IV TRANSFORMADAS DE FOURIER

IV.1 DA SÉRIE PARA A TRANSFORMADA

Seja a série de Fourier de uma função com período $T=2L$,

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{in\pi x/L} \quad , \quad c_n = \frac{1}{2L}\int_{-L}^{L} dx\, f(x)\, e^{-in\pi x/L}$$

O que desejamos agora é considerar uma função que não seja necessariamente periódica, ou seja, estender o intervalo para todo $\mathbb{R}$ tomando $L \to \infty$. Para isso vamos denotar

$$K = K(n) = \frac{n\pi}{L} \quad , \quad \Delta K = K(n+1) - K(n) = \frac{\pi}{L}$$

Então:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{iK(n)x} = \sum_{n=-\infty}^{+\infty} \left(\frac{c_n L}{\pi}\right) e^{iK(n)x} \Delta K$$

com:

$$\frac{c_n L}{\pi} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\, f(x)\, e^{-iK(n)x}$$

Vamos agora pensar em n como função de K, ou seja, $n = n(K) = KL/\pi$. Então

$$\frac{C_n L}{\pi} = C_{KL/\pi} \cdot \frac{L}{\pi} = C_L(K)$$

e

$$f(x) = \sum_{\frac{KL}{\pi}=-\infty}^{+\infty} C_L(K) e^{iKx} \Delta K \quad , \quad C_L(K) = \frac{1}{2\pi} \int_{-L}^{L} dx\, f(x) e^{-iKx}$$

Agora, uma vez que $\frac{\Delta K \cdot L}{\pi} = 1$, para $L \to \infty$ temos $\Delta K \to 0$ e a soma acima torna-se uma integral de Riemann. Portanto, para $L \to \infty$:

$$f(x) = \int_{-\infty}^{+\infty} C(K) e^{iKx} dK \quad , \quad C(K) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx\, f(x) e^{-iKx}$$

Finalmente, para tornar essas expressões "simétricas", vamos definir $F(K) = \sqrt{2\pi}\, C(-K)$. Assim

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(K) e^{-iKx} dK \\ F(K) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{iKx} dx \end{aligned}}$$

$F(K)$ = transformada de Fourier de $f(x)$

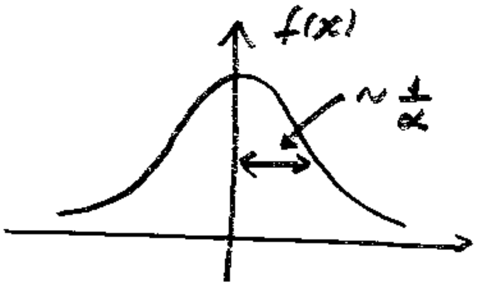
$f(x)$ = transformada de Fourier inversa de $F(K)$.

$F(K) = \mathcal{F}\{f(x)\}$

$f(x) = \mathcal{F}^{-1}\{F(K)\}$

EX

$$f(x) = N e^{-\alpha x^2} \quad (\alpha > 0)$$



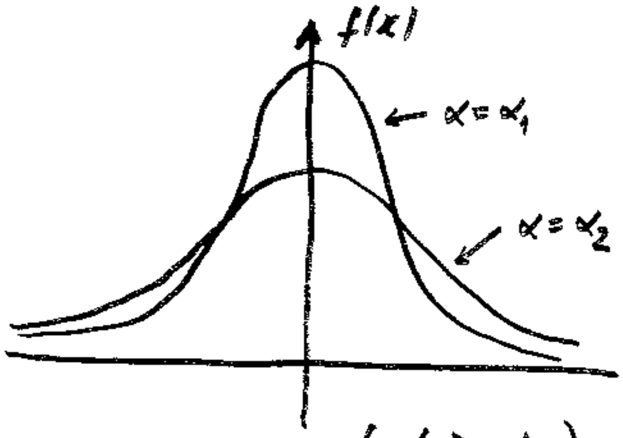
$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \left(N e^{-\alpha x^2}\right) e^{ikx} = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \; e^{-\alpha x^2 + ikx} =$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \; e^{-\left(\sqrt{\alpha}\, x - ik/2\sqrt{\alpha}\right)^2 - k^2/4\alpha} = \frac{N e^{-k^2/4\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \, e^{-\left(\sqrt{\alpha}\, x - \frac{ik}{2\sqrt{\alpha}}\right)^2}$$

