The Beta Burr XII Distribution with Applications to Lifetime Data

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Abstract

For the first time, a five-parameter distribution, so-called the beta Burr XII distribution, is defined and investigated. The new distribution contains as special sub-models some well-known distributions discussed in the literature, such as the logistic, Weibull and Burr XII distributions, among several others. We derive its moment generating function. As a special case, we obtain the moment generating function of the Burr XII distribution, which seems to be a new result. Moments, mean deviations, Bonferroni and Lorenz curves and reliability are provided. We derive two representations for the moments of the order statistics. The method of maximum likelihood is proposed for estimating the model parameters. We obtain the observed information matrix. An application to a real data set demonstrates that the new distribution can provide a better fit than other classical models. We hope that this generalization may attract wider applications in reliability, biology and lifetime data analysis.

Keywords: Beta distribution; Burr XII distribution; Maximum likelihood; Observed information matrix; Order statistic; Weibull distribution.

1 Introduction

Burr system of distributions was constructed in 1942 by Irving W. Burr. Since the corresponding density functions have a wide variety of shapes, this system is useful for approximating histograms,

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particularly when a simple mathematical structure for the fitted cumulative distribution function (cdf) is required. Other applications include simulation, quantal response, approximation of distributions, and development of non-normal control charts. A number of standard theoretical distributions are limiting forms of Burr distributions. Rodriguez (1977) showed that the Burr coverage area on a specific plane is occupied by various well-known and useful distributions, including the normal, log-normal, gamma, logistic and extreme-value type-I distributions. The Burr XII (BXII) distribution, having logistic and Weibull as special sub-models, is a very popular distribution for modeling lifetime data and for modeling phenomenon with monotone failure rates. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, it does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub shaped and the unimodal failure rates that are common in reliability and biological studies. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive anti-mode. Unimodal failure rates can be observed in course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

The cdf and the reliability function of the BXII can be written in closed form, thus it simplifies the computation of the percentiles and the likelihood function for censored data. This distribution has algebraic tails which are effective for modeling failures that occur with lesser frequency than with those models based on exponential tails. Hence, it represents a good model for modeling failure time data (Zimmer et al., 1998). Shao (2004a) discussed maximum likelihood estimation for the three-parameter BXII distribution. Shao et al. (2004b) studied models for extremes using the extended three parameter BXII distribution with application to flood frequency analysis. According to Soliman (2005), this distribution covers the curve shape characteristics for a large number of distributions. The versatility and flexibility of the BXII distribution turns it quite attractive as a tentative model for data whose underlying distribution is unknown. Wu et al. (2007) studied the estimation problems for this distribution based on progressive type II censoring with random removals, where the number of units removed at each failure time has a discrete uniform distribution. Silva et al. (2008) proposed a location-scale regression model based on this distribution, refereed to as the log-Burr XII regression model, for lifetime data analysis as a feasible alternative to the log-logistic regression model.

Many models were introduced in the literature by extending the Weibull and exponential distributions. For example, the exponentiated Weibull (EW) (Mudholkar et al., 1995, 1996), the additive Weibull (Xie and Lai, 1995), the extended Weibull (Xie et al., 2002), the modified Weibull (MW) (Lai et al., 2003), the beta exponential (BE) (Nadarajah and Kotz, 2005), the beta Weibull (BW) (Lee et al., 2007), the extended flexible Weibull (Bebbington et al., 2007), the generalized modified Weibull (GMW) (Carrasco et al., 2008) and the generalized inverse Weibull (Gusmão et al., 2009) distributions.

In this article, we propose a new distribution with five parameters, refereed to as the beta Burr XII (BBXII) distribution, which contains as special sub-models the beta log-logistic (BLL), BW, BXII, Weibull and log-logistic (LL) distributions, among others. Our distribution due to its flexibility in accommodating different forms of the risk function is an important model that can be used in a variety of problems in modeling lifetime data. The BBXII distribution is not only convenient for modeling comfortable unimodal-shaped failure rates but it is also suitable for testing goodness-of-fit of some special sub-models such as the BLL, BW and Weibull distributions.

The rest of the paper is organized as follows. In Section 2, we define the BBXII distribution and present some special sub-models. In Section 3, we derive expansions for its probability density function (pdf) and cdf. We show that the density function of the BBXII distribution can be expressed as a mixture of BXII density functions. General expansions for the moments are also given in this section. In Section 4, we give an expansion for the moment generating function (mgf). In Section 5, we derive the mean deviations, Bonferroni and Lorenz curves and the reliability. Section 6 is devoted to order statistics, their moments and L-moments (Hosking, 1986). In Section 7, we discuss maximum likelihood estimation and calculate the elements of the observed information matrix. in Section 8, we give an application of the BBXII distribution to a real data set to show that it can yield a better fit than some other known models. Section 9 ends with some concluding remarks.

2 Model Definition

If G denotes the baseline cdf, Eugene et al. (2002) defined a generalized class of distributions by

$$F(x) = I_{G(x)}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x)} w^{a-1}(1-w)^{b-1} dw, \tag{1}$$

for $a > 0$ and $b > 0$. Here, $I_y(a, b) = B_y(a, b)/B(a, b)$ is the incomplete beta function ratio, $B_y(a, b) = \int_0^y w^{a-1}(1-w)^{b-1} dw$ is the incomplete beta function and $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function, where $\Gamma(\cdot)$ is the gamma function. This class of generalized distributions has been receiving considerable attention over the last years, in particular after the work of Jones (2004). In fact, based on equation (1), Eugene et al. (2002), Nadarajah and Kotz (2004), Nadarajah and Gupta (2004) and Nadarajah and Kotz (2005) proposed the beta normal, beta Gumbel, beta Fréchet and beta exponential distributions, respectively. More recently, Barreto-Souza et al. (2009) and Pescim et al. (2010) introduced the beta generalized exponential and beta generalized half-normal distributions, respectively. Another distribution that happens to belong to equation (1) is the beta logistic distribution, which has been around for over 20 years (Brown et al., 2002), even if it did not originate directly from this equation.

The density function corresponding to (1) is given by

$$f(x) = \frac{g(x)}{B(a, b)} G(x)^{a-1}\{1 - G(x)\}^{b-1}, \tag{2}$$

where $g(x) = dG(x)/dx$ is the density function of the baseline distribution. The density $f(x)$ will be most tractable when the functions $G(x)$ and $g(x)$ have simple analytic expressions as is the cases of the BXII and Weibull distributions. Except for some special choices for $G(x)$, equation (2) will be difficult to deal with in generality.

We are motivated to introduce the BBXII distribution because of the generalizations discussed in Section 1, the wide usage of the BXII distribution and the fact that the current generalization provides means of its continuous extension to still more complex situations. Zimmer et al. (1998) introduced the three parameter BXII distribution with cdf and pdf (for $x > 0$) given by

$$G(x; s, k, c) = 1 - \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k} \tag{3}$$

and

$$g(x; s, k, c) = ck \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k-1} \frac{x^{c-1}}{s^{c}}, \tag{4}$$

respectively, where $k > 0$ and $c > 0$ are shape parameters and $s > 0$ is a scale parameter.

If $c > 1$, the density function is unimodal with mode at

$$\frac{x}{s} = \left(\frac{c-1}{ck+1}\right)^{\frac{1}{c}},$$

and is L-shaped if $c = 1$. If $ck > q$, the qth moment about zero of the BXII distribution is given by

$$\mu'_q = s^q k B\left(k - \frac{q}{c}, \frac{q}{c} + 1\right).$$

The BBXII distribution is defined by taking $G(x)$ in equation (1) to be the cdf (3) of the BXII distribution and its cdf (for $x > 0$) is given by

$$F(x) = I_{1-[1+(x/s)^c]^{-k}}(a, b) = \frac{1}{B(a, b)} \int_0^{1-[1+(x/s)^c]^{-k}} \omega^{a-1}(1-\omega)^{b-1} d\omega. \tag{5}$$

The BBXII density function can be written from equations (2)-(4) as

$$f(x) = \frac{ck(x)^{c-1}}{s^c B(a, b)} \left\{1 - \left[1 + (x/s)^c\right]^{-k}\right\}^{a-1} \left[1 + (x/s)^c\right]^{-(kb+1)}. \tag{6}$$

The importance of the BBXII distribution is that it contains as special sub-models several well-known distributions. Clearly, the BXII distribution is a particular case for $a = b = 1$. For $b = 1$, it becomes the exponentiated Burr XII (EBXII) distribution, which is also not known in the literature. For $s = m^{-1}$ and $k = 1$, it reduces to a new distribution referred to as the beta log-logistic (BLL) distribution. For $a = b = 1$, $s = m^{-1}$ and $k = 1$, it becomes the LL distribution, where the survival function is $S(t; m, c) = [1 + (tm)^c]^{-1}$. For $a = c = 1$ and $a = b = c = 1$, it reduces to the beta Pareto type II (BPII) and Pareto type II (PII) distributions, respectively. If $k \to \infty$, it is identical to the BW distribution. If $k \to \infty$ in addition to $a = b = 1$, it becomes the Weibull distribution.

