A NOTE ON THE NUMBER OF NODAL SOLUTIONS OF AN ELLIPTIC EQUATION WITH SYMMETRY

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ABSTRACT. We consider the semilinear problem

\(- \Delta u + \lambda u = |u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega\)

where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain and \(2 < p < 2^* = 2N/(N-2)\). We show that if \(\Omega\) is invariant by a nontrivial orthogonal involution then, for \(\lambda > 0\) sufficiently large, the equivariant topology of \(\Omega\) is related with the number of solutions which change sign exactly once.

1. INTRODUCTION

Consider the problem

\((P_\lambda)\)

\[- \Delta u + \lambda u = |u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain and \(2 < p < 2^* = 2N/(N-2)\). It is well known that it possesses infinitely many solutions. However, when we require some properties of the sign of the solutions, the problem seems to be more complicated. In the paper [1], Benci and Cerami showed that, if \(\lambda\) is sufficiently large, then \((P_\lambda)\) has at least \(\text{cat}(\Omega)\) positive solutions, where \(\text{cat}(\Omega)\) denotes the Lusternik-Schnirelmann category of \(\Omega\) in itself. Since the work [1], multiplicity results for \((P_\lambda)\) involving the category have been intensively studied (see [2, 3, 4] for subcritical, and [5, 6, 7] for critical nonlinearities).

In the aforementioned works, the authors considered positive solutions. In [8], Bartsch obtained infinite nodal solutions for \((P_\lambda)\), that is, solutions which change sign. Motivated by this work and for a recent paper of Castro and Clapp [9], we are interested in relating the topology of \(\Omega\) with the number of solutions which change sign exactly once. This means that the solution \(u\) is such that \(\Omega \setminus u^{-1}(0)\) has exactly two connected components, \(u\) is positive in one of them and negative in the other. We deal with the problem

\[(P^\tau_\lambda)\]

\[
\begin{align*}
- \Delta u + \lambda u &= |u|^{p-2}u, & \text{in } \Omega, \\
\quad u &= 0, & \text{on } \partial \Omega, \\
\quad u(\tau x) &= -u(x), & \text{for all } x \in \Omega,
\end{align*}
\]

where \(\tau : \mathbb{R}^N \to \mathbb{R}^N\) is a linear orthogonal transformation such that \(\tau \neq \text{Id}\), \(\tau^2 = \text{Id}\), and \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain such that \(\tau \Omega = \Omega\). Our main result can be stated as follows.

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Theorem 1.1. For any \( p \in (2, 2^*) \) fixed there exists \( \overline{\lambda} = \overline{\lambda}(p) \) such that, for all \( \lambda > \overline{\lambda} \), the problem \((P_\lambda^*)\) has at least \( \tau\text{-cat}_\Omega(\Omega \setminus \Omega^\tau) \) pairs of solutions which change sign exactly once.

Here, \( \Omega^\tau = \{ x \in \Omega : \tau x = x \} \) and \( \tau\text{-cat} \) is the \( G_\tau \)-equivariant Lusternik-Schnirelmann category for the group \( G_\tau = \{ \text{Id}, \tau \} \). There are several situations where the equivariant category turns out to be larger than the nonequivariant one. The classical example is the unit sphere \( S^{N-1} \subset \mathbb{R}^N \) with \( \tau = -\text{Id} \). In this case \( \text{cat}(S^{N-1}) = 2 \) whereas \( \tau\text{-cat}(S^{N-1}) = N \). Thus, as an easy consequence of Theorem 1.1 we have

Corollary 1.2. Let \( \Omega \) be symmetric with respect to the origin and such that \( 0 \notin \Omega \). Assume further that there is an odd map \( \varphi : S^{N-1} \to \Omega \). Then, for any \( p \in (2, 2^*) \) fixed there exists \( \lambda = \lambda(p) \) such that, for all \( \lambda > \lambda \), the problem \((P_\lambda)\) has at least \( N \) pairs of odd solutions which change sign exactly once.

The above results complement those of [9] where the authors considered the critical semilinear problem

\[
-\Delta u = \lambda u + |u|^{2^*-2}u, \quad u \in H^1_0(\Omega), \quad u(\tau x) = -u(x) \quad \text{in} \quad \Omega,
\]

and obtained the same results for \( \lambda > 0 \) small enough. It also complement the aforementioned works that deal only with positive solutions. We finally note that Theorem 1.1 also holds if \( \lambda \geq 0 \) is fixed and the exponent \( p \) is sufficiently close to \( 2^* \) (see Remark 3.2).

