_0^*)$ when $\varepsilon \to 0$.

We provide an application of Theorem 1 in the following corollary.

Corollary 3. *If $f(t, x, \dot{x}) = \cos(\omega t)$, then the differential equation (3), for $\varepsilon \neq 0$ sufficiently small, has one periodic solution $x(t, x)$ such that*

$$
(x(0, \varepsilon), \dot{x}(0, \varepsilon)) \to \left(0, \frac{1}{\bar{b}} \right),
$$

when $\varepsilon \to 0$.

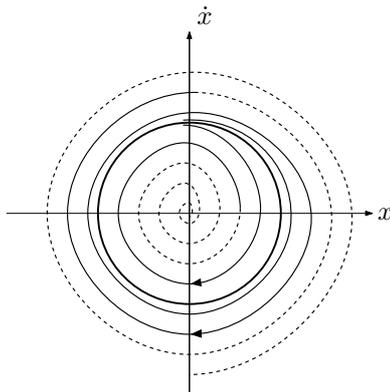The Corollaries 2 and 3 will be proved in section 6.

FIGURE 5.   Limit Cycle.

2.2. **Autonomous perturbation.** Consider the equation of motion of the damped pendulum with small autonomous perturbation (3), with $f(t, x, \dot{x}) = f(x, \dot{x})$ given by

$$f(x, \dot{x}) = c_1 x + c_2 \dot{x} + c_3 x^3 + c_4 \dot{x}^3,$$

where $c_1, c_3 \in \mathbb{R}$, $c_2 < \bar{b}$ and $c_4 > 0$. For $\varepsilon \neq 0$ sufficiently small, this system has one periodic solution, represented in the Figure 5.

For this system the functions $\mathcal{F}_1(x, y)$ and $\mathcal{F}_2(x, y)$ are given by

$$\mathcal{F}_1(x, y) = \frac{\pi x \left(4c_2\omega^4 - 4\bar{b}\omega^4\right)}{4\omega^5} - \frac{\pi c_1 y}{\omega^3} - \frac{3\pi c_3 y^3}{4\omega^5} - \frac{3\pi c_3 x^2 y}{4\omega^3} + \frac{3}{4}\pi c_4 \omega x^3 + \frac{3\pi c_4 x y^2}{4\omega},$$

$$\mathcal{F}_2(x, y) = \frac{\pi y \left(4c_2\omega^2 - 4\bar{b}\omega^2\right)}{4\omega^3} + x \left(\frac{\pi c_1}{\omega} + \frac{3\pi c_3 y^2}{4\omega^3}\right) + \frac{3\pi c_3 x^3}{4\omega} + \frac{3}{4}\pi c_4 \omega x^2 y + \frac{3\pi c_4 y^3}{4\omega},$$

then the system of equations (5) has a unique solution, and this solutions is the trivial solution $(x_0, y_0) = (0, 0)$. It implies that the Theorem 1 is not efficient in finding periodic solution in the autonomous perturbation case.

Given $r > 0$ and $0 \leq \varphi < 2\pi$, we define the real functions

$$(7) \qquad \mathcal{G}(r) = -\bar{b}\omega\pi r + \int_0^{2\pi} \sin(\varphi) f(r\cos(\varphi), \omega r \sin(\varphi)) d\varphi,$$

A zero r_0^* of the equation

$$(8) \qquad\qquad\qquad \mathcal{G}(r_0) = 0,$$

such that $d_{r_0}\mathcal{G}(r_0^*) \neq 0$, is called a *simple zero* of system (8).

Our second main result on the periodic solutions of the damped pendulum with small perturbation (3) is the following

Theorem 4. *Assume that the function f of the damped pendulum with small perturbation with equations of motion (2) is smooth and independent of t, then for $\varepsilon \neq 0$ sufficiently small and for every simple zero $r_0^* \neq 0$ of the equation (8), the damped pendulum with small autonomous perturbation (3) has a periodic solution $x(t, \varepsilon)$, such that for all $t \in \mathbb{R}$, $|(x(t, \varepsilon), \dot{x}(t, \varepsilon))| \to r_0^*$, when $\varepsilon \to 0$.*

Theorem 4 is proved in section 4. Its proof is based in the averaging theory for computing periodic solutions, which has been explored in section 3.

We provide an application of Theorem 4 in the following corollary, which will be proved in the section 6.

Corollary 5. *If $f(x, \dot{x}) = c_1 x + c_2 \dot{x} + c_3 x^3 + c_4 \dot{x}^3$ with $c_1, c_3 \in \mathbb{R}$, $c_2 < \bar{b}$ and $c_4 > 0$, then the differential equation (3), for $\varepsilon \neq 0$ sufficiently small, has one periodic solution $x(t, x)$ such that*

$$|(x(t, \varepsilon), \dot{x}(t, \varepsilon))| \to \frac{2}{\omega} \sqrt{\frac{b - c_2}{3 c_4}},$$

when $\varepsilon \to 0$.