Some of the possible shapes of the density (6) for selected parameter values, including well-known distributions, are illustrated in Figure 1. The BBXII hazard rate function is given by (for

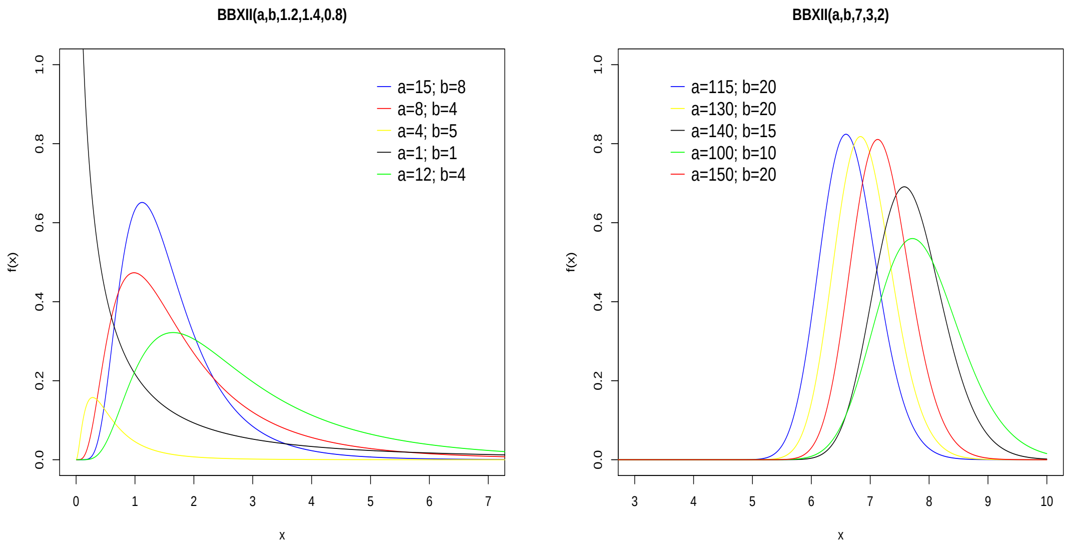


Figure 1: Plots of the density function (6) for some parameter values.

$x > 0$)

$$h(x) = \frac{ck(x)^{c-1}\left\{1 - \left[1 + (x/s)^c\right]^{-k}\right\}^{a-1}\left[1 + (x/s)^c\right]^{-(kb+1)}}{s^c B(a,b)\left[1 - I_{\left\{1-\left[1+(x/s)^c\right]^{-k}\right\}}(a,b)\right]}. \tag{7}$$

Some different BBXII hazard rate shapes are illustrated in Figure 2 for selected parameter values.

Let X be a random variable with density function (6), say $X \sim BBXII(a,b,s,k,c)$, and V a beta random variable with parameters a and b. We can generate X following the BBXII distribution by inverting (5) yielding $X = s\left[(1-V)^{-\frac{1}{k}} - 1\right]^{\frac{1}{c}}$.

3 Properties

We hardly need to emphasize the necessity and importance of moments in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution

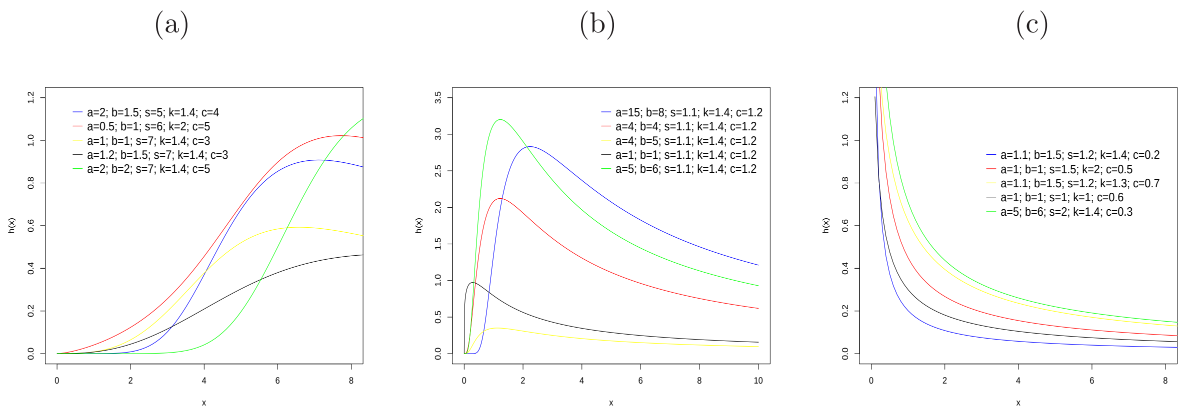


Figure 2: Plots of the hazard rate function (7) for some parameter values. (a) Increasing (b) Upside-down bathtub (c) Decreasing.

can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis).

Theorem 1: If $X \sim BBXII(a, b, s, k, c)$, we have the following approximations:

1.1 For $a > 0$ and $b > 0$ real non-integers, we have the double mixture form

$$f(x) = \sum_{j,r=0}^{\infty} w_{j,r} g(x; s, k(r+1), c), \tag{8}$$

where $g(x; s, k(r+1), c)$ represents the BXII density function with scale parameter s and shape parameters c and $k(r+1)$, and the coefficients are given by

$$w_{j,r} = w_{j,r}(a, b) = \frac{(-1)^{j+r}\Gamma(b)\Gamma(a+j)}{B(a,b)\Gamma(b-j)\Gamma(a+j-r)j!(r+1)!}. \tag{9}$$

Equation (8) shows that the BBXII density is an infinite linear combination of BXII density functions. If b is an integer, the index j stops at $b-1$. If a is an integer, r stops at $a+j-1$.

1.2 For $a > 0$ and $b > 0$ real non-integers, we have

$$F(x) = \sum_{j,r=0}^{\infty} w_{j,r} G(x; s, k(r+1), c), \tag{10}$$

where $w_{j,r}$ is given by (9).

1.3 The nth moment of the BBXII distribution is given by

$$\mu_n' = \sum_{j,r=0}^{\infty} w_{j,r} s^n k B\Big[k(r+1) - \frac{n}{c}, \frac{n}{c} + 1\Big]. \tag{11}$$

Proof 1.1:
First, if $|z| < 1$ and $b > 0$ is real non-integer, we have the series representation

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j) j!} z^j. \tag{12}$$

Using (12), the BBXII cumulative function (5) for $b > 0$ real non-integer can be written as

$$F(x) = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j) j!} \int_0^{1-\left[1+(x/s)^c\right]^{-k}} \omega^{a+j-1} d\omega.$$

After integration, we obtain

$$F(x) = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b) \left\{1 - \left[1+(x/s)^c\right]^{-k}\right\}^{a+j}}{\Gamma(b-j) j! (a+j)}. \tag{13}$$

Differentiating (13) yields

$$f(x) = \sum_{j=0}^{\infty} \frac{ck(x)^{c-1}\left[1+(x/s)^c\right]^{-k-1}}{B(a,b)s^c} \frac{(-1)^j \Gamma(b) \left\{1 - \left[1+(x/s)^c\right]^{-k}\right\}^{a+j-1}}{\Gamma(b-j) j!}.$$

The series expansion (12) for $\left\{1 - \left[1+(x/s)^c\right]^{-k}\right\}^{a+j-1}$ in the last equation gives

$$f(x) = \sum_{j=0}^{\infty} \frac{ck(x)^{c-1}\left[1+(x/s)^c\right]^{-k-1}}{B(a,b)s^c} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j) j!} \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(a+j)}{\Gamma(a+j-r) r!} \left[1+(x/s)^c\right]^{-kr}$$

and after some algebraic manipulation, we obtain

$$f(x) = \sum_{j,r=0}^{\infty} \frac{(-1)^{j+r} \Gamma(b) \Gamma(a+j)}{B(a,b) \Gamma(b-j) \Gamma(a+j-r) j! r!} ck \left[1 + \left(\frac{x}{s}\right)^c\right]^{-k(r+1)-1} \frac{x^{c-1}}{s^c}.$$

Finally, we can obtain

$$f(x) = \sum_{j,r=0}^{\infty} w_{j,r} g(x; s, k(r+1), c)$$

where $w_{j,r}$ is given by (9) and $g(x; s, k(r+1), c)$ is the $BXII(s, k(r+1), c)$ density function with scale parameter c and shape parameters s and $k(r+1)$. $\blacksquare$

Proof 1.2:

Using Theorem 1.1 we obtain (10) by simple integration.

Proof 1.3:

The nth moment of X for $b > 0$ real non-integer comes from Theorem 1.1

$$E(X^n) = \sum_{j,r=0}^{\infty} w_{j,r} E(X^n_{s,k(r+1),c}), \tag{14}$$

where $X_{s,k(r+1),c} \sim BXII(s, k(r+1), c)$. Using the results in Zimmer et al. (1998), we obtain

$$\mu'_n = ks^n \sum_{j,r=0}^{\infty} w_{j,r} B[k(r+1) - nc^{-1}, nc^{-1} + 1]. \blacksquare$$

Plots of the skewness and kurtosis for some choices of b as function of a, and for some choices of a as function of b, are shown in Figures 3 and 4, respectively.

4 Moment Generating Function

Let $X \sim BBXII(a, b, s, k, c)$. The mgf of X, say $M(t) = E[\exp(tX)]$, is given by

$$M(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k),$$

where $E(X^k)$ follows from equation (14).