2. Notations and Some Technical Results

Throughout this paper, we denote by \( H \) the Hilbert space \( H^1_0(\Omega) \) endowed with the norm \( \| u \| = \left\{ \int_\Omega |\nabla u|^2 \, dx \right\}^{1/2} \). The involution \( \tau \) of \( \Omega \) induces an involution of \( H \), which we also denote by \( \tau \), in the following way: for each \( u \in H \) we define \( \tau u \in H \) by

\[
(\tau u)(x) = -u(\tau x). \tag{2.1}
\]

We denote by \( H^\tau = \{ u \in H : \tau u = u \} \) the subspace of \( \tau \)-invariant functions.

Let \( E_\lambda : H \to \mathbb{R} \) be given by

\[
E_\lambda(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + \lambda u^2) \, dx - \frac{1}{p} \int_\Omega |u|^p \, dx,
\]

and its associated Nehari manifold

\[
N_\lambda = \{ u \in H \setminus \{ 0 \} : \langle E_\lambda'(u), u \rangle = 0 \} = \{ u \in H \setminus \{ 0 \} : \| u \|^2 + \lambda |u|^2_2 = |u|^p_2 \}
\]

where \( |u|_s \) denote the \( L^s(\Omega) \)-norm for \( s \geq 1 \). In order to obtain \( \tau \)-invariant solutions, we will look for critical points of \( E_\lambda \) restricted to the \( \tau \)-invariant Nehari manifold

\[
N^\tau_\lambda = \{ u \in N_\lambda : \tau u = u \} = N_\lambda \cap H^\tau,
\]
by considering the following minimization problems

\[ m_\lambda = \inf_{u \in \mathcal{N}_\lambda} E_\lambda(u) \text{ and } m_\lambda^* = \inf_{u \in \mathcal{N}_\lambda^*} E_\lambda(u). \]

For any \( \tau \)-invariant bounded domain \( D \subset \mathbb{R}^N \) we define \( E_{\lambda,D}, \mathcal{N}_{\lambda,D}, \mathcal{N}_{\lambda,D}^* \), \( m_{\lambda,D} \) and \( m_{\lambda,D}^* \) in the same way by taking the above integrals over \( D \) instead \( \Omega \). For simplicity, we use only \( m_{\lambda,r} \) and \( m_{\lambda,r}^* \) to denote \( m_{\lambda,B_r(0)} \) and \( m_{\lambda,B_r(0)}^* \) respectively.

**Lemma 2.1.** For any \( \lambda \geq 0 \), we have that \( 2m_\lambda \leq m_\lambda^* \).

**Proof.** Note that, if \( u \in H^\tau \) is positive in some subset \( A \subset \Omega \), we can use (2.1) to conclude that \( u \) is negative in \( \tau(A) \). Thus, for any given \( u \in \mathcal{N}_\lambda^* \), we have that \( u^+, u^- \in \mathcal{N}_\lambda \), where \( u^\pm = \max\{ \pm u, 0 \} \). Hence \( E_\lambda(u) = E_\lambda(u^+) + E_\lambda(u^-) \geq 2m_\lambda \), and the result follows. □

**Lemma 2.2.** If \( u \) is a critical point of \( E_\lambda \) restricted to \( \mathcal{N}_\lambda^* \), then \( E_\lambda^*(u) = 0 \) in the dual space of \( H \).

**Proof.** By the Lagrange multiplier rule, there exits \( \theta \in \mathbb{R} \) such that

\[ \langle E'_\lambda(u) - \theta J'_\lambda(u), \phi \rangle = 0, \]

for all \( \phi \in H^\tau \), where \( J_\lambda(u) = \|u\|^2 + \lambda|u|^2 - |u|^p \). Since \( u \in \mathcal{N}_\lambda^* \), we have

\[ 0 = \langle E'_\lambda(u), u \rangle - \theta \langle J'_\lambda(u), u \rangle = \theta(p - 2)|u|^p. \]

This implies \( \theta = 0 \) and therefore \( \langle E'_\lambda(u), \phi \rangle = 0 \) for all \( \phi \in H^\tau \). The result follows from the principle of symmetric criticality [10] (see also [11, Theorem 1.28]). □

By standard regularity theory we know that if \( u \) is a solution of \( (P_\lambda) \), then it is of class \( C^1 \). We say it changes sign \( k \) times if the set \( \{ x \in \Omega : u(x) \neq 0 \} \) has \( k + 1 \) connected components. By (2.1), if \( u \) is a nontrivial solution of problem \( (P_\lambda^*) \) then it changes sign an odd number of times.