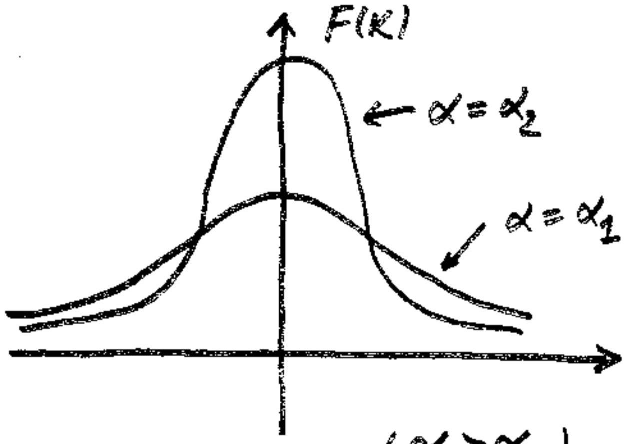
$$= \frac{N e^{-k^2/4\alpha}}{\sqrt{2\pi}} \, \frac{1}{\sqrt{\alpha}} \underbrace{\int_{-\infty}^{+\infty} du \, e^{-u^2}}_{\sqrt{\pi}} = \frac{N}{\sqrt{2\alpha}} \, e^{-k^2/4\alpha}$$

$$\underbrace{f(x) = N e^{-\alpha x^2}}_{\text{gaussiana}} \quad \longleftrightarrow \quad F(k) = \underbrace{N' e^{-\alpha' x^2}}_{\text{gaussiana}}$$

$$N' = \frac{N}{\sqrt{2\alpha}}, \quad \alpha' = \frac{1}{4\alpha}$$



$(\alpha_1 > \alpha_2)$



$(\alpha_1 > \alpha_2)$

Apenas para confirmar, vamos calcular $\mathcal{F}^{-1}$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \left(\frac{N}{\sqrt{2\alpha}} e^{-k^2/4\alpha} \right) e^{-ikx}$$

$$= \frac{N}{2\sqrt{\pi\alpha}} \underbrace{\int_{-\infty}^{+\infty} dk\, e^{-ikx} e^{-\left(\frac{1}{4\alpha}\right)k^2}}_{\frac{\sqrt{\pi}}{\sqrt{\frac{1}{4\alpha}}} e^{-(-x)^2/4\left(\frac{1}{4\alpha}\right)}} = \frac{N}{2\sqrt{\pi\alpha}} \sqrt{\pi}\, 2\sqrt{\alpha}\, e^{-x^2/\frac{1}{\alpha}} = N e^{-\alpha x^2}$$

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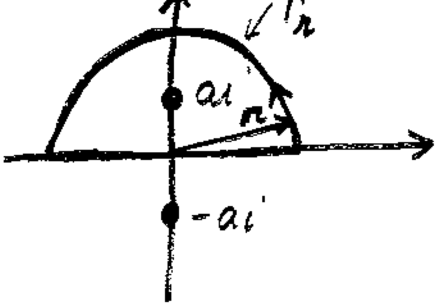
EX $\quad f(x) = \dfrac{a}{x^2+a^2} \quad (a>0)$

$$F(k) = \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, \frac{e^{ikx}}{x^2+a^2} = \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, \frac{e^{ikx}}{(x+ai)(x-ai)}$$

i) $k>0$

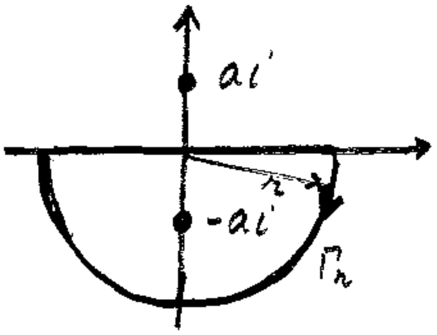


$$\operatorname*{Res}_{z\to ai} \frac{e^{ikz}}{(z+ai)(z-ai)} = \frac{e^{ikai}}{2ai} = \frac{e^{-ka}}{2ai}$$

$$\oint = \int_{\Gamma_r} + \int_{-r}^{r}, \quad \lim_{r\to\infty} \int_{\Gamma_r} = 0$$