Another representation for $M(t)$ can be obtained from equation (15) as an infinite weighted sum

$$M(t) = \sum_{j,r=0}^{\infty} w_{j,r} M_{r+1}(t), \tag{15}$$

where $M_{r+1}(t)$ is the mgf of the $BurrXII(s, k(r+1), c)$ distribution and $w_{j,r}$ is defined by (9). We provide a simple representation for the mgf $M_{BXII}(t)$ of the $BXII(s, k, c)$ distribution. We can write for $t < 0$

$$M_{BXII}(t) = ck \int_0^{\infty} \exp(yst) y^{c-1} (1 + y^c)^{-(k+1)} dy.$$

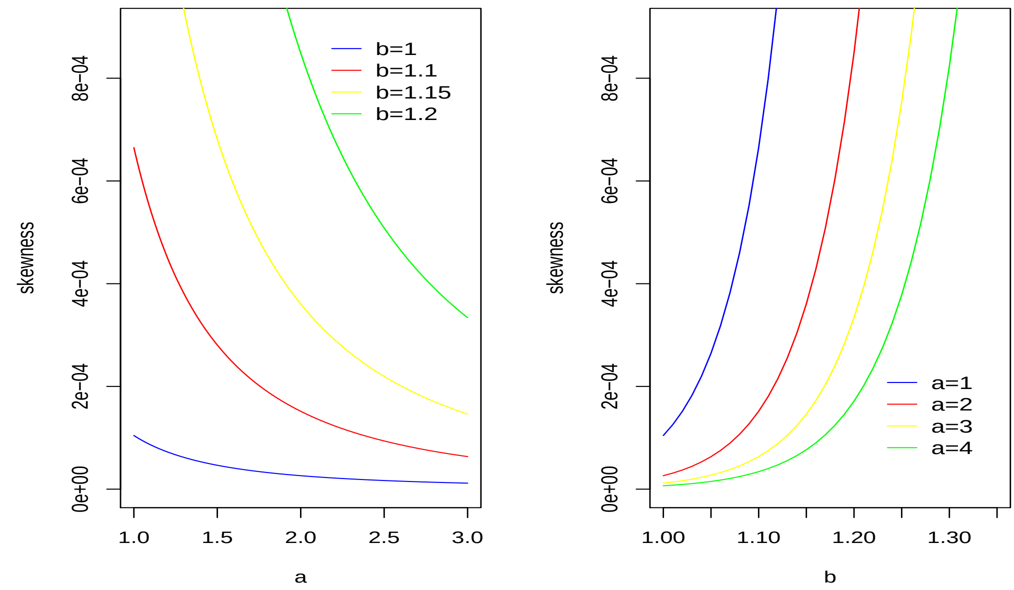


Figure 3: Skewness of the BBXII distribution as function of a for selected values of b and as function of b for some values of a.

Now, we need the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+t)\prod_{j=1}^{n}\Gamma(1-a_j-t)}{\prod_{j=n+1}^{p}\Gamma(a_j+t)\prod_{j=m+1}^{p}\Gamma(1-b_j-t)} x^{-t}dt,$$

where $\mathrm{i}=\sqrt{-1}$ is the complex unit and L denotes an integration path; see Section 9.3 in Gradshteyn and Ryzhik (2000) for a description of this path. The Meijer G-function contains as particular cases many integrals with elementary and special functions (Prudnikov *et al.*, 1986).

We now assume that $c = m/k$, where m and k are positive integers. This condition is not restrictive since every positive real number can be approximated by a rational number. Using the integral (25) given in Appendix 1, we conclude for $t < 0$ that

$$M_{BXII}(t) = mI\left(-st, \frac{m}{k}-1, \frac{m}{k}, -k-1\right). \tag{16}$$

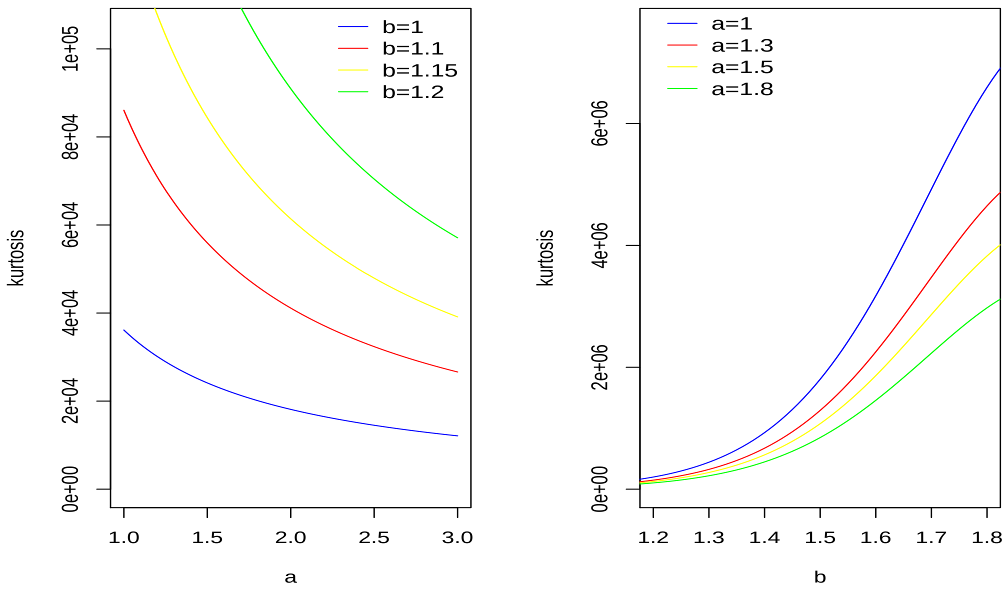


Figure 4: Kurtosis of the BBXII distribution as function of a for selected values of b and as function of b for some values of a.

Hence, from equation (15), the mgf of the $BBXII(a, b, s, k, c)$ distribution for $t < 0$ follows as

$$M(t) = m \sum_{j,r=0}^{\infty} w_{j,r} I\left(-st, \frac{m}{k(r+1)} - 1, \frac{m}{k(r+1)}, -k(r+1) - 1\right). \tag{17}$$

Equation (17) is the main result of this section. The characteristic function (chf) is simply $\phi(t) = M(\mathrm{i}t)$, which holds for $t < 1$.

For the special cases $c = 1$ and $c = 2$, we can obtain simple expressions for $M_{BXII}(t)$ and consequently for $M(t)$ using equations (1) (on page 16) and (2) (on page 20) of the book by Prudnikov *et al.* (1992). We have for $c = 1$ and $t < 0$

$$M_{BXII}(t) = k(-st)^k \exp(-st)\Gamma(-k, -st),$$

where $\Gamma(v, x) = \int_x^\infty t^{v-1} \exp(-t)dt$ is the complementary incomplete gamma function. For $c = 2$ and $t < 0$, we obtain

$$M_{BXII}(t) = {}_1F_2\left(1; \frac{1}{2}; 1-k; \frac{s^2t^2}{4}\right) + \frac{st}{2} B\left(2, k - \frac{1}{2}\right) {}_1F_2\left(1; \frac{3}{2}; k + \frac{7}{2}; \frac{-s^2t^2}{4}\right) + \frac{\Gamma(-2k)}{(-st)^{-2k}},$$

where

$$ {}_1F_2(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k (c)_k} \frac{x^k}{k!} $$

is a generalized hypergeometric function and $(a)_k = a(a+1)\dots(a+k-1)$ denotes the ascending factorial.

5 Other Measures

In this section, we calculate the following measures: means deviations, Bonferroni and Lorenz curves and the reliability for the BBXII distribution.

5.1 Means Deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median – defined by

$$ \delta_1(X) = \int_0^{\infty} |x-\mu|\, f(x)dx \ \text{ and } \ \delta_2(X) = \int_0^{\infty} |x-M|\, f(x)dx, $$

respectively, where $\mu = E(X)$ and $M = Median(X)$ denotes the median. Here, M is obtained as the solution of the non-linear equation $I_{1-[1+(x/s)^c]^{-k}}(a,b) = 1/2$. We define $T(q) = \int_q^{\infty} x f(x)dx$, which is calculated below. The measures $\delta_1(X)$ and $\delta_2(X)$ follow from the relationships

$$ \begin{aligned} \delta_1(X) &= \int_0^{\mu} (\mu - x) f(x)dx + \int_{\mu}^{\infty} (x-\mu) f(x)dx \\ &= 2\mu F(\mu) - 2\mu + 2T(\mu) \end{aligned} $$

and

$$ \begin{aligned} \delta_2(X) &= MF(M) - \int_0^{M} x f(x)dx - M\{1 - F(M)\} + \int_M^{\infty} x f(x)dx \\ &= 2T(M) - \mu. \end{aligned} $$

Clearly, $F(M)$ and $F(\mu)$ are easily calculated from equation (5). We have from equation (8)

$$ T(q) = cks \sum_{j,r=0}^{\infty} w_{j,r}(a,b)(r+1) \int_{q/s}^{\infty} y^c (1+y^c)^{-k(r+1)-1} dy. $$

Setting $u = y^c$, we obtain

$$ T(q) = ks \sum_{j,r=0}^{\infty} w_{j,r}(a,b)(r+1) \int_{(q/s)^c}^{\infty} u^{1/c} (1+u)^{-k(r+1)-1} du. $$

We now define the following integral for $q > 0$ and $k > 0$ which is calculated using Maple

$$\begin{aligned} J(q,r,k) &= \int_q^\infty u^r (1+u)^{-k} du \\ &= -\left[\frac{{}_2F_1\left[(k,r+1);(2+r);-q\right]q^{r+1}}{(r+1)} + \frac{\Gamma(k-r-1)\pi\csc(\pi r)}{\Gamma(k)\Gamma(-r)}\right], \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function defined by

$${}_3F_2(a,b;d;x) = \frac{\Gamma(d)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^\infty \frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(d+j)} \frac{x^j}{j!}.$$

Hence,

$$T(q) = ks \sum_{j,r=0}^\infty w_{j,r}(r+1) J\left(\left(\frac{q}{s}\right)^c, \frac{1}{c}, k(r+1)+1\right). \tag{18}$$

Equation (18) is the main result of this sub-section from which $\delta_1(X)$ and $\delta_2(X)$ are immediately calculated.