3. Averaging Theory

We present in this section a basic result known as *First Order Averaging Theorem*. For a general introduction to averaging theory see for instance the book of Sanders and Verhulst [4].

We consider the differential equation

$$(9) \qquad\qquad \dot{X} = \varepsilon\, F_1(s, X) + \varepsilon^2\, R(s, X, \varepsilon)$$

where the smooth functions $F_1 : \mathbb{R} \times U \to \mathbb{R}^n$ and $R : \mathbb{R} \times U \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are T-periodic in the first variable and U is an open subset of $\mathbb{R}^n$.

We define the averaged system associated with system (9) as

$$(10) \qquad\qquad \dot{Y}(t) = \varepsilon f_1(Y(s)),$$

where $f_1 : U \to \mathbb{R}^n$ is given by

$$(11) \qquad\qquad f_1(Z) = \int_0^T F_1(s, Z) ds.$$

We present, in the following theorem, the conditions for the singularities, of the system (10), be associated with the periodic solutions of the differential system (9).

Theorem 6. *Assume that for $a \in U$ with $f_1(a) = 0$, there exist a neighbourhood V of a such that $f_1(Z) \neq 0$ for all $Z \in \bar{V} - \{a\}$ and $\det(df_1(a)) \neq 0$. Then, for $|\varepsilon| > 0$ sufficiently small, there exist a T-periodic solution $X(t, \varepsilon)$ of the system (9) such that $X(0, \varepsilon) \to a$ as $\varepsilon \to 0$.*

Theorem 6 can be announced with weaker hypothesis using Brower degree. For a proof of Theorem 6 see [5].

4. Proof of Theorem 1

In order to use the Theorem 6 in the proof of Theorem 1, we have to modify the equation (3). Let's take $y = \dot{x}$. If we denote

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix} \quad \text{and} \quad F(t, \mathbf{x}) = \begin{pmatrix} 0 \\ -\bar{b}y + f(t, x, y) \end{pmatrix},$$

then the equation (3) can be written in a matrix form

$$(12) \qquad\qquad \dot{\mathbf{x}} = A\mathbf{x} + \varepsilon F(t, \mathbf{x}).$$

We define a new variable $\mathbf{y}(t) \in \mathbb{R}^2$ by

$$\mathbf{y}(t) = e^{-At}\mathbf{x}(t),$$

where e^{At} is the matrix of fundamental solution of the unperturbed differential system (12), i.e., $\varepsilon = 0$. Note that $y(0) = x(0)$.

The system (12) is written in the new variable $\mathbf{y}$ as

(13) $$\dot{\mathbf{y}} = \varepsilon e^{-At}\tilde{F}(t, \mathbf{y}),$$

with $\tilde{F}(t, \mathbf{y}) = F(t, e^{At}\mathbf{y})$.

Now, we are ready to prove the Theorem 1.

Proof of Theorem 1. Assume that the smooth function $f(t, \mathbf{x})$ is T-periodic in t, then the smooth function $\tilde{F}(t, \mathbf{y})$ is also T-periodic in t. We shall apply Theorem 6 of the section 3 to the differential equation (13). We note that equation (13) can be written as equation (9) taking

$$X = \mathbf{y}, \quad s = t, \quad F_1(t, \mathbf{y}) = e^{-At}\tilde{F}(t, \mathbf{y}) \quad \text{and} \quad R(t, \mathbf{y}, \varepsilon) = 0.$$

Given $\mathbf{z} = (x_0, y_0) \in \mathbb{R}^2$, we define the averaged function $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(\mathbf{z}) = \int_0^T F_1(s, \mathbf{z})ds.$$

Proceeding with calculations we have

$$f_1(\mathbf{z}) = \left(\left(\frac{\bar{b}\sin(2T\omega)}{4\omega} - \frac{\bar{b}T}{2}\right)x_0 + \frac{\bar{b}\sin^2(T\omega)}{2\omega^2}y_0 - \int_0^T \frac{\sin(s\omega)f(s, e^{As}\mathbf{z})}{\omega}\,ds \right.,$$

$$\frac{\bar{b}\sin^2(T\omega)}{2}x_0 + \left(-\frac{\bar{b}\sin(2T\omega)}{4\omega} - \frac{\bar{b}T}{2}\right)y_0 + \left.\int_0^T \cos(s\omega)f(s, e^{As}\mathbf{z})\,ds \right),$$

$$= (\mathcal{F}_1(\mathbf{z}), \mathcal{F}_2(\mathbf{z})).$$

Then, by Theorem 6 we have that for every simple zero $(x_0^*, y_0^*) \in V$ of the system (5), there exist a periodic solution $\mathbf{y}(t, \varepsilon)$ of the system (13) such that

$$\mathbf{y}(0, \varepsilon) \to (x_0^*, y_0^*) \quad \text{when } \varepsilon \to 0.$$