$$\therefore\ F(k) = \frac{a}{\sqrt{2\pi}}\, 2\pi i\, \frac{e^{-ka}}{2ai} = \sqrt{\frac{\pi}{2}}\, e^{-ka} \qquad (k>0)$$

ii) $K<0$



$$\operatorname*{Res}_{z\to -ai} \frac{e^{iKz}}{(z+ai)(z-ai)} = \frac{e^{iK(-ai)}}{-2ai} = \frac{e^{Ka}}{-2ai}$$

$$\oint = \int_{\Gamma_r} + \int_{-r}^{r}, \quad \lim_{r\to\infty} \int_{\Gamma_r} = 0$$

$$\therefore F(K) = \frac{a}{\sqrt{2\pi}} (-2\pi i)\left(\frac{e^{Ka}}{-2ai}\right) = \sqrt{\frac{\pi}{2}}\, e^{-(-K)a} = \sqrt{\frac{\pi}{2}}\, e^{-|K|a}$$

$(K<0)$

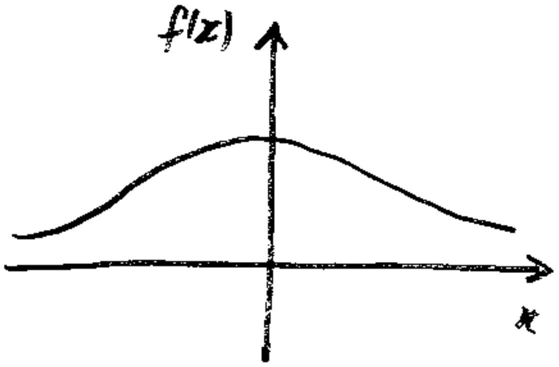
iii) $K=0$

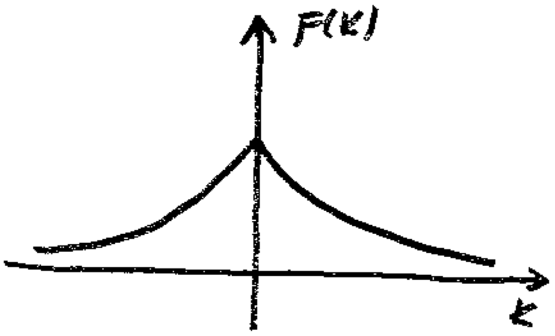
$$F(K) = \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \frac{1}{x^2+a^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d(x/a)}{(x/a)^2+1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{du}{1+u^2}$$

$$= \frac{1}{\sqrt{2\pi}} \tan^{-1} u \Big|_{-\infty}^{+\infty} = \frac{1}{\sqrt{2\pi}}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{\sqrt{2\pi}} = \sqrt{\frac{\pi}{2}}$$

$(K=0)$

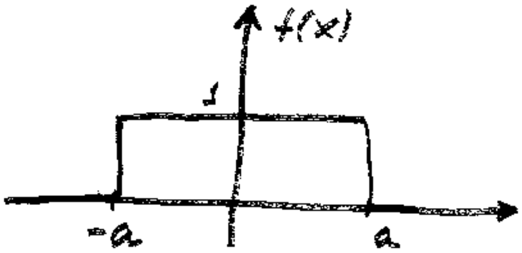
$$\therefore F(K) = \sqrt{\frac{\pi}{2}}\, e^{-|K|a}$$





EX

$$f(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases} \qquad (a>0)$$



$$F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, f(x)\, e^{iKx} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} dx\, e^{iKx} = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{iKx}}{iK} \right|_{-a}^{a} =$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{iKa} - e^{-iKa}}{iK} \right] = \sqrt{\frac{2}{\pi}}\, \frac{1}{K} \left[\frac{e^{iKa} - e^{-iKa}}{2i} \right] = \sqrt{\frac{2}{\pi}}\, \frac{\sin Ka}{K}$$

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EX

$$f(x) = \delta(x)$$

$$F(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, \delta(x)\, e^{iKx} = \frac{1}{\sqrt{2\pi}}\, e^{0} = \frac{1}{\sqrt{2\pi}}$$

$$\therefore\ F(K) = \frac{1}{\sqrt{2\pi}}$$

Se recorrêssemos às sequências delta:

$$F(K) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, \phi_n(x)\, e^{iKx} = \lim_{n\to\infty} \mathcal{F}\{\phi_n(x)\}$$