5.2 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. They are defined by

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x) dx \;\text{ and }\; L(p) = \frac{1}{\mu}\int_0^q x f(x) dx,$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$. From $\int_0^q x f(x) = \mu - T(q)$, we have

$$B(p) = \frac{1}{p} - \frac{T(q)}{p\mu} \;\text{ and }\; L(p) = 1 - \frac{T(q)}{\mu}.$$

Some expansions for the inverse of the beta incomplete function $q = I_w^{-1}(a,b)$ in terms of the quantity w (here $w = 1 - [1 + (p/s)^c]^{-k}$) can be found in wolfram website[1].

5.3 Reliability

In the context of reliability, the stress-strength model describes the life of a component which has a random strength X_1 that is subjected to a random stress X_2. The component fails at the instant

[1] http://functions.wolfram.com/06.23.06.0004.01

that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X_1 > X_2$. Hence, $R = Pr(X_2 < X_1)$ is a measure of component reliability. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures, and the aging of concrete pressure vessels. In the area of stress-strength models there has been a large amount of work as regards estimation of the reliability R when X_1 and X_2 are independent random variables belonging to the same univariate family of distributions. We derive the reliability R when X_1 and X_2 have independent $BBXII(a_1, b_1, s, k_1, c)$ and $BBXII(a_2, b_2, s, k_2, c)$ distributions with the same shape parameter c. The reliability is given by

$$R = \int_0^\infty f_1(x)F_2(x)dx.$$

The cdf of X_2 and density of X_1 are obtained from Theorem 1

$$F_2(x) = \sum_{j,r=0}^{\infty} w_{j,r}(a_2, b_2)G(x; s, k_2(r+1), c)$$

and

$$f_1(x) = \sum_{p,q=0}^{\infty} w_{p,q}(a_1, b_1)g(x; s, k_1(q+1), c).$$

Hence,

$$R = \sum_{j,r,p,q=0}^{\infty} ck_1(q+1)w_{p,q}(a_1, b_1)w_{j,r}(a_2, b_2)s^{-c}I(c, s, k_1, k_2, r, q),$$

where

$$I(c, s, k_1, k_2, r, q) \quad = \quad \int_0^\infty \frac{x^{c-1}}{s^c}\left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-k_2(r+1)}\right\}\left[1 + \left(\frac{x}{s}\right)^c\right]^{-k_1(q+1)-1} dx.$$

Setting $u = 1 + (x/s)^c$, we have

$$I(c, s, k_1, k_2, r, q) \quad = \quad \frac{ck_2(r+1) - 1}{ck_1(q+1) + ck_2(r+1)},$$

and then we can obtain R as desired.

6 Order Statistics and Moments

Moments of order statistics play an important role in quality control testing and reliability, where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictors are often based on moments of order statistics. We now derive an explicit expression for the density of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$, in a random sample of size n from the BBXII distribution. It is well-known that

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x) F(x)^{i-1} \left[1 - F(x)\right]^{n-i},$$

for $i = 1, \ldots, n$. Using the binomial expansion in the last equation, we readily obtain

$$f_{i:n}(x) \quad = \quad \sum_{l=0}^{n-i} \frac{(-1)^l \binom{n-i}{l} f(x)}{B(i, n-i+1)} F(x)^{i+l-1}. \tag{19}$$

Substituting (6) and (13) into equation (19), the density $f_{i:n}(x)$ for $b > 0$ real non-integer is

$$\begin{aligned} f_{i:n}(x) \quad &= \quad \sum_{l=0}^{n-i} \frac{(-1)^l \binom{n-i}{l} \Gamma(b)^{i+l-1} ck(x)^{c-1} \left[1 + (x/s)^c\right]^{(-k-1)} \left\{1 - \left[1 + (x/s)^c\right]^{(-k)}\right\}^{a(i+l)-1}}{s^c B(a,b)^{i+l} B(i, n-i+1) \left\{\left[1 + (x/s)^c\right]^{-k}\right\}^{-(b-1)}} \\ &\times \left[\sum_{j=0}^{\infty} \frac{(-1)^j \left\{1 - \left[1 + (x/s)^c\right]^{-k}\right\}^j}{\Gamma(b-j) j! (a+j)}\right]^{i+l-1}. \end{aligned}$$

Using the identity $\left(\sum\limits_{i=0}^{\infty} a_i\right)^k = \sum\limits_{m_1=0}^{\infty} \ldots \sum\limits_{m_k=0}^{\infty} a_{m_1} \ldots a_{m_k}$ for k positive integer in the last equation, we can write

$$f_{i:n}(x) \quad = \quad \sum_{l=0}^{n-i} \sum_{m_1=0}^{\infty} \cdots \sum_{m_{i+l-1}=0}^{\infty} \delta_{i,l} f_{i,l}(x), \tag{20}$$

where

$$f_{i,l}(x) \quad = \quad \frac{ck(x)^{c-1} \left[1 + (x/s)^c\right]^{-k-1} \left\{1 - \left[1 + (x/s)^c\right]^{-k}\right\}^{\left[a(i+l) + \sum_{j=1}^{i+l-1} m_j\right] - 1}}{s^c \left\{\left[1 + (x/s)^c\right]^{-k}\right\}^{(1-b)} B\left[a(i+l) + \sum_{j=1}^{i+l-1} m_j, b\right]}.$$

and the coefficients $\delta_{i,l}$ are given by

$$\delta_{i,l} = \frac{(-1)^{l + \sum_{j=1}^{i+l-1} m_j} \binom{n-i}{l} \Gamma(b)^{i+l-1} B\left[a(i+l) + \sum_{j=1}^{i+l-1} m_j, b\right]}{B(a,b)^{i+l} B(i, n-i+1) \prod_{j=1}^{i+l-1} \Gamma(b - m_j) m_j! (a + m_j)}.$$

Here, $f_{i,l}(x)$ is the density function of the $BBXII\left(a(i+l) + \sum_{j=1}^{i+l-1} m_j, b, s, k, c\right)$ distribution. Equation (20) is an important result in the applications since it gives the density function of

BBXII order statistics as a mixture of BBXII densities. Several mathematical properties for the BBXII order statistics (mgf, ordinary, inverse and factorial moments) can be derived from this mixture form. Thus, from equation (20), the tth ordinary moment of the BBXII order statistics (valid for $b>0$ real non-integer) is given by

$$\mathrm{E}(X_{i:n}^t)=\sum_{l=0}^{n-i}\sum_{m_1=0}^{\infty}\ldots\sum_{m_{i+l-1}=0}^{\infty}\delta_{i,l}\mathrm{E}(X_{i,l}^t), \tag{21}$$

where $X_{i,l}\sim BBXII(a(i+l)+\sum_{j=1}^{i+l-1}m_j,b,s,k,c)$. For $b>0$ integer, the indices $m_1,\ldots,m_{i+l-1}$ stop at $b-1$. Clearly, $\mathrm{E}(X_{i,l}^t)$ can be calculated directly from equation (14) with the parameters of this BBXII distribution.

An alternative expression to (21) can be derived using a result due to Barakat and Abdelkader (2004). We have

$$\mathrm{E}(X_{i:n}^t)=t\sum_{p=n-i+1}^{n}(-1)^{p-n+i-1}\binom{p-1}{n-i}\binom{n}{p}\int_0^\infty x^{t-1}S(x)^p dx, \tag{22}$$

where $S(x)=1-F(x)$ is the BBXII survival function.
Using the binomial expansion for $[1-F(x)]^p$ in equation (22), it follows

$$[1-F(x)]^p=\sum_{l=0}^{p}(-1)^l\binom{p}{l}F(x)^l.$$

We can rewritten the integral in equation (22) as

$$\int_0^\infty x^{t-1}S(x)^p dx=\sum_{l=0}^{p}(-1)^l\binom{p}{l}\int_0^\infty x^{t-1}F(x)^l dx. \tag{23}$$

Substituting (13) for $F(x)$ in equation (23), we obtain

$$\begin{aligned}F(x)^l &= \frac{\Gamma(b)^l}{B(a,b)^l}\left[\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(b-j)j!(a+j)}\left\{1-\left[1+(x/s)^c\right]^{-k}\right\}^{a+j}\right]^l,\\ &= \frac{\Gamma(b)^l}{B(a,b)^l}\left\{1-\left[1+(x/s)^c\right]^{-k}\right\}^{al}\left[\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(b-j)j!(a+j)}\left\{1-\left[1+(x/s)^c\right]^{-k}\right\}^j\right]^l.\end{aligned}$$