**Lemma 2.3.** If \( u \) is a solution of problem \( (P_\lambda^*) \) which changes sign \( 2k - 1 \) times, then \( E_\lambda(u) \geq km_\lambda^* \).

**Proof.** The set \( \{ x \in \Omega : u(x) > 0 \} \) has \( k \) connected components \( A_1, \ldots, A_k \). Let \( u_i(x) = u(x) \) if \( x \in A_i \cup \tau A_i \) and \( u_i(x) = 0 \), otherwise. We have that

\[ 0 = \langle E'_\lambda(u), u_i \rangle = \int_{\Omega} (\nabla u \nabla u_i + \lambda uu_i - |u|^{p-2}uu_i) \, dx = \|u_i\|^2 + \lambda|u_i|^2 - |u_i|^p. \]

Thus, \( u_i \in \mathcal{N}_\lambda^* \) for all \( i = 1, \ldots, k \), and \( E_\lambda(u) = E_\lambda(u_1) + \cdots + E_\lambda(u_k) \geq km_\lambda^* \), as desired. □

We recall now some facts about equivariant Lusternik-Schnirelmann theory. An involution on a topological space \( X \) is a continuous function \( \tau_X : X \to X \) such that \( \tau_X^2 \) is the identity map of \( X \). A subset \( A \) of \( X \) is called \( \tau_X \)-invariant if \( \tau_X(A) = A \). If \( X \) and \( Y \) are topological spaces equipped with involutions \( \tau_X \) and \( \tau_Y \) respectively, then an equivariant map is a continuous function
f : X → Y such that f ◦ τX = τY ◦ f. Two equivariant maps \( f_0, f_1 : X \to Y \) are equivariantly homotopic if there is an homotopy \( \Theta : X \times [0, 1] \to Y \) such that \( \Theta(x, 0) = f_0(x), \Theta(x, 1) = f_1(x) \) and \( \Theta(\tau_X(x), t) = \tau_Y(\Theta(x, t)) \), for all \( x \in X, t \in [0, 1] \).

**Definition 2.4.** The equivariant category of an equivariant map \( f : X \to Y \), denoted by \((\tau_X, \tau_Y)\)-cat\((f)\), is the smallest number \( k \) of open invariant subsets \( X_1, \ldots, X_k \) of \( X \) which cover \( X \) and which have the property that, for each \( i \in \{ \text{cat} \tau \} \)

\[ \Theta : X_1, \ldots, X_k \to X \text{ is an homotopy} \]

\[ \Theta(x, 0) = x, \Theta(x, 1) = \tau_X(x) \]

\[ \text{for every } x \in X, t \in [0, 1]. \]

If no such covering exists we define \((\tau_X, \tau_Y)\)-cat\((f)\) = \( \infty \).

The following properties can be verified.

**Lemma 2.5.** (i) If \( f : X \to Y \) and \( h : Y \to Z \) are equivariant maps then

\[ (\tau_X, \tau_Z)\)-cat\((h \circ f) \leq \tau_Z\)-cat\((Y) \).

(ii) If \( f_0, f_1 : X \to Y \) are equivariantly homotopic, then \((\tau_X, \tau_Y)\)-cat\((f_0) = (\tau_X, \tau_Y)\)-cat\((f_1) \).

Let \( V \) be a Banach space, \( M \) be a \( C^1 \)-manifold of \( V \) and \( I : V \to \mathbb{R} \) a \( C^1 \)-functional. We recall that \( I \) restricted to \( M \) satisfies de Palais-Smale condition at level \( c \) \((\text{PS})_c \) for short) if any sequence \( (u_n) \subset M \) such that \( I(u_n) \to c \) and \( \|I'(u_n)\| \to 0 \) contains a convergent subsequence. Here we denote by \( \|I'(u)\| \) the norm of the derivative of the restriction of \( I \) to \( M \) (see [11, Section 5.3]).