Tomando, por exemplo, $\phi_n(x) = \frac{n}{\sqrt{\pi}}\, e^{-n^2x^2}$, temos

$$\mathcal{F}\{\phi_n(x)\} = \frac{n}{\sqrt{\pi}}\, \frac{1}{\sqrt{2(n^2)}}\, e^{-K^2/4n^2} = \frac{1}{\sqrt{2\pi}}\, e^{-K^2/4n^2}$$

$$\therefore\ F(K) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}}\, e^{-K^2/4n^2} \quad \therefore\ F(K) = \frac{1}{\sqrt{2\pi}}$$

Uma outra alternativa é:

$$\phi_n(x) = \begin{cases} \frac{n}{2}, & |x| \leq \frac{1}{n} \\ 0, & |x| > \frac{1}{n} \end{cases}$$

de modo que

$$\mathcal{F}\{\phi_n(x)\} = \frac{n}{2} \cdot \sqrt{\frac{2}{\pi}} \frac{\sin K(\frac{1}{n})}{K} = \frac{n}{\sqrt{2\pi}} \frac{\sin K/n}{K}$$

$$F(K) = \lim_{n\to\infty} \mathcal{F}\{\phi_n(x)\} = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \frac{\sin(K/n)}{(K/n)} = \frac{1}{\sqrt{2\pi}} \quad \therefore F(K) = \frac{1}{\sqrt{2\pi}}$$

Consequentemente para a transformada inversa temos

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dK \left(\frac{1}{\sqrt{2\pi}}\right) e^{-iKx}$$

ou seja

$$\boxed{\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dK \, e^{-iKx}}$$

que é um resultado de grande importância.

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IV.2 FÓRMULA INTEGRAL DE FOURIER

Na seção I.5, pg. 19, estudamos o teorema de Fourier para séries. Considerando um período $T = 2L$, esse teorema garante que a série

$$f(x) = \frac{1}{2L}\int_{-L}^{L} d\xi\, f(\xi) + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} d\xi\, f(\xi) \cos\frac{n\pi}{L}(\xi - x)$$

$$= \lim_{N\to\infty} \int_{-L}^{L} d\xi\, f(\xi)\, \frac{\sin (N+1/2)\pi(\xi-x)/L}{2L \sin \pi(\xi-x)/2L}$$

converge para

$$\frac{1}{2}\left[f(x+0) + f(x-0)\right]$$

quando $f(x)$ é contínua por partes e com derivadas laterais em $(-L, L)$ e com período $2L$.

Vamos agora considerar o limite $L \to \infty$. Para isso vamos supor que exista a integral

$$\int_{-\infty}^{+\infty} dx\, |f(x)| < +\infty$$

Nesse caso, temos

$$\lim_{L\to\infty} \frac{1}{2L}\int_{-\infty}^{+\infty} d\xi\, f(\xi) = 0$$

e agora devemos ver o que acontece com a série.

Definindo, como na seção anterior,

$$K = \frac{n\pi}{L} \quad , \quad \Delta K = \frac{\pi}{L}$$

temos

$$\frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} d\xi\, f(\xi)\cos\frac{n\pi}{L}(\xi - x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\Delta K\int_{-L}^{L} d\xi\, f(\xi)\cos K(\xi - x)$$

Para $L \to \infty$ temos $\Delta K \to 0$ e a soma pode ser tomada por uma integral de Riemann de $K=0$ até $K=+\infty$ (pois $K=0$ para $L\to\infty$ e n fixo). Logo:

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} dK \int_{-\infty}^{+\infty} d\xi\, f(\xi)\cos K(\xi - x)$$

Essa é a fórmula integral de Fourier. Apesar da natureza heurística das argumentações acima, iremos mostrar agora que ela é de fato válida. Para isso precisamos estender o lema 4 da pg 29 de modo a incluir a integral ao longo de toda reta, ou seja, precisamos do seguinte:

• **LEMA**: Seja F uma função contínua por partes e com derivadas laterais à esquerda e à direita para todo intervalo real e tal que exista a integral $\int_{-\infty}^{+\infty} |F(x)|\,dx < +\infty$. Então

$$\lim_{K\to\infty} \int_{-\infty}^{+\infty} F(x) \frac{\sin K(x-x_0)}{x-x_0}\,dx = \pi \cdot \frac{F(x_0+0)+F(x_0-0)}{2}$$