Equation (22) can be rewritten as

$$\sum_{l=0}^{p}\frac{(-1)^l\binom{p}{l}[\Gamma(b)]^l}{[B(a,b)]^l}\int_0^\infty x^{t-1}\left\{1-\left[1+(x/s)^c\right]^{-k}\right\}^{al}\left[\sum_{j=0}^{\infty}a_j\left\{1-\left[1+(x/s)^c\right]^{-k}\right\}^j\right]^l dx, \tag{24}$$

where

$$a_j = \frac{(-1)^j}{\Gamma(b-j)j!(a+j)}.$$

Using the identity $\left(\sum_{k=0}^{\infty} a_k x^k\right)^n = \sum_{k=0}^{\infty} c_{k,n} x^k$ for n positive integer (see Gradshteyn and Ryzhik, 2000) in equation (24), we readily obtain

$$\left[\sum_{j=0}^{\infty} a_j \{1 - [1 + (x/s)^c]^{-k}\}^j\right]^l = \sum_{j=0}^{\infty} c_{j,l} \{1 - [1 + (x/s)^c]^{-k}\}^j,$$

$$c_{0,l} = a_0^l \qquad \text{and} \qquad c_{j,l} = (j a_0)^{-1} \sum_{q=1}^{j} (lq - j + q) a_q c_{j-q,l}.$$

Therefore, we can rewrite the integral in (22) as

$$\sum_{l=0}^{p} \sum_{j=0}^{\infty} \frac{(-1)^l \binom{p}{l} \Gamma(b)^l c_{j,l}}{B(a,b)^l} \int_0^{\infty} x^{t-1} \{1 - [1 + (x/s)^c]^{-k}\}^{al+j} dx,$$

Using the series expansion

$$\sum_{l=0}^{p} \sum_{j,v=0}^{\infty} \frac{(-1)^{l+v} \binom{p}{l} \Gamma(b)^l \Gamma(al+j+1) c_{j,l}}{B(a,b)^l \Gamma(al+j+1-v) v!} \int_0^{\infty} x^{t-1} [1 + (x/s)^c]^{-kv} dx$$

and setting $u = (x/s)^c$, we obtain

$$\sum_{l=0}^{p} \sum_{j,v=0}^{\infty} \frac{(-1)^{l+v} \binom{p}{l} \Gamma(b)^l \Gamma(al+j+1) c_{j,l} s^t}{B(a,b)^l \Gamma(al+j+1-v) v! c} \left[\frac{\Gamma(kv - \frac{1}{c}) \Gamma(\frac{1}{c})}{\Gamma(kv)}\right].$$

Finally, equation (22) reduces to

$$\begin{aligned} \mathrm{E}(X_{i:n}^t) &= t \sum_{p=n-i+1}^{n} (-1)^{p-n+i-1} \binom{p-1}{n-i} \binom{n}{p} \sum_{l=0}^{p} \sum_{j,v=0}^{\infty} \frac{(-1)^{l+v} \binom{p}{l} [\Gamma(b)]^l \Gamma(al+j+1) c_{j,l} s^t}{[B(a,b)]^l \Gamma(al+j+1-v) v! c} \\ &\quad \times \left[\frac{\Gamma(kv - \frac{t}{c}) \Gamma(\frac{t}{c})}{\Gamma(kv)}\right]. \end{aligned}$$

The L-moments (Hosking, 1990) are expectations of certain linear combinations of order statistics and represent the basis of a general theory which covers the summarization and description of theoretical probabilities. They exist whenever the mean of the distribution exists, even though some higher moments may not exist and are relatively robust to the effects of outliers. They are defined by

$$\lambda_{r+1} = (r+1)^{-1} \sum_{k=0}^{r} (-1)^k \binom{r}{k} E(X_{r+1-k:r+1}),\ r = 0, 1, \ldots$$

The first four L-moments are: $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$. From the previous expansion for the moments of the order statistics, we can obtain expansions for the L-moments of the BBXII distribution as linear functions of the means of suitable BBXII distributions.

7 Maximum Likelihood Estimation

Let T_i be a random variable following (6) with the vector $\boldsymbol{\theta} = (a, b, s, k, c)^T$ of parameters. The data encountered in survival analysis and reliability studies are often censored. A very simple random censoring mechanism that is often realistic is one in which each individual i is assumed to have a lifetime X_i and a censoring time C_i, where X_i and C_i are independent random variables. Suppose that the data consist of n independent observations $x_i = \min(X_i, C_i)$ for $i = 1, \cdots, n$. The distribution of C_i does not depend on any of the unknown parameters of T_i. Parametric inference for such data are usually based on likelihood methods and their asymptotic theory. The censored log-likelihood $l(\boldsymbol{\theta})$ for the model parameters is

$$\begin{aligned} l(\boldsymbol{\theta}) &= r\big[\log(c) + \log(k) - c\log(s)\big] - r\log\big[B(a,b)\big] + (a-1)\sum_{i\in F}\log(u_i) \\ &\quad + \Big(\frac{kb+1}{k}\Big)\sum_{i\in F}\log(1-u_i) + \sum_{i\in C}\log(q_i), \end{aligned}$$

where $u_i = 1 - \big[1 + \big(\frac{t_i}{s}\big)^c\big]^{-k}$, $q_i = 1 - I_{u_i}(a,b)$, r is the number of failures and F and C denote the uncensored and censored sets of observations, respectively.

The score functions for the parameters a, b, s, k and c are given by

$$U_a(\boldsymbol{\theta}) = -r[\psi(a) - \psi(a+b)] + \sum_{i\in F}\log(u_i) + \sum_{i\in C}\frac{\Big\{\psi(a) - \psi(a+b)B_{u_i}(a,b) - [\dot{B}_{u_i}(a,b)]_a\Big\}}{B(a,b)q_i},$$

$$U_b(\boldsymbol{\theta}) = -r[\psi(b) - \psi(a+b)] + \sum_{i\in F}\log(1-u_i) + \sum_{i\in C}\frac{\Big\{\psi(b) - \psi(a+b)B_{u_i}(a,b) - \big[\dot{B}_{u_i}(a,b)\big]_b\Big\}}{B(a,b)q_i},$$

$$
\begin{aligned}
U_s(\boldsymbol{\theta}) &= \frac{-rc}{s} + (a-1)\sum_{i\in F}\left(\frac{t_i^c}{s^{c+1}}\right)\left[\frac{-kc(1-u_i)^{\frac{k+1}{k}}}{u_i} + \frac{c(kb+1)}{(1-u_i)^{-1/k}}\right] \\
&\quad + \sum_{i\in C}\frac{c(u_i)^{a-1}(1-u_i)^{\frac{kb+1}{k}}t_i^c}{q_i B(a,b)s^{c+1}},
\end{aligned}
$$

$$
U_k(\boldsymbol{\theta}) = \frac{r}{k} - \frac{1}{k}\sum_{i\in F}\log(1-u_i)\left[\frac{(a-1)(1-u_i)}{(u_i)} - b\right] + \frac{1}{k}\sum_{i\in C}\frac{(u_i)^{a-1}(1-u_i)^b\log(1-u_i)}{B(a,b)q_i},
$$

and

$$
\begin{aligned}
U_c(\boldsymbol{\theta}) &= \frac{r}{c} - r\log(s) + \sum_{i\in F}\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right)\left[\frac{(a-1)k(1-u_i)^{\frac{k+1}{k}}}{u_i} - \frac{(kb+1)}{(1-u_i)^{-1/k}}\right] \\
&\quad -k\sum_{i\in C}\frac{(u_i)^{a-1}(1-u_i)^{\frac{kb+1}{k}}}{B(a,b)q_i}\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right).
\end{aligned}
$$

where

$$
[\dot{B}_{u_i}(a,b)]_a = \sum_{j=0}^{\infty} w_j J(u_i, a+j-1, 1), \quad [\dot{B}_{u_i}(a,b)]_b = \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} w_j^* \frac{u_i^{a+j+n}}{a+j+n},
$$

$$
w_j = \frac{(-1)^j\Gamma(b)}{\Gamma(b-j)j!} \quad \text{and} \quad w_j^* = \frac{(-1)^{j+1}\Gamma(b)}{\Gamma(b-j)j!n}.
$$

Here, $\psi(.)$ is the digamma function, the quantities u_i and q_i were defined before and $J(a,p,k)$ is an integral defined and calculated in Appendix 2.

The maximum likelihood estimate (MLE) $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is obtained by solving the nonlinear likelihood equations $U_a(\boldsymbol{\theta}) = 0$, $U_b(\boldsymbol{\theta}) = 0$, $U_s(\boldsymbol{\theta}) = 0$, $U_k(\boldsymbol{\theta}) = 0$ and $U_c(\boldsymbol{\theta}) = 0$. These equations cannot be solved analytically and statistical software can be used to solve the equations numerically. We can use iterative techniques such as a Newton-Raphson type algorithm to obtain the estimate $\widehat{\boldsymbol{\theta}}$. We employed the programming matrix language Ox (Doornik, 2007).

For interval estimation of (a,b,s,c,k) and hypothesis tests on these parameters, we obtain the observed information matrix since its expectation requires numerical integration. The 5×5 observed information matrix $J(\boldsymbol{\theta})$ is

$$
J(\boldsymbol{\theta}) = -\begin{pmatrix}
\mathbf{L}_{aa} & \mathbf{L}_{ab} & \mathbf{L}_{as} & \mathbf{L}_{ac} & \mathbf{L}_{ak} \\
. & \mathbf{L}_{bb} & \mathbf{L}_{bs} & \mathbf{L}_{bc} & \mathbf{L}_{bk} \\
. & . & \mathbf{L}_{ss} & \mathbf{L}_{sc} & \mathbf{L}_{sk} \\
. & . & . & \mathbf{L}_{cc} & \mathbf{L}_{ck} \\
. & . & . & . & \mathbf{L}_{kk}
\end{pmatrix},
$$

whose elements are given Appendix 2.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \text{ is } N_5(\mathbf{0}, I(\boldsymbol{\theta})^{-1}),$$

where $I(\boldsymbol{\theta})$ is the expected information matrix. This asymptotic behavior is valid if $I(\boldsymbol{\theta})$ is replaced by $J(\widehat{\boldsymbol{\theta}})$, i.e., the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$. The multivariate normal $N_5(\mathbf{0}, J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution can be used to construct approximate confidence intervals and confidence regions for the individual parameters and for the hazard rate and survival functions.