It has proved the existence of a periodic solution $\mathbf{x}(t, \varepsilon) = (x(t, \varepsilon), y(t, \varepsilon))$ of the system (12) and consequently of the equation (3). As $\mathbf{x}(0) = \mathbf{y}(0)$ it follows that

$$\begin{pmatrix} x(0, \varepsilon) \\ y(0, \varepsilon) \end{pmatrix} \to \begin{pmatrix} x_0^* \\ y_0^* \end{pmatrix}.$$

when $\varepsilon \to 0$. Hence Theorem 1 is proved. $\qquad\square$

5. Proof of Theorem 4

Rescaling the time by $\tau = \omega t$, the equation (3) becomes

(14) $$x'' = -x + \varepsilon\bar{f}(t, x, x')$$

where now the prime denotes derivative with respect to the new time τ and

$$\bar{f}(t, x, x') = \frac{1}{a}\left(-\bar{b}\omega x' + f(t, x, \omega x')\right).$$

In order to use the Theorem 6 in the proof of Theorem 4, we have to modify the equation (3). Take $y = x'$, then the second order differential equation (14) can be written as the system os first order differential equations

$$
\begin{aligned}
x' &= y, \\
y' &= -x + \varepsilon \bar{f}(t, x, y).
\end{aligned}
$$

(15)

In polar coordinates

$$
\begin{aligned}
x &= r\cos(\varphi), \quad r^2 = x^2 + y^2, \\
y &= r\sin(\varphi), \quad \tan(\varphi) = \frac{y}{x},
\end{aligned}
$$

with $r > 0$ and $0 \leq \varphi < 2\pi$, the system (15) becomes

$$
\begin{aligned}
r' &= \varepsilon \sin(\varphi) g(r, \varphi), \\
\varphi' &= -1 + \varepsilon \frac{\cos(\varphi) g(r, \varphi)}{r},
\end{aligned}
$$

(16)

where $g(r, \varphi) = \bar{f}(r\cos(\varphi), r\sin(\varphi))$.

Using φ as the new independent variable, the system (16) becomes

$$
\frac{dr}{d\varphi} = \frac{\varepsilon r \sin(\varphi) g(r, \varphi)}{\varepsilon \cos(\varphi) g(r, \varphi) - r}.
$$

Expanding the above expression in Taylor's series near $\varepsilon = 0$, we have

$$
(17) \qquad \frac{dr}{d\varphi} = -\varepsilon \sin(\varphi) g(r, \varphi) - \varepsilon^2 \tan(\varphi) \sum_{n=2}^{\infty} \frac{(\cos(\varphi) g(r, \varphi))^n}{r^{n-1}} \varepsilon^{n-2}.
$$

Now, we are ready to prove the Theorem 4.

Proof of Theorem 4. Assume that the smooth function $f(t, x, \dot{x})$ doesn't depend of the variable t, i.e, $f(t, x, \dot{x}) = f(x, \dot{x})$, then the smooth function $g(r, \varphi)$ also doesn't depend of the variable t t. We shall apply Theorem 6 of the section 3. We note that the equation (17) can be written as equation (9) taking

$$
X = r, \quad s = \varphi, \quad F_1(\varphi, r) = -\sin(\varphi) g(r, \varphi) \quad \text{and}
$$

$$
R(\varphi, r, \varepsilon) = -\tan(\varphi) \sum_{n=2}^{\infty} \frac{(\cos(\varphi) g(r, \varphi))^n}{r^{n-1}} \varepsilon^{n-2},
$$

Observe that the function $F_1(\varphi, r)$ is 2π–periodic in φ, then the hypothesis, about equation (9) of Theorem 6, are satisfied.

Given $r > 0$, we define the averaged function $f_1 : \mathbb{R} \to \mathbb{R}$

$$
f_1(r) = \int_0^{2\pi} F_1(\varphi, r) d\varphi.
$$

Proceeding with calculations we have

$$
\begin{aligned}
f_1(r) &= -\int_0^{2\pi} \sin(\varphi)g(r,\varphi)d\varphi, \\
&= -\int_0^{2\pi} \sin(\varphi)\left(\bar{f}(r\cos(\varphi), r\sin(\varphi))\right)d\varphi, \\
&= -\int_0^{2\pi} \sin(\varphi)\frac{1}{a}\left(-\bar{b}\omega r\sin(\varphi) + f(r\cos(\varphi), \omega r\sin(\varphi))\right)d\varphi, \\
&= \frac{\bar{b}\omega}{a}r\int_0^{2\pi}\sin^2(\varphi)d\varphi - \frac{1}{a}\int_0^{2\pi}\sin(\varphi)(r\cos(\varphi), \omega r\sin(\varphi))d\varphi, \\
&= \frac{\bar{b}\omega\pi}{a}r - \frac{1}{a}\int_0^{2\pi}\sin\varphi(r\cos(\varphi), \omega r\sin(\varphi))d\varphi, \\
&= -\frac{1}{a}\mathcal{G}(r).
\end{aligned}
$$