DEM: Vamos denotar

$$G(x,x_0;K) = F(x)\frac{\sin K(x-x_0)}{x-x_0} = KF(x)\frac{\sin K(x-x_0)}{K(x-x_0)}$$

$$= KF(x)\,S(K(x-x_0))$$

Vimos durante a demonstração do lema 2 (pg. 24) que $|S(x)| \leq 1$ para $x \in \mathbb{R}$; logo:

$$|G(x,x_0;K)| \leq K|F(x)|$$

e

$$\int_{-\infty}^{+\infty} |G(x,x_0;K)|\,dx \leq K \int_{-\infty}^{+\infty} |F(x)|\,dx < +\infty$$

de modo que existe a integral $\int_{-\infty}^{+\infty} G(x,x_0;K)\,dx$. Seja

$$H(x_0;K) = \int_{-\infty}^{+\infty} G(x,x_0;K)\,dx - \frac{\pi}{2}\left[F(x_0+0)+F(x_0-0)\right]$$

Agora:

$$|H(x_0;K)| \leq \underbrace{\left|\int_{-\infty}^{a} G(x,x_0;K)\,dx\right|}_{|I_1|} + \underbrace{\left|\int_{a}^{b} G(x,x_0;K)\,dx - \frac{\pi}{2}\left[F(x_0+0)+F(x_0-0)\right]\right|}_{|I_2|} + \underbrace{\left|\int_{b}^{\infty} G(x,x_0;K)\,dx\right|}_{|I_3|}$$

Para I_1:

$$|I_1| \leq \int_{-\infty}^{a} |G(x,x_0;K)|\,dx = \int_{-\infty}^{a}\left|F(x)\frac{\sin K(x-x_0)}{x-x_0}\right|dx \leq \int_{-\infty}^{a}\frac{|F(x)|}{|x-x_0|}\,dx$$

Mas: $|x-x_0| = |x_0-x| \geq |x_0-a|$ para $-\infty < x \leq a$; logo:

$$|I_1| \leq \frac{1}{|x_0-a|}\int_{-\infty}^{a}|F(x)|\,dx$$

Como $I_1 \to 0$ para $a \to -\infty$, existe $a = a(\epsilon)$ tal que

$$|I_1| = \left|\int_{-\infty}^{a(\epsilon)} G(x,x_0;K)\,dx\right| < \frac{\epsilon}{3}$$

para $\forall \epsilon > 0$. Da mesma forma existe $b = b(\epsilon)$ tal que, $\forall \epsilon > 0$,

$$|I_3| = \left|\int_{b(\epsilon)}^{\infty} G(x,x_0;K)\,dx\right| < \frac{\epsilon}{3}.$$

Podemos ainda notar que tanto $|I_1|$ quanto $|I_3|$ não dependem de K. Já quanto a $|I_2|$, o lema 4 da pag. 29 diz que existe K_0 tal que

$$|I_2| = \left| \int_{a(\epsilon)}^{b(\epsilon)} G(x, x_0; K)\,dx - \frac{\pi}{2}\left[F(x_0+0) + F(x_0-0)\right] \right| < \frac{\epsilon}{3} \quad \text{para } K > K_0 .$$

Com isso:

$$|H(x_0; K)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \text{para } K > K_0,$$

ou seja,

$$\lim_{K\to\infty} H(x_0; K) = 0,$$

provando assim o lema. ✔

Agora podemos provar o seguinte:

☞ <u>TEOREMA</u>: Seja $f(x)$ contínua por partes e com derivadas laterais à esquerda e à direita em todo intervalo real e tal que $\int_{-\infty}^{+\infty} |f(x)|\,dx < +\infty$. Então vale a chamada <u>Fórmula</u> <u>integral</u> <u>de</u> <u>Fourier</u>:

$$\boxed{\frac{1}{\pi}\int_0^{+\infty} dK \int_{-\infty}^{+\infty} d\xi\, f(\xi)\cos K(\xi - x) = \frac{1}{2}\left[f(x+0) + f(x-0)\right]}$$

para $-\infty < x < +\infty$.