We can compute the maximized unrestricted and restricted log-likelihoods to obtain likelihood ratio (LR) statistics for testing some BBXII sub-models. For example, we can use LR statistics to check if the fitted BBXII distribution for a given data set is statistically "superior" to the fitted BXII, BW, EBXII and BLL distributions. In any case, hypothesis testing of the type $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ *versus* $H : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ can be performed using LR statistics. For example, the LR statistic for testing $H_0 : b = 1$ *versus* $H : b \neq 1$, which is equivalent to compare the EBXII and BBXII distributions, is

$$w = 2\{l(\widehat{a}, \widehat{b}, \widehat{s}, \widehat{k}, \widehat{c}) - l(\widetilde{a}, 1, \widetilde{s}, \widetilde{k}, \widetilde{c})\},$$

where $\widehat{a}$, $\widehat{a}$, $\widehat{s}$, $\widehat{k}$ and $\widehat{c}$ are the MLEs under H and $\widetilde{a}$, $\widetilde{s}$, $\widetilde{k}$ and $\widetilde{c}$ are the estimates under H_0.

8 Application

In this section the application of the BBXII distribution to a real data set on cancer recurrence is discussed. The data are part of a study on cutaneous melanoma (a type of malignant cancer) for the evaluation of postoperative treatment performance with a high dose of a certain drug (interferon alfa-2b) in order to prevent recurrence. Patients were included in the study from 1991 to 1995, and follow-up was conducted until 1998. The data were collected by Ibrahim et al. (2001) and present the survival times, X, as the time until the patient's death. The original sample size was $n = 427$ patients, 10 of whom did not present a value for explanatory variable tumor thickness. When such cases were removed, a sample of size $n = 417$ patients was retained. The percentage of censored observations was 56%. In many applications, there is a qualitative information about the failure rate function shape, which can help in selecting a particular model. In this context, a device called the total time on test (TTT) plot (Aarset, 1987) is useful. The TTT plot is obtained by plotting $\mathsf{G}(r/n) = [(\sum_{i=1}^{r} T_{i:n}) + (n-r)T_{r:n}]/(\sum_{i=1}^{n} T_{i:n})$, where $r = 1, \ldots, n$ and $T_{i:n}$, $i = 1, \ldots, n$, are the order statistics of the sample, against r/n (Mudholkar et al., 1996).

Figure 5a shows that the TTT-plot for these data has first a concave shape and then a convex shape. It indicates a unimodal shaped hazard rate function. Hence, the BBXII distribution could be in principle an appropriate model for fitting such data. Table 1 lists the MLEs (the corresponding standard errors are in parentheses) of the parameters from the fitted BBXII, BXII and LL models and the values of the following statistics: AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and CAIC (Consistent Akaike Information Criterion). The computations were performed using the subroutine NLMixed in SAS. These results indicate that the BGHN model has the lowest values for the AIC, BIC and CAIC statistics among the fitted models, and therefore it could be chosen as the best model. In order to assess if the model is appropriate, Figure 5b plots

Table 1: MLEs of the model parameters for the myelogenous leukemia data, the corresponding SE (given in parentheses) and the measures AIC, BIC and CAIC.

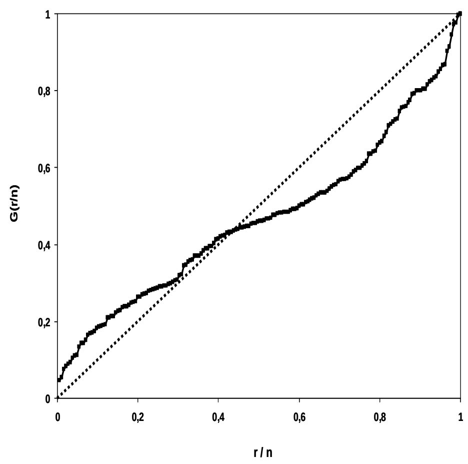
Model	c	k	s	a	b	AIC	CAIC	BIC
BBXII	14.3884	0.4159	1.7729	0.1177	0.0489	1054.9	1055.0	1075.0
	(0.5014)	(0.1124)	(0.1122)	(0.0112)	(0.0094)			
BXII	2.5664	0.1466	0.8734	1	1	1058.3	1058.4	1070.4
	(0.3509)	(0.0325)	(0.1237)	-	-			
	c	m						
LL	1.2669	0.2109				1086.3	1086.5	1094.4
	(0.0807)	(0.0172)						
	α	γ	a	b				
BW	143624	0.05463	195.73	254.6		1079.2	1079.3	1095.3
	(0.0001)	(0.0004)	(0.4306)	(0.4291)				
Weibull	6.9437	1.0509	1	1		1102.1	1102.1	1110.2
	(0.5611)	(0.0691)	-	-				

the empirical and estimated survival functions of the BBXII, BXII, LL, beta Weibull (BW) and Weibull distributions. Further, Figure 5c plots the estimated hazard rate function. We conclude that the BBXII distribution provides a good fit to these data.

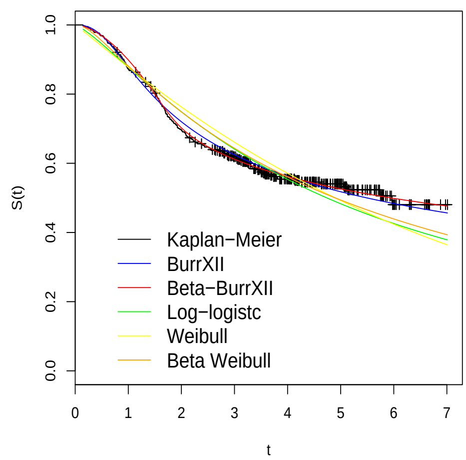
9 Conclusions

We define and study a five parameter lifetime distribution, referred to as the beta Burr XII (BBXII) distribution, which extends several distributions widely used in the lifetime literature. The new model is much more flexible than the Burr XII (BXII), beta Weibull (BW) and log-logistic (LL) sub-models. It is useful to model lifetime data with a unimodal shaped hazard rate function. We provide a mathematical treatment of the distribution including the mean deviations, Bonferroni and Lorenz curves and expansions for the densities of the order statistics. We derive explicit closed form expressions for the moments and moment generating function which hold in generality for any parameter values. We give linear functions to compute the moments of the order statistics. The estimation of parameters is approached by the method of maximum likelihood and

(a)



(b)



(c)

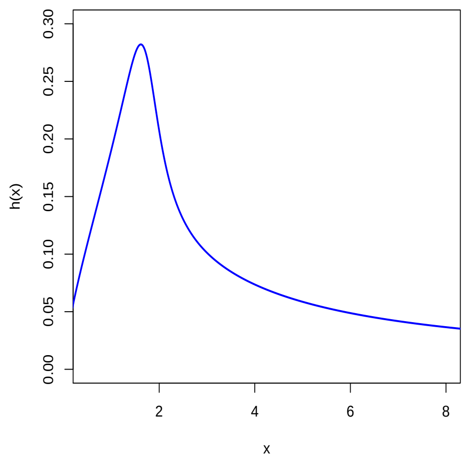


Figure 5: (a) TTT-plot on myelogenous leukemia data. (b) Estimated survival functions from the fitted BXII, BBXII, LL, BW and Weibull distributions and the empirical survival for myelogenous leukemia data. (c) Plot of the estimated hazard rate function for myelogenous leukemia data.

the observed information matrix is derived. One application of the BBXII distribution shows that it could provide a better fit than other statistical models used in lifetime data analysis.

Appendix 1

We have the following result which holds for m and k positive integers, $\mu > -1$ and $p > 0$ (Prudnikov et al., 1992, page 21)

$$\begin{aligned} I\left(p, \mu, \frac{m}{k}, \nu\right) &= \int_0^\infty \exp(-px)\, x^\mu\, (1 + x^{\frac{m}{k}})^\nu dx \\ &= \frac{k^{-\nu} m^{\mu+\frac{1}{2}}}{(2\pi)^{\frac{(m-1)}{2}} \Gamma(-\nu) p^{\mu+1}} \times \\ & \quad G^{k,k+m}_{k+m,k}\left(\frac{m^m}{p^m} \,\middle|\, \begin{array}{l} \Delta(m,-\mu), \Delta(k, \nu+1) \\ \Delta(k, 0) \end{array} \right), \end{aligned} \tag{25}$$

where $\Delta(k, a) = \frac{a}{k}, \frac{a+1}{k}, \cdots, \frac{a+k}{k}$.