Then, by Theorem 6, we have that for every simple zero r_0^* for the equation (8), there exist a periodic solution $r(\varphi, \varepsilon)$ of the equation (17) such that

$$
r(0, \varepsilon) \to r_0^* \quad \text{quando } \varepsilon \to 0.
$$

It has proved the existence of a periodic solution $x(t, \varepsilon)$ of the equation (3) such that

$$
|(\dot{x}(t, \varepsilon), x(t, \varepsilon))| \to r_0^*,
$$

for all $t \in \mathbb{R}$, when $\varepsilon \to 0$. Hence Theorem 4 is proved. $\qquad \square$

6. Proofs of Corollaries

Demonstração do Corolário 2. We have that $f(t, x, \dot{x})$ is T–periodic with $T = (p/q)(2\pi/\omega)$, then $\bar{T} = qT$ is also a period of the function $f(t, x, \dot{x})$. Since $\sin(2\bar{T}\omega) = 0$, it follows

$$
\mathcal{F}_1(\mathbf{z}) = -\frac{\bar{b}T}{2}x_0 - \frac{1}{\omega}\int_0^T \sin(s\omega)f(s, e^{As}\mathbf{z})\,ds,
$$

$$
\mathcal{F}_2(\mathbf{z}) = -\frac{\bar{b}T}{2}y_0 + \int_0^T \cos(s\omega)f(s, e^{As}\mathbf{z})\,ds.
$$

Note that the systems (6) is equivalent to the system

$$
\mathcal{F}_1(\mathbf{z}) = 0, \quad \mathcal{F}_2(\mathbf{z}) = 0,
$$

because both equations only differs in a non–zero multiplicative constant. Hence, applying the Theorem 1, the result of the corollary 2 follows. $\qquad \square$

Proof of Corollary 3. We have that $f(t, x, \dot{x})$ is T–periodic, with $T = 2\pi/\omega$, then we can apply the Corollary 2. Proceeding with the calculations we get

$$
\tilde{\mathcal{F}}_1(\mathbf{z}) = \frac{\bar{b}\pi}{\omega}x,
$$

$$
\tilde{\mathcal{F}}_2(\mathbf{z}) = \frac{\bar{b}\pi}{\omega}y - \frac{\pi}{\omega},
$$

with

$$
(x_0^*, y_0^*) = \left(0, \frac{1}{\bar{b}}\right)
$$

as a solution of the system (6). Since

$$\det\left(\frac{\partial(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)}{\partial \mathbf{z}}\right) = \left(\frac{\bar{b}\pi}{\omega}\right)^2,$$

it follows that (x_0^*, y_0^*) is a simple zero of system (6). Hence, applying the Corollary 2, the result of the corollary 3 follows. $\qquad \square$

Proof of Corollary 5. Proceeding with calculations we get

$$\mathcal{G}(r) = \frac{3}{4}\omega^3\pi c_4 r^3 - (b - c_2)\omega\pi r,$$

a polynomial in r, whose zeros are

$$0 \quad \text{and} \quad \pm\frac{2}{\omega}\sqrt{\frac{b - c_2}{3c_4}}.$$

Let's take

$$r_0^* = 2\sqrt{\frac{b - c_2}{3c_4}},$$

since

$$\frac{d\mathcal{G}}{r_0}(r_0^*) = 2\omega\pi(b - c_2) \neq 0,$$

it follows that r_0^* is a simple zero of system (8). Hence, applying the Theorem 4, the result of the corollary 5 follows. $\qquad \square$

7. CONCLUSIONS

The averaging theory is a collection of techniques to study the persistence of periodic orbits of a dynamical system under small perturbations.

In this paper, we present some of this techniques and we use it to give conditions for the existence of periodic solutions of the equation of motion of the perturbed damped pendulum. This problem is divided in two cases: non–autonomous perturbation and autonomous perturbation. The same technique is used in both cases, differing only in the preliminary treatment in each case before apply the first order averaging theorem.

For the first case, the equation of motion is written as a system of two first order differential equation, which the independent variable is the time. For the second case, using polar coordinates, we approximate the equation of motion by a first order differential equation independent of the time.

We get, in both cases, systems of equations that have their simple zeros associated with periodic solutions of the equation of motion of the perturbed damped pendulum. These relations are explicit in the theorems that we prove here.

8. ACKNOWLEDGEMENTS

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