DEM: Do lema anterior:

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{\pi}\lim_{K\to\infty}\int_{-\infty}^{+\infty} d\xi\, f(\xi)\,\frac{\sin K(\xi-x)}{\xi-x}$$

$$= \frac{1}{\pi}\lim_{K\to\infty}\int_{-\infty}^{+\infty} d\xi\, f(\xi)\int_{0}^{K} d\alpha\, \cos\alpha(\xi-x)$$

Notando que

$$\left|\int_{-\infty}^{+\infty} d\xi\, f(\xi)\cos\alpha(\xi-x)\right| \leq \int_{-\infty}^{+\infty} d\xi\, |f(\xi)| = M < +\infty$$

de modo que pelo critério M de Weierstrass a integral $\int_{-\infty}^{+\infty} d\xi\, f(\xi)\cos\alpha(\xi-x)$ é absolutamente convergente para todo α, podemos inverter a ordem de integração para obter

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{\pi}\lim_{K\to\infty}\int_{0}^{K} d\alpha\int_{-\infty}^{+\infty} d\xi\, f(\xi)\cos\alpha(\xi-x)$$

$$= \frac{1}{\pi}\int_{0}^{\infty} d\alpha\int_{-\infty}^{+\infty} d\xi\, f(\xi)\cos\alpha(\xi-x)$$

✓

Vamos agora escrever a fórmula integral de Fourier de uma outra forma. Usando $\cos\phi = (e^{i\phi} + e^{-i\phi})/2$,

$$\frac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{\pi}\lim_{a\to\infty}\int_0^a dK \int_{-\infty}^{+\infty} d\xi\, f(\xi)\left[\frac{e^{iK(\xi-x)}+e^{-iK(\xi-x)}}{2}\right]$$

$$= \frac{1}{2\pi}\lim_{a\to\infty}\int_0^a dK \int_{-\infty}^{+\infty} d\xi\, f(\xi)\, e^{iK(\xi-x)} + \frac{1}{2\pi}\lim_{a\to\infty}\int_0^a dK \int_{-\infty}^{+\infty} d\xi\, f(\xi)\, e^{-iK(\xi-x)}$$

$$\| \leftarrow \boxed{K \leftrightarrow -K}$$

$$= \frac{1}{2\pi}\lim_{a\to\infty}\int_0^a dK \int_{-\infty}^{+\infty} d\xi\, f(\xi)\, e^{iK(\xi-x)} + \frac{1}{2\pi}\lim_{a\to\infty}\int_{-a}^0 dK \int_{-\infty}^{+\infty} d\xi\, f(\xi)\, e^{iK(\xi-x)}$$

$$= \frac{1}{2\pi}\lim_{a\to\infty}\int_{-a}^a dK \int_{-\infty}^{+\infty} d\xi\, f(\xi)\, e^{iK(\xi-x)} = \frac{1}{2\pi}\lim_{a\to\infty}\int_{-a}^a dK\, e^{-iKx} \int_{-\infty}^{+\infty} d\xi\, f(\xi)\, e^{iK\xi}$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^a dK\, F(K)\, e^{-iKx}$$

o que mostra que na transformada inversa a integral deve ser interpretada no sentido do <u>valor principal de Cauchy</u>:

$$\boxed{\mathcal{F}^{-1}[F(K)] = \frac{1}{\sqrt{2\pi}}\, PV\int_{-\infty}^{+\infty} dK\, F(K)\, e^{-iKx} = \frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^{a} dK\, F(K)\, e^{-iKx}}$$

EX

$$f(x) = \begin{cases} 0 & , x<0 \\ e^{-x} & , x>0 \end{cases}$$

$$\mathcal{F}[f(x)](K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{iKx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} e^{iKx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-x} e^{iKx}}{iK-1} \right|_0^{\infty} = \frac{1}{\sqrt{2\pi}} \frac{1}{1-iK} = \frac{1}{\sqrt{2\pi}} \frac{1+iK}{1+K^2} = F(K)$$

$$\mathcal{F}^{-1}[F(K)](x) = \,?$$

$$\mathcal{F}^{-1}[F(K)](0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dK \frac{1}{\sqrt{2\pi}} \frac{1+iK}{1+K^2} = \frac{1}{2\pi} \left[\int_{-\infty}^{+\infty} dK \frac{1}{1+K^2} + i \int_{-\infty}^{+\infty} dK \frac{K}{1+K^2} \right]$$

e vemos claramente na integral $\int_{-\infty}^{+\infty} dK \frac{K}{1+K^2} = \frac{1}{2} \ln(1+K^2) \Big|_{-\infty}^{+\infty}$ que devemos interpretar as integrais em termos do valor principal de Cauchy.