Appendix 2

The following integrals can be easily calculated from the integral given by Prudnikov et al. (1986, vol 1, Section 2.6.3, integral 1),

$$J(a, p, 1) = \int_0^a x^p \log(x) dx = \frac{a^{p+1}}{(p+1)^2}[(p+1)\log(a) - 1].$$

and

$$J(a, p, 2) = \int_0^a x^p \log^2(x) dx = \frac{a^{p+1}}{(p+1)^3}\{2 - (p+1)\log(a)[2 - (p+1)\log(a)]\}.$$

Another integral

$$H(a, p) = \int_0^a x^p \log(x) \log(1-x) dx$$

is easily expressed in terms of a sum involving $J(a, p, 1)$ from the expansion

$$\log(1-x) = \sum_{n=1}^{\infty} (-1) \frac{x^n}{n},$$

which converges for $x \in (0, 1)$. We have

$$H(a, p) = -\sum_{n=1}^{\infty} \frac{a^{p+n+1}}{n(p+n+1)^2}[(p+n+1)\log(a) - 1].$$

The elements of the observed information matrix $J(\boldsymbol{\theta})$ for the parameters (a, b, s, k, c) are

$$\begin{aligned}
\mathbf{L}_{aa}(\boldsymbol{\theta}) &= -r[\psi'(a) - \psi'(a+b)] \\
&+ \frac{-1}{B(a,b)} \sum_{i \in C} \Big\{ q_i \Big[\psi(a) - \psi(a+b) \Big(\psi(a) - \psi(a+b) B_{u_i}(a,b) - [\dot{B}_{u_i}(a,b)]_a \Big) \\
&+ \Big(\psi'(a) - \psi'(a+b) B_{u_i}(a,b) - \psi(a+b)[\dot{B}_{u_i}(a,b)]_a - [\ddot{B}_{1-u_i^{-k}}(a,b)]_{bb} \Big) \Big] \\
&+ \Big[\frac{1}{[B(a,b)]} \Big(\psi(a) - \psi(a+b) B_{u_i}(a,b) - [\dot{B}_{u_i}(a,b)]_a \Big) \\
&\times \Big([\psi(a) - \psi(a+b)] B_{u_i}(a,b) - [\dot{B}_{u_i}(a,b)]_a \Big) \Big] \Big\} \times \frac{1}{q_i},
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}_{ab}(\boldsymbol{\theta}) &= -r\psi'(a+b) + \frac{-1}{B(a,b)} \sum_{i \in C} \Big\{ q_i \Big[\psi(b) - \psi(a+b) \Big(\psi(a) - \psi(a+b) B_{u_i}(a,b) - [\dot{B}_{u_i}(a,b)]_a \Big) \\
&+ \Big(-\psi'(a+b) B_{u_i}(a,b) - \psi(a+b)[\dot{B}_{u_i}(a,b)]_b - [\ddot{B}_{1-u_i^{-k}}(a,b)]_{bb} \Big) \Big] \\
&+ \Big[\frac{1}{[B(a,b)]} \Big(\psi(a) - \psi(a+b) B_{u_i}(a,b) - [\dot{B}_{u_i}(a,b)]_a \Big) \\
&\times \Big([\psi(b) - \psi(a+b)] B_{u_i}(a,b) - [\dot{B}_{u_i}(a,b)]_b \Big) \Big] \Big\} \times \frac{1}{q_i^2},
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}_{as}(\boldsymbol{\theta}) &= \sum_{i \in F} \frac{-kc(1-u_i)^{\frac{k+1}{k}} t_i^c}{s^{c+1}(u_i)} + \frac{1}{B(a,b)} \sum_{i \in C} \Big\{ q_i \Big[\Big(\frac{kc}{s} \Big) (1-u_i)^{\frac{k+1}{k}} \Big(\frac{t_i}{s} \Big)^c (u_i)^{a-1} \big(\psi(a+b) + \log(u_i) \big) \Big] \\
&- \Big[\Big(\psi(a) - \psi(a+b) B_{u_i}(a,b) - [\dot{B}_{u_i}(a,b)]_a \Big) \\
&\times \Big(\frac{1}{B(a,b)} \Big(\frac{kc}{s} \Big) (1-u_i)^{\frac{kb+1}{k}} \Big(\frac{t_i}{s} \Big)^c (u_i)^{a-1} \Big) \Big] \Big\} \times \frac{1}{q_i^2},
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}_{ak}(\boldsymbol{\theta}) &= \sum_{i \in F} - \frac{(1-u_i)\log(1-u_i)}{k(u_i)} + \frac{1}{B(a,b)} \sum_{i \in C} \Big\{ q_i \Big[(1-u_i)^b \log(1-u_i)^{\frac{1}{k}} (u_i)^{a-1} \left(\psi(a+b) + \log(u_i) \right) \Big] \\
&- \Big[\Big(\psi(a) - \psi(a+b) B_{u_i}(a,b) - [\dot{B}_{u_i}(a,b)]_a \Big) \\
&\times \Big(\frac{1}{B(a,b)} (1-u_i)^b \log(1-u_i)^{\frac{1}{k}} (u_i)^{a-1} \Big) \Big] \Big\} \times \frac{1}{q_i^2},
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}_{ac}(\boldsymbol{\theta}) &= \sum_{i \in F} \frac{k(1-u_i)^{\frac{k+1}{k}}}{(u_i)} \Big(\frac{t_i}{s} \Big)^c \log \Big(\frac{t_i}{s} \Big) + \frac{1}{B(a,b)} \sum_{i \in C} \Big\{ q_i \Big[-k(1-u_i)^{\frac{kb+1}{k}} \Big(\frac{t_i}{s} \Big)^c \log \Big(\frac{t_i}{s} \Big) (u_i)^{a-1} \\
&\times \left(\psi(a+b) + \log(u_i) \right) \Big] - \Big[\Big(\psi(a) - \psi(a+b) B_{u_i}(a,b) - [\dot{B}_{u_i}(a,b)]_a \Big) \\
&\times \Big(\frac{-k(1-u_i)^{\frac{kb+1}{k}}}{B(a,b)} \Big(\frac{t_i}{s} \Big)^c \log \Big(\frac{t_i}{s} \Big) (u_i)^{a-1} \Big) \Big] \Big\} \times \frac{1}{q_i^2},
\end{aligned}$$