Let \( \tau_a : V \to V \) be the antipodal involution \( \tau_a(u) = -u \) on the vector space \( V \). Equivariant Lusternik-Schnirelmann category provides a lower bound for the number of pairs \( \{u, -u\} \) of critical points of an even functional, as stated in the following abstract result (see [12, Theorem 1.1], [13, Theorem 5.7]).

**Theorem 2.6.** Let \( I : M \to \mathbb{R} \) be an even \( C^1 \)-functional on a complete symmetric \( C^{1,1} \)-submanifold \( M \) of some Banach space \( V \). Assume that \( I \) is bounded below and satisfies \((\text{PS})_c \) for all \( c \leq d \).

Then, if \( I^d = \{ u \in M : I(u) \leq d \} \), the functional \( I \) has at least \( \tau_a\)-cat\(_{I_0}\)(I\(^d\)) antipodal pairs \( \{u, -u\} \) of critical points with \( I(\pm u) \leq d \).

3. Proofs of the results

Given \( r > 0 \), we define the sets

\[ \Omega^+_r = \{ x \in \Omega : \operatorname{dist}(x, \Omega) < r \} \quad \text{and} \quad \Omega^-_r = \{ x \in \Omega : \operatorname{dist}(x, \partial \Omega \cup \Omega^r) \geq r \}. \]
Throughout the rest of the paper we fix $r > 0$ sufficiently small in such way that the inclusion maps $\Omega^{-}_r \hookrightarrow \Omega \setminus \Omega^r$ and $\Omega \hookrightarrow \Omega^+_r$ are equivariant homotopy equivalences. Without loss of generality we suppose that $B_r(0) \subset \Omega$.

We now note that, in [1], Benci and Cerami considered the minimization problem

$$\tilde{m}_\lambda = \inf \left\{ \int_\Omega (|\nabla u|^2 + \lambda u^2) \, dx : u \in H, \int_\Omega |u|^p \, dx = 1 \right\}.$$

An easy calculation show that

$$m_\lambda = \left( \frac{p-2}{2p} \right) \frac{\tilde{m}_\lambda^{p/(p-2)}}{\lambda}. \quad \text{Therefore, if we denote by } \beta : H \setminus \{0\} \to \mathbb{R}^N \text{ the barycenter map given by }$$

$$\beta(u) = \frac{\int_\Omega x \cdot |\nabla u(x)|^2 \, dx}{\int_\Omega |\nabla u(x)|^2 \, dx},$$

we can rephrase [1, Lemma 3.4] as

**Lemma 3.1.** For any fixed $p \in (2, 2^*)$ there exist $\overline{\lambda} = \overline{\lambda}(p)$ such that,

(i) $m_{\lambda,r} < 2m_\lambda,$

(ii) if $u \in N_\lambda$ and $E_\lambda(u) \leq m_{\lambda,r}$, then $\beta(u) \in \Omega^+_r$,

for all $\lambda > \overline{\lambda}$.

We are now ready to present the proof of our main result.

**Proof of Theorem 1.1.** Let $p \in (2, 2^*)$ and $\overline{\lambda}$ be given by the Lemma 3.1. For any $\lambda > \overline{\lambda}$, since $2 < p < 2^*$, the even functional $E_\lambda$ satisfies the Palais-Smale condition at any level $c \in \mathbb{R}$. Thus, we can apply Theorem 2.6 to obtain $\tau_a$-cat$(N^r_\lambda \cap E^{2m_\lambda,r}_\lambda)$ pairs $\pm u_i$ of critical points of $E_\lambda$ restricted to $N^r_\lambda$ verifying

$$E_\lambda(\pm u_i) \leq 2m_{\lambda,r} < 4m_\lambda \leq 2m^{r}_\lambda,$$

where we used Lemma 3.1(i) and Lemma 2.1. It follows from Lemmas 2.2 and 2.3 that $\pm u_i$ are solutions of $(P^{r}_\lambda)$ which change sign exactly once.

It suffices now to check that

$$\tau_a \text{-cat}(N^r_\lambda \cap E^{2m_\lambda,r}_\lambda) \geq \tau \text{-cat}_{\Omega}(\Omega \setminus \Omega^r).$$

With this aim, we claim that there exist two maps

$$\Omega^{-}_r \xrightarrow{\alpha_\lambda} N^r_\lambda \cap E^{2m_\lambda,r}_\lambda \xrightarrow{\gamma_\lambda} \Omega^+_r$$

such that $\alpha_\lambda(\tau x) = -\alpha_\lambda(x)$, $\gamma_\lambda(-u) = \tau \gamma_\lambda(u)$, and $\gamma_\lambda \circ \alpha_\lambda$ is equivariantly homotopic to the inclusion map $\Omega^{-}_r \hookrightarrow \Omega^+_r$.