$$\int_{-\infty}^{+\infty} dK \frac{1}{1+K^2} = \lim_{a\to\infty} \int_{-a}^{a} dK \frac{1}{1+K^2} = \lim_{a\to\infty} \left[\tan^{-1} a - \tan^{-1}(-a) \right] =$$

$$= \lim_{a\to\infty} 2 \tan^{-1} a = 2 \cdot \frac{\pi}{2} = \pi$$

$$\int_{-\infty}^{+\infty} dK \frac{K}{1+K^2} = \lim_{a\to\infty} \int_{-a}^{a} dK \frac{K}{1+K^2} = \lim_{a\to\infty} \left[\frac{1}{2} \ln(1+a^2) - \frac{1}{2} \ln(1+(-a)^2) \right] = 0$$

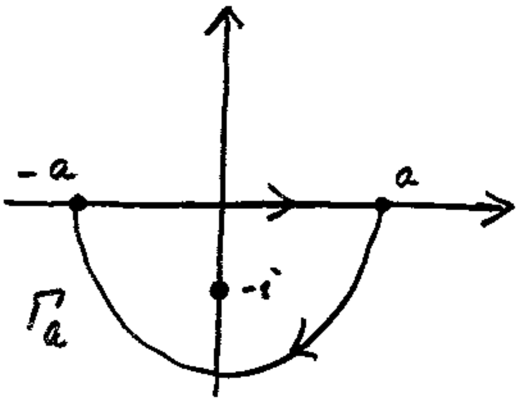
Logo:

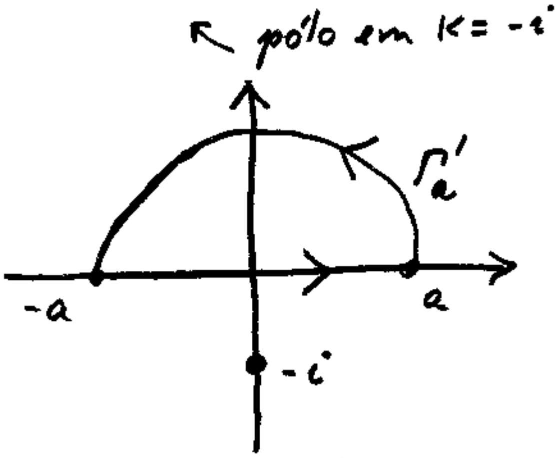
$$\mathcal{F}^{-1}[F(K)](0) = \frac{1}{2\pi} \cdot \pi = \frac{1}{2} = \frac{1}{2}\left[\lim_{x\to 0^+} f(x) + \lim_{x\to 0^-} f(x) \right] = \frac{1}{2}(1+0) = \frac{1}{2}$$

Já para $x \neq 0$,

$$\mathcal{F}^{-1}[F(K)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dK \frac{1}{\sqrt{2\pi}} \frac{1+iK}{1+K^2} e^{-iKx} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dK \frac{e^{-iKx}}{1-iK}$$

$$= \frac{1}{2\pi} \lim_{a\to\infty} \int_{-a}^{a} dK \frac{e^{-iKx}}{1-iK}$$





$$\lim_{a\to\infty} \int_{\Gamma_a} dK \frac{e^{-iKx}}{1-iK} = 0 \quad \text{se } x>0$$

$$\lim_{a\to\infty} \int_{\Gamma_a'} dK \frac{e^{-iKx}}{1-iK} = 0 \quad \text{se } x<0$$

$$\therefore x>0 \Rightarrow \mathcal{F}^{-1}[F(K)](x) = \frac{1}{2\pi}(-2\pi i) \underbrace{\operatorname*{Res}_{K=-i} \frac{e^{-iKx}}{1-iK}}_{ie^{-x}} = (-i).(i)e^{-x} = e^{-x}$$

$$x<0 \Rightarrow \oint = 0 \Rightarrow \mathcal{F}^{-1}[F(K)](x) = 0$$