$$
\begin{aligned}
\mathbf{L}_{bb}(\boldsymbol{\theta}) \quad = \quad & -r[\psi'(b)-\psi'(a+b)] + \frac{-1}{B(a,b)}\sum_{i\in C}\Big\{q_i\Big[\psi(b)-\psi(a+b)\Big(\psi(b)-\psi(a+b)B_{u_i}(a,b)-[\dot{B}_{u_i}(a,b)]_b\Big) \\
& +\Big(\psi'(b)-\psi'(a+b)B_{u_i}(a,b)-\psi(a+b)[\dot{B}_{u_i}(a,b)]_b-[\ddot{B}_{1-u_i^{-k}}(a,b)]_{bb}\Big)\Big] \\
& +\Big[\frac{1}{[B(a,b)]}\Big(\psi(b)-\psi(a+b)B_{u_i}(a,b)-[\dot{B}_{u_i}(a,b)]_b\Big) \\
& \times\Big([\psi(b)-\psi(a+b)][\dot{B}_{u_i}(a,b)]_b-[\ddot{B}_{1-u_i^{-k}}(a,b)]_{bb}\Big)\Big]\Big\}\times\frac{1}{q_i^2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}_{bs}(\boldsymbol{\theta}) \quad = \quad & \sum_{i\in F}\frac{kc}{s(1-u_i)^{\frac{-1}{k}}}\left(\frac{t_i}{s}\right)^c+\frac{1}{B(a,b)}\sum_{i\in C}\Big\{q_i\Big[\Big(\frac{kc}{s}\Big)(1-u_i)^{\frac{k+1}{k}}\left(\frac{t_i}{s}\right)^c(u_i)^{a-1}\big(\psi(a+b)+\log(1-u_i)\big)\Big] \\
& -\Big[\Big(\psi(b)-\psi(a+b)B_{u_i}(a,b)-[\dot{B}_{u_i}(a,b)]_a\Big)\times\Big(\frac{1}{B(a,b)}\Big(\frac{kc}{s}\Big)(1-u_i)^{\frac{kb+1}{k}}\left(\frac{t_i}{s}\right)^c(u_i)^{a-1}\Big)\Big]\Big\} \\
& \times\frac{1}{q_i^2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}_{bk}(\boldsymbol{\theta}) \quad = \quad & \frac{1}{k}\sum_{i\in F}\log(1-u_i)+\frac{1}{B(a,b)}\sum_{i\in C}\Big\{q_i\Big[(1-u_i)^b\log(1-u_i)^{\frac{1}{k}}(u_i)^{a-1}\big(\psi(a+b)+\log(1-u_i)\big)\Big] \\
& -\Big[\Big(\psi(b)-\psi(a+b)B_{u_i}(a,b)-[\dot{B}_{u_i}(a,b)]_b\Big)\times\Big(\frac{1}{kB(a,b)}(1-u_i)^b\log(1-u_i)(u_i)^{a-1}\Big)\Big]\Big\}\times\frac{1}{q_i^2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}_{bc}(\boldsymbol{\theta}) \quad = \quad & \sum_{i\in F}\frac{-k}{(1-u_i)^{\frac{-1}{k}}}\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right)+\frac{1}{B(a,b)}\sum_{i\in C}\Big\{q_i\Big[-k(1-u_i)^{\frac{kb+1}{k}}\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right)(u_i)^{a-1} \\
& \times(\psi(a+b)+\log(1-u_i))\Big]-\Big[\Big(\psi(b)-\psi(a+b)B_{u_i}(a,b)-[\dot{B}_{u_i}(a,b)]_b\Big) \\
& \times\Big(\frac{-k(1-u_i)^{\frac{kb+1}{k}}}{B(a,b)}\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right)(u_i)^{a-1}\Big)\Big]\Big\}\times\frac{1}{q_i^2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}_{ss}(\boldsymbol{\theta}) &= \frac{rc}{s^2} - ck(a-1) + \sum_{i\in F}\left\{\left[\frac{-t^c(u_i)}{s^{c+2}}\left((c+1)(1-u_i)^{\frac{k+1}{k}} + c\left(\frac{t_i}{s}\right)^c\right)\right] + \left[\frac{kc}{s^2}(1-u_i)^{\frac{2(k+1)}{k}}\left(\frac{t_i}{s}\right)^{2c}\right]\right\} \\
&\quad \times\frac{1}{u_i^2} + c(kb+1)\sum_{i\in F}\left\{\left[\frac{-t^c}{s^{c+2}}\left((c+1)(1-u_i)^{\frac{-1}{k}} - c\left(\frac{t_i}{s}\right)^c\right)\right]\right\}\times\frac{1}{(1-u_i)^{\frac{-2}{k}}} \\
&\quad +\frac{c}{B(a,b)}\sum_{i\in C}\left\{q_i\left[\frac{-kc}{s^2}(1-u_i)^{b+1+\frac{2}{k}}\left(\frac{t_i}{s}\right)^{2c}\right]\right. \\
&\quad +\left[(1-u_i)^{\frac{kb+1}{k}}\left((u_i)^{a-1}(1-u_i)^{\frac{1}{k}}(-kb-1)\left(\frac{t_i}{s}\right)^{2c}\left(\frac{-c}{s^2}\right) - \frac{t^c(c+1)}{s^{c+2}}\right)\right] \\
&\quad \left. -\left[(1-u_i)^{\frac{2(kb+1)}{k}}(u_i)^{2(a-1)}\left(\frac{t_i}{s}\right)^{2c}\frac{kc}{s^2B(a,b)}\right]\right\}\times\frac{1}{q_i^2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}_{sk}(\boldsymbol{\theta}) &= \frac{-ct^c(a-1)}{s^{c+1}}\sum_{i\in F}\left\{(1-u_i)^{\frac{k+1}{K}}\left[(u_i) + (u_i)\log(1-u_i) + (u_i)\log(1-u_i)\right]\right\}\times\frac{1}{u_i^2} \\
&\quad +b\sum_{i\in F}\frac{ct^c}{s^{c+1}(1-u_i)^{\frac{-1}{K}}} + \frac{ct^c}{s^{c+1}B(a,b)}\sum_{i\in C}\left\{q_i\left[(u_i)^{a-1}(1-u_i)^{\frac{k+1}{K}}\log(1-u_i)^{\frac{-1}{k}}\right.\right. \\
&\quad \left.\left.\times\left((u_i)^{-1}(a-1)(1-u_i)^{-k} - b\right)\right] + \left[\frac{1}{B(a,b)}(u_i)^{2(a-1)}(1-u_i)^{\frac{2bk+1}{k}}\log(1-u_i)^{\frac{-1}{k}}\right]\right\}\times\frac{1}{q_i^2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}_{sc}(\boldsymbol{\theta}) &= \frac{-r}{s} + k(a-1)\sum_{i\in F}\frac{t_i^c}{s^{c+1}}\left\{u_i\left[(1-u_i)^{\frac{k+1}{k}}\left(c(1-u_i)^{\frac{1}{k}}(-k-1)\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right) + 1 + c\log(t_i) + c\log(s)\right)\right]\right. \\
&\quad \left. -\left[(1-u_i)^{\frac{2(k+1)}{k}}ck\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right)\right]\right\}\times\frac{1}{u_i^2} \\
&\quad +(kb+1)\sum_{i\in F}\frac{t_i^c}{s^{c+1}}\left\{\left[(1-u_i)^{\frac{-1}{k}}(1 + c\log(t_i) + c\log(s))\right]\right. \\
&\quad \left. -\left[c\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right)\right]\right\}\times\frac{1}{(1-u_i)^{\frac{-2}{k}}} \\
&\quad +\frac{1}{B(a,b)}\sum_{i\in C}\frac{t_i^c}{s^{c+1}}\left\{\left[\left((u_i)^{a-2}(a-1)(1-u_i)^{b+1+\frac{2}{k}}ck\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right)\right)\right.\right. \\
&\quad \left. +\left((u_i)^{a-1}(1-u_i)^{\frac{kb+1}{k}}\left(c(1-u_i)^{\frac{1}{k}}(-kb-1)\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right) + 1 + c\log(t_i) + c\log(s)\right)\right)\right]\times q_i \\
&\quad \left. +\left[(u_i)^{2(a-1)}(1-u_i)^{\frac{2(bk+1)}{k}}\frac{ck}{B(a,b)}\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right)\right]\right\}\times\frac{1}{q_i^2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}_{kk}(\boldsymbol{\theta}) &= \frac{-r}{k^2} + \frac{2(a-1)}{k}\sum_{i\in F}\left[(1-u_i)\log(1-u_i)\right]\times\frac{1}{u_i^2} + \frac{1}{kB(a,b)}\sum_{i\in C}\log(1-u_i) \\
&\quad \Big\{q_i\left[(u_i)^{a-1}(1-u_i)^b\log(1-u_i)^{\frac{-1}{k}}\left((u_i)^{-1}(a-1)(1-u_i)-b\right)\right] \\
&\quad + \left[\frac{2(u_i)^{2(a-1)}}{kB(a,b)}(1-u_i)^{2b}\log(1-u_i)\right]\Big\}\times\frac{1}{q_i^2},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{L}_{kc}(\boldsymbol{\theta}) &= (a-1)\sum_{i\in F}\log\left(\frac{t_i}{s}\right)\left\{\left[u_i(1-u_i)^{\frac{k+1}{k}}\left(\frac{t_i}{s}\right)^c(\log(1-u_i)+1)\right]+\left[(1-u_i)^{\frac{2k+1}{k}}\log(1-u_i)\left(\frac{t_i}{s}\right)\right]\right\} \\
&\quad \times\frac{1}{u_i^2} - b\sum_{i\in F}\left(\frac{t_i}{s}\right)\log\left(\frac{t_i}{s}\right)(1-u_i)^{\frac{1}{k}} \\
&\quad + \frac{1}{kB(a,b)}\sum_{i\in C}\log(1-u_i)\left\{\left[(u_i)^{a-1}(1-u_i)^{\frac{kb+1}{k}}\left(\frac{t_i}{s}\right)\log\left(\frac{t_i}{s}\right)\times\right.\right. \\
&\quad \left.\left((a-1)(u_i)^{-1}(1-u_i)\log(1-u_i)^{-1}+1+b\log(1-u_i)\right)\right]\times q_i \\
&\quad \left. - \left[\frac{k}{B(a,b)}(u_i)^{2(a-1)}(1-u_i)^{\frac{2bk+1}{k}}\log(1-u_i)\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right)\right]\right\}\times\frac{1}{q_i^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{L}_{cc}(\boldsymbol{\theta}) &= \frac{-r}{c^2} + k(a-1)\sum_{i\in F}\log\left(\frac{t_i}{s}\right)\left\{u_i\left[(1-u_i)^{\frac{k+1}{k}}\left(\frac{t_i}{s}\right)^c\left(\frac{t_i}{s}\right)\left((1-u_i)^{\frac{1}{k}}(-k-1)\left(\frac{t_i}{s}\right)+1\right)\right]\right. \\
&\quad \left. - \left[(1-u_i)^{\frac{2(1-k)}{k}}\left(\frac{t_i}{s}\right)^{2c}\log\left(\frac{t_i}{s}\right)\right]\right\}\times\frac{1}{u_i^2} \\
&\quad -(kb+1)\sum_{i\in F}\log\left(\frac{t_i}{s}\right)\left\{\left[\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right)(1-u_i)^{\frac{-1}{k}} - \left(\frac{t_i}{s}\right)^{2c}\right]\right\}\times\frac{1}{(1-u_i)^{\frac{-2}{k}}} \\
&\quad -\frac{1}{B(a,b)}\sum_{i\in C}\log\left(\frac{t_i}{s}\right)\left\{q_i\left[(u_i)^{a-1}(1-u_i)^{\frac{kb+1}{k}}\left(\frac{t_i}{s}\right)^c\log\left(\frac{t_i}{s}\right)\left(k(a-1)(u_i)^{-1}(1-u_i)^{\frac{k+1}{k}}\left(\frac{t_i}{s}\right)^c+\right.\right.\right. \\
&\quad \left.\left.(1-u_i)^{\frac{1}{k}}(-kb-1)\left(\frac{t_i}{s}\right)^c+1\right)\right]\times+\left[\frac{k}{B(a,b)}(u_i)^{2(a-1)}(1-u_i)^{\frac{2(bk+1)}{k}}\left(\frac{t_i}{s}\right)^{2c}\log\left(\frac{t_i}{s}\right)\right]\right\}\times\frac{1}{q_i^2},
\end{aligned}
$$

where

$$
[\ddot{B}_{1-u_i^{-k}}(a,b)]_{aa} = \sum_{j=0}^{\infty} w_j J(1-u_i^{-k}, a+j-1, 2),
$$

$$
[\ddot{B}_{1-u_i^{-k}}(a,b)]_{ab} = \sum_{j=0}^{\infty} w_j H(1-u_i^{-k}, a+j-1, 1),
$$

$$[\ddot{B}_{1-u_i^{-k}}(a,b)]_{bb} = \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} w_j^* \frac{(1-u_i^{-k})^{a+j+n}-1}{a+j+n},$$

$\psi'(.)$ is the derivative of the digamma function, u_i, q_i, w_j and w_j^* are defined in Section 7.

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