Assuming the claim and recalling that the maps $\Omega^{-}_r \hookrightarrow \Omega \setminus \Omega^r$ and $\Omega \hookrightarrow \Omega^+_r$ are equivariant homotopy equivalences, we can use Lemma 2.5 to get

$$\tau_a \text{-cat}(N^r_\lambda \cap E^{2m_\lambda,r}_\lambda) \geq \tau \text{-cat}_{\Omega^+_r}(\Omega^-_r) = \tau \text{-cat}_{\Omega}(\Omega \setminus \Omega^r).$$
In order to prove the claim we follow [9]. Let $v_\lambda \in \mathcal{N}_{\lambda, B_r(0)}$ be a positive radial function such that $E_{\lambda, B_r(0)}(v_\lambda) = m_{\lambda,r}$. We define $\alpha_\lambda : \Omega^- \rightarrow E_{\lambda}^\ast \cap E_{\lambda}^{2m_{\lambda,r}}$ by

$$\alpha_\lambda(x) = v_\lambda(x) - v_\lambda(\cdot - \tau x). \quad (3.1)$$

It is clear that $\alpha_\lambda(\tau x) = -\alpha_\lambda(x)$. Furthermore, since $v_\lambda$ is radial and $\tau$ is an isometry, we have that $\alpha_\lambda(x) \in H^\ast$. Note that, for every $x \in \Omega^-$, we have $|x - \tau x| \geq 2r$ (if this is not true, then $\bar{x} = (x + \tau x)/2$ satisfies $|x - \bar{x}| < r$ and $\tau \bar{x} = \bar{x}$, contradicting the definition of $\Omega^-$). Thus, we can check that $E_\lambda(\alpha_\lambda(x)) = 2m_{\lambda,r}$ and $\alpha_\lambda(x) \in \mathcal{N}_{\lambda}^\ast$. All this considerations show that $\alpha_\lambda$ is well defined.

Given $u \in \mathcal{N}_{\lambda}^\ast \cap E_{\lambda}^{2m_{\lambda,r}}$ we can use (2.1) and the $\tau$-invariance of $\Omega$ to conclude that $u^+ \in \mathcal{N}_{\lambda}$ and $2E_\lambda(u^+) = E_\lambda(u) \leq 2m_{\lambda,r}$. Hence, $u^+ \in \mathcal{N}_{\lambda}^\ast \cap E_{\lambda}^{m_{\lambda,r}}$ and it follows from Lemma 3.1(ii) that $\gamma_\lambda : \mathcal{N}_{\lambda}^\ast \cap E_{\lambda}^{2m_{\lambda,r}} \rightarrow \Omega_\lambda^+$ given by $\gamma_\lambda(u) = \beta(u^+)$ is well defined. A simple calculation shows that $\gamma_\lambda(-u) = \tau \gamma_\lambda(u)$. Moreover, using (3.1) and the fact that $v_\lambda$ is radial we get

$$\gamma_\lambda(\alpha_\lambda(x)) = \frac{\int_{B_r(x)} y \cdot |\nabla v_\lambda(y - x)|^2 \, dy}{\int_{B_r(x)} |\nabla v_\lambda(y - x)|^2 \, dy} = \frac{\int_{B_r(0)} (y + x) \cdot |\nabla v_\lambda(y)|^2 \, dy}{\int_{B_r(0)} |\nabla v_\lambda(y)|^2 \, dy} = x,$$

for any $x \in \Omega^-$. This concludes the proof. \qed

**Remark 3.2.** Arguing along the same lines of the above proof and using a version of Lemma 4.2 in [1] instead of Lemma 3.1, we can check that Theorem 1.1 also holds if $\lambda \geq 0$ is fixed and the exponent $p$ is sufficiently close to $2^\ast$.

**Proof of Corollary 1.2.** Let $\tau : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be given by $\tau(x) = -x$. It is proved in [9, Corollary 3] that our assumptions imply $\tau\text{-cat}(\Omega) \geq N$. Since $0 \notin \Omega$, $\Omega^\tau = \emptyset$. It suffices now to apply Theorem 1.1. \qed

**References